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On the Dynamic Characteristic of Thermoelastic Waves in Thermoelastic Plates with Thermal Relaxation Times

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ABSTRACT

In this paper, analysis for the propagation of general anisotropic media of finite thickness with two thermal relaxation times is studied. Expression of displacements, temperature, thermal stresses, and thermal gradient for most general anisotropic thermoelastic plates of finite thickness are obtained in the analysis. The calculation is then carried forward for slightly more specialized case of a monoclinic plate. Dispersion relations for symmetric and antisymmetric wave modes are obtained. Thermoelastic plates of higher symmetry are contained implicitly in the analysis. Numerical solution of the frequency equation for a representative plate of assigned thickness is carried out, and the dispersion curves for the few lower modes are presented. Coupled thermoelastic thermal motions of the medium are found dispersive and coupled with each other due to the thermal and anisotropic effects. Some special cases have also been deduced and discussed.

Keywords: Anisotropic; Generalized thermoelasticity; Thermal relaxation times; Symmetric; Antisymmetric

1 INTRODUCTION

STUDIES of the propagation of elastic waves have long been of interest to researchers [1-6] in the fields of geophysics, acoustics, and nondestructive evaluation [7-10]. Fiber-reinforced composites have extremely advantageous engineering properties and applications including their relatively low weight, high stiffness and damage tolerance that make them very significant for aerospace and other recent structural applications. This interest has been encouraged by the recent expansion of the use of composite materials which are anisotropic in nature have wide variety of applications [11]. In comparison with the reasonable rich literature on the interaction of plane harmonic waves in anisotropic media, very little work is available on the response of such media to concentrated source loadings.

Classical theory [12, 13] of thermal conduction, based on the Fourier law, implies an immediate response to a temperature gradient and leads to a parabolic differential equation for the evolution of the temperature. In contrast, when thermal relaxation effects are taken into account in the constitutive equation describing the heat flux, as, for instance, in the Maxwell Cattaneo equation, one has a hyperbolic equation which implies a finite speed for heat transport. Extensive literature survey on the subject can be found in the review articles by [14, 15]. While several models have been developed to incorporate the hyperbolic heat conduction equation into thermoelasticity theory [16], two models which are considered landmarks in the field of generalized thermoelasticity are described here.

A single time constant to dictate the relaxation of thermal propagation, as well as the rate of change of strain rate and the rate of change of heat generation introduced in [17]. This coupled generalized thermoelasticity model as

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described by Lord and Shulman, henceforth referred to as the LS Model, and on the other hand, [18] (referred to as the GL theory) extended the coupled theory of thermoelasticity by introducing two thermal relaxation times in the constitutive equations, in which the thermal and thermo-mechanical relaxations are governed by two different time constants. It can also be seen that not only does the mechanical field depend on the heat flux but also on the temporal gradient of heat flux. Therefore, the GL model predicts a finite wave propagation speed for both the thermal and mechanical fields. In comparison to the classical theory, there is a strong coupling between the thermal and mechanical fields.

Generalized theory of thermoelasticity extended to anisotropic heat conducting elastic materials by [19] and [20], several others authors [21-27] have considered the propagation of generalized thermoelastic waves in plates with one or two thermal relaxation times. The propagation of thermoelastic waves in anisotropic periodically laminated composites [28], considering a single thermal relaxation time is studied. Wave propagation in plates of general anisotropic media in generalized thermoelasticity considering with a single thermal relaxation time is studied in [29] and this work is extended to study the propagation of waves in layered anisotropic media [30]. Thermoelastic wave's problems in thin plates considered with one or two thermal relaxation times by [31-32]. The generalized coupled thermoporoelasticity model of hollow and solid spheres under radial symmetric loading condition is considered by [33].

In this manuscript, analysis for the propagation of generalized thermoelastic waves in thermoelastic plates of general anisotropic media of finite thickness in Lord and Shulman and Green and Lindasy theories of thermoelasticity is considered and studied. Expressions for displacements, temperature, thermal stresses, and thermal gradient for the case of most general anisotropic thermoelastic plates of finite thickness are obtained. Calculations are then carried forward for a more specialized case of a monoclinic plate and dispersion relations for this case in closed form and separate the mathematical conditions for symmetric and antisymmetric are obtained. Thermoelastic plates of orthotropic, transversely isotropic, cubic and isotropic are contained implicitly in the analysis Numerical solution of the frequency equations for plate of assigned thickness is carried out, and the dispersion curves for the few lower modes are presented for a representative thermoelastic plate. Coupled thermoelastic thermal motions of the medium are found dispersive and coupled with each other due to the thermal and anisotropic effects. Some special cases such as coupled thermoelasticity and classical have also been deduced and discussed.

2 MATHEMATICAL FORMULATION

Consider an infinite, generally anisotropic thermoelastic plate at uniform temperature T_0, having thickness $2d$, whose normal is aligned with the x_3- axis of a reference Cartesian coordinate system $x_i = (x_1, x_2, x_3)$. The mid-plane of the plate is chosen to coincide with the $x_1 - x_2$ plane. The basic field equation of generalized thermoelasticity with two thermal relaxation times in the absence of body forces and heat sources of the plate are [27]

$$\left[C_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} - \beta_{ij}\frac{\partial T}{\partial x_j}\right] - \tau_1 \beta_{ij}\frac{\partial \dot{T}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2}, \tag{1}$$

$$K_{ij}\frac{\partial^2 T}{\partial x_i \partial x_j} - \rho C_e\left(\frac{\partial T}{\partial t} + \tau_0\frac{\partial^2 T}{\partial t^2}\right) = T_0\beta_{ij}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial t}\right), \tag{2}$$

$$\beta_{ij} = C_{ijkl}\alpha_{kl}\ , \qquad i, j, k, l = 1, 2, 3. \tag{3}$$

Parameters in Eqs. (1) and (2) are: ρ is the density, t is the time, u_i is the displacement in the x_i direction, K_{ij} are the thermal conductivities, C_e and τ_0 are respectively the specific heat at constant strain, and thermal relaxation time, σ_{ij} and e_{ij} are the stress and strain tensor respectively; β_{ij} are thermal moduli; α_{ij} is the thermal expansion tensor; T is temperature; and the fourth order tensor of the elasticity C_{ijkl} satisfies the (Green) symmetry conditions:

$$C_{ijkl} = C_{klij} = C_{ijlk} = C_{jikl}, \qquad K_{ij} = K_{ji}, \qquad \beta_{ij} = \beta_{ji}. \tag{4}$$

The parameter τ_1 and τ_0 are the thermal-mechanical relaxation time and the thermal relaxation time of the GL theory and satisfy the inequality $\tau_1 \geq \tau_0 \geq 0$.

The thermal conductivity tensor K_{ij} is symmetric and positive-definite. The thermoelastic coupling tensor β_{ij} is non-singular. The specific heat C_e at constant strain is positive. The isothermal linear elasticities are positive-definite in the sense that

$$C_{ijkl}\, e_{ij}\, e_{kl} > 0. \tag{5}$$

3 ANALYSIS

Assume that solutions of Eqs. (1) and (2) are expressed by

$$(u_j, T) = (U, V, W, \Theta)\exp\left[\mathrm{i}\xi(n_1x_1 + n_2x_2 + n_3x_3 - ct)\right], \qquad \mathrm{i} = \sqrt{-1}, \qquad j = 1,2,3. \tag{6}$$

where ξ is the wave number, c is the phase velocity ($=\omega/\xi$), ω is the circular frequency, U, V, W and Θ are the constants related to the amplitudes of displacement and temperature, n_1, n_2, n_3 are the components of the unit vector giving the direction of propagation.

Substituting Eq. (6) into Eqs. (1) and (2), we have

$$(\Gamma_{ik} - \rho c^2\delta_{ik})U_k + c\beta_{ij}(\mathrm{i}\omega^{-1} + \tau_1)n_j\Theta = 0, \tag{7}$$

$$cT_0\beta_{ij}n_jU_j + (\mathrm{K}_{ij}n_in_j - \rho c^2 C_e\tau)\Theta = 0, \tag{8}$$

where

$$U_j = (U, V, W), \qquad \tau = i\omega^{-1} + \tau_0, \qquad \tau^* = i\omega^{-1} + \tau_1, \tag{9}$$

and δ_{ik} is the Kronecker delta, and Γ_{ik} is the isothermal acoustic tensor defined as follows:

$$\Gamma_{ik} = \Gamma_{ki} = C_{ijkl}n_jn_l. \tag{10}$$

Now Eqs. (7) and (8) provide a non-trivial solution for U_j and Θ if the determinant of their coefficients vanishes. This leads to

$$\rho c^2 \det\left[\bar{\Gamma}_{ik} - \rho c^2\delta_{ik}\right] + \iota\omega\left\{KC_e^{-1}\det[\Gamma_{ik} - \rho c^2\delta_{ik}] - \tau_0\rho c^2\det\left[\bar{\Gamma}_{ik} - \rho c^2\delta_{ik}\right]\right\} = 0, \tag{11}$$

where

$$K = K_{ij}n_jn_l, \qquad \bar{\Gamma}_{ik} = \frac{\tau}{\tau *}\Gamma_{ik} + \frac{T_0\beta_{ip}\beta_{kq}n_pn_q}{\rho C_e}. \tag{12}$$

Are the effective thermal conductivity for linear heat flow in the direction of **n** and the isentropic acoustical tensor [5] and in $\det[A_{ik} - \rho c^2\delta_{ij}]$,

$$A_{ik} = \Gamma_{ik} \quad \text{or} \quad \bar{\Gamma}_{ik}. \tag{13}$$

For an infinite anisotropic body, when n_i are given, four phase velocities can be obtained by Eq. (11). For the finite thickness plate, we can obtain n_3 from Eq. (11), when n_1 and n_2 are given. On the other hand, this equation can be written as bi-quadratic polynomial equation in n_3. Designate roots of this Eq. (11) as α_m^2 $(m = 1,2,3,4) = n_3^2$ and write the displacements and temperature as follows:

$$\begin{aligned} U &= \sum_{m=1}^{4} (U^{(2m-1)} E_m^+ + U^{(2m)} E_m^-), \\ V &= \sum_{m=1}^{4} (V^{(2m-1)} E_m^+ + V^{(2m)} E_m^-), \\ W &= \sum_{m=1}^{4} (W^{(2m-1)} E_m^+ + W^{(2m)} E_m^-), \\ \Theta &= \sum_{m=1}^{4} (\Theta^{(2m-1)} E_m^+ + \Theta^{(2m)} E_m^-), \end{aligned} \tag{14}$$

where $E_m^+ = \exp(i\xi\alpha_m n_3)$, $E_m^- = \exp(-i\xi\alpha_m n_3)$, $(m = 1,2,3,4)$ and $U^{(i)}, V^{(i)}, W^{(i)}$ and $\Theta^{(i)}$, $(i = 1,2,...8)$ are disposable constants. The sets of eight disposable constants for $U^{(i)}, V^{(i)}, W^{(i)}$ and $\Theta^{(i)}$ are not independent as they are coupled through the equation of motion and the equation of heat conduction. Using Eqs. (7) and (8) we obtain

$$\begin{aligned} U &= \sum_{m=1}^{4} q_{1(m)} (V^{(2m-1)} E_m^+ + V^{(2m)} E_m^-), \\ V &= \sum_{m=1}^{4} q_{2(m)} (V^{(2m-1)} E_m^+ + V^{(2m)} E_m^-), \\ W &= \sum_{m=1}^{4} q_{3(m)} (V^{(2m-1)} E_m^+ - V^{(2m)} E_m^-), \\ \Theta &= \sum_{m=1}^{4} q_{4(m)} (V^{(2m-1)} E_m^+ + V^{(2m)} E_m^-), \end{aligned} \tag{15}$$

where

$$q_{1(m)} = 1, \qquad q_{2(m)} = \frac{\Delta_1}{\Delta}, \qquad q_{3(m)} = \frac{\Delta_2}{\Delta}, \qquad q_{4(m)} = \frac{\Delta_3}{\Delta},$$

$$\Delta_1 = \begin{vmatrix} \Gamma_{11} - \rho c^2 & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{12} & \Gamma_{23} & \Gamma_{14} \\ \Gamma_{13} & \Gamma_{33} - \rho c^2 & \Gamma_{34} \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} \Gamma_{12} & \Gamma_{11} - \rho c^2 & \Gamma_{14} \\ \Gamma_{22} - \rho c^2 & \Gamma_{12} & \Gamma_{24} \\ \Gamma_{23} & \Gamma_{13} & \Gamma_{34} \end{vmatrix}, \tag{16}$$

$$\Delta_3 = \begin{vmatrix} \Gamma_{12} & \Gamma_{13} & \Gamma_{11} - \rho c^2 \\ \Gamma_{22} - \rho c^2 & \Gamma_{23} & \Gamma_{12} \\ \Gamma_{23} & \Gamma_{33} - \rho c^2 & \Gamma_{13} \end{vmatrix}, \qquad \Delta = \begin{vmatrix} \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ \Gamma_{23} & \Gamma_{33} - \rho c^2 & \Gamma_{34} \end{vmatrix}. \tag{17}$$

Now Eqs. (15) and (16) satisfy Eq. (7) and contain eight undefined constants. From Eqs. (15) and strain tensors can be expressed as follows:

$$e_{ik} = \sum_{m=1}^{4} p_{ik(m)} E_m \exp[\imath\xi(n_1 x_1 + n_2 x_2 - ct)], \qquad \text{where} \qquad E = \exp(i\xi\alpha_m x_3) \tag{18}$$

$$p_{ik(m)} = \frac{i\xi(n_k q_{i(m)} + n_i q_{k(m)})}{2}, \qquad (i,k = 1,2,3) \tag{19}$$

The stress tensor and temperature gradient are

$$\sigma_{ik} = \sum_{m=1}^{4} r_{ik(m)} \left[V^{(2m-1)} E_m^{+} + (-1)\delta_{ik} V^{(2m)} E_m^{-} \right], \qquad \text{where} \qquad \delta_{ik} = \begin{cases} -1 & \text{if } i = k \\ \ \ 1 & \text{if } i \neq k \end{cases} \tag{20}$$

$$\frac{\partial T}{\partial x_3} = \sum_{m=1}^{4} \Omega_{(m)} \left[V^{(2m-1)} E_m^{+} - V^{(2m)} E_m^{-} \right], \tag{21}$$

where

$$r_{ik(m)} = C_{ijkl} p_{pq(m)}, (i,k,p,q = 1,2,3), \qquad \Omega_l = \iota \xi \alpha_l \Theta_l. \tag{22}$$

It should also be noted that triclinic materials (and indeed all materials) are, of course, invariant under the identity operation.

4 MATERIAL SYSTEMS OF ADVANCED SYMMETRY

Monoclinic materials are materials having one plane of mirror symmetry. For a heat conducting monoclinic materials having x_1-x_2 as a plane of mirror symmetry, and x_1-x_3 as the plane of incidence, then via the equation of motion and heat conduction for monoclinic plate, solutions of Eqs. (6) with $(n_1, 0, n_3) = (n_1, 0, \alpha)$, where $n_1 = \sin\theta$, θ is an angle of incidence and α is still an unknown parameters, U, V, W and Θ are respectively the amplitudes of the displacements u_1, u_2, u_3 and temperature T. Although solutions are explicitly independent of x_2, an implicit dependence is contained in the transformation and the transverse displacement component u_2 is non-vanishing in Eqs. (7). The choice of solutions leads to four coupled equations

$$M_{mn}(\alpha) U_n = 0, \qquad m,\ n = 1,2,3,4 \tag{23}$$

where

$$\begin{aligned}
&M_{11} = n_1^2 - \zeta^2 + c_2\alpha^2, \quad M_{12} = c_4 n_1^2 + c_5\alpha^2, \quad M_{13} = c_7 n_1 \alpha, \quad M_{14} = n_1, \\
&M_{22} = c_3 n_1^2 - \zeta^2 + c_6\alpha^2, \quad M_{23} = c_8 n_1 \alpha, \quad M_{24} = \bar{\beta}_6 n_1, \quad M_{33} = c_2 n_1^2 - \zeta^2 + c_1\alpha^2, \\
&M_{34} = \bar{\beta}_3 \alpha, \quad M_{41} = \varepsilon_1 \tau_g \omega_1^* \zeta^2 n_1, \quad M_{42} = \varepsilon_1 \tau_g \bar{\beta}_6 \omega_1^* \zeta^2 n_1, \quad M_{43} = \varepsilon_1 \tau_g \omega_1^* \zeta^2 \bar{\beta}_3 \alpha, \\
&M_{44} = n_1^2 - \tau \omega_1^* \zeta^2 + \bar{K}\alpha^2,
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&c_1 = \frac{c_{33}}{c_{11}}, \quad c_2 = \frac{c_{55}}{c_{11}}, \quad c_3 = \frac{c_{66}}{c_{11}}, \quad c_4 = \frac{c_{16}}{c_{11}}, \quad c_5 = \frac{c_{45}}{c_{11}}, \\
&c_6 = \frac{c_{44}}{c_{11}}, \quad c_7 = \frac{(c_{13} + c_{55})}{c_{11}}, \quad c_8 = \frac{(c_{36} + c_{45})}{c_{11}}, \\
&\bar{\beta}_6 = \frac{\beta_6}{\beta_1}, \quad \bar{\beta}_3 = \frac{\beta_3}{\beta_1}, \quad \bar{K} = \frac{K_3}{K_1}, \quad \varepsilon_1 = \frac{T_0 \beta_1^2}{\rho C_e c_{11}}, \\
&\omega_1^* = \frac{C_e c_{11}}{K_1}, \quad \zeta^2 = \frac{c^2 \rho}{c_{11}}, \quad \tau_g = \tau_1 + \frac{i}{\xi c}, \quad \tau = \tau_0 + \frac{i}{\xi c}.
\end{aligned} \tag{25}$$

For Eq. (23) have a non-trivial solution if the determinant of the coefficients of U, V, W and Θ vanishes, which yields an algebraic equation relating α to c. We obtain an eighth-degree polynomial equation in α which can be written as

$$C_0\alpha^8 + C_1\alpha^6 + C_2\alpha^4 + C_3\alpha^2 + C_4 = 0, \tag{26}$$

where C_0, C_1, C_2 , C_3 and C_4 are coefficients of polynomial. Eq. (26) admits eight solutions for α (having the properties)

$$\alpha_2 = -\alpha_1, \qquad \alpha_4 = -\alpha_3, \qquad \alpha_6 = -\alpha_5, \qquad \alpha_8 = -\alpha_7. \tag{27}$$

For each α_m (m=1, 2, 3...8), we can use the relations (28) and the corresponding ratios $q_{1(m)}, q_{2(m)}, q_{3(m)}$ and $q_{4(m)}$ are

$$q_{1(m)} = 1, q_{2(m)} = \frac{R_1(\alpha_m)}{R(\alpha_m)}, \qquad q_{3(m)} = \frac{R_2(\alpha_m)}{R(\alpha_m)}, \qquad q_{4(m)} = \frac{R_3(\alpha_l)}{R(\alpha_l)}. \tag{28}$$

Specializing the equation for thermoelastic monoclinic material, we have

$$\begin{aligned} r_{33(m)} &= \mathrm{i}\xi[(c_7 - c_2)n_1 + (c_8 - c_5)q_{2(m)}n_1 + c_1\alpha_m q_{3(m)} + c_{11}^{-1}c\tau_g\beta_3 q_{4(m)}], \\ r_{13(m)} &= \mathrm{i}\xi[c_2(\alpha_m + q_{3(m)}n_1) + c_5\alpha_m q_{2(m)}], \\ r_{23(m)} &= \mathrm{i}\xi[c_5(\alpha_m + q_{3(m)}n_1) + c_6\alpha_m q_{2(m)}], \\ \Omega_m &= \mathrm{i}\xi\alpha_m q_{4(m)}, \ \ m = \ 1, \ 2, \ 3, \ ... \ 8. \end{aligned} \tag{29}$$

Incorporating Eqs. (27) in (29) and inspecting the resulting relations, we conclude that monoclinic symmetry implies the further restrictions.

$$q_{2(m+1)} = q_{2(m)}, \qquad q_{3(m+1)} = -q_{3(m)}, \qquad q_{4(m+1)} = q_{4(m)}, \qquad m = 1,3,5,7 \tag{30}$$

$$r_{33(\mathrm{m}+1)} = r_{33(.\mathrm{m})}, \quad r_{13(\mathrm{m}+1)} = -r_{13(\mathrm{m})}, \quad r_{23(\mathrm{m}+1)} = -r_{23(\mathrm{m})} \text{ and } \Omega_{m+1} = -\Omega_m. \tag{31}$$

5 DISPERSION RELATIONS

The dispersion relation associated with the plate is now derived from Eqs. (20) and (21) by applying traction free boundary conditions $\sigma_{13} = \sigma_{23} = \sigma_{33} = \partial T / \partial x_3 = 0$ at $x_3 = \pm d$ on each free surfaces,

$$\begin{aligned} &\sum_{m=1}^{4}(r_{33(m)}, r_{23(m)}, r_{13(m)}, \Omega_{(m)})(V^{(2m-1)}\mathrm{e}^{i\xi\alpha_m d} + V^{(2m)}\mathrm{e}^{-i\xi\alpha_m d}) = 0, \\ &\sum_{m=1}^{4}(r_{33(m)}, r_{23(m)}, r_{13(m)}, \Omega_{(m)})(V^{(2m-1)}\mathrm{e}^{-i\xi\alpha_m d} + V^{(2m)}\mathrm{e}^{i\xi\alpha_m d}) = 0, \end{aligned} \tag{32}$$

On simplifying above equations we have

$$\begin{aligned} &\sum_{m=1}^{4} r_{33(m)}(\tilde{V}_m^+ C_m + i\tilde{V}_m^- S_m) = 0, \qquad \sum_{m=1}^{4} r_{33(m)}(\tilde{V}_m^+ C_m - i\tilde{V}_m^- S_m) = 0, \\ &\sum_{m=1}^{4} r_{23(m)}(\tilde{V}_m^- C_m + i\tilde{V}_m^+ S_m) = 0, \qquad \sum_{m=1}^{4} r_{23(m)}(\tilde{V}_m^- C_m - i\tilde{V}_m^+ S_m) = 0, \\ &\sum_{m=1}^{4} r_{13(m)}(\tilde{V}_m^- C_m + i\tilde{V}_m^+ S_m) = 0, \qquad \sum_{m=1}^{4} r_{13(m)}(\tilde{V}_m^- C_m - i\tilde{V}_m^+ S_m) = 0, \\ &\sum_{m=1}^{4} \Omega_{(m)}(\tilde{V}_m^- C_m + i\tilde{V}_m^+ S_m) = 0, \qquad \sum_{m=1}^{4} \Omega_{(m)}(\tilde{V}_m^- C_m - i\tilde{V}_m^+ S_m) = 0. \end{aligned} \tag{33}$$

The symmetry of the plate allows us to simplify the system of eight homogeneous equations in eight unknowns into two systems of four equations in four unknowns, which on employing straight forward algebraic manipulations, yield the following relations associated with the plate

$$\sum_{m=1}^{4} r_{33(m)} \tilde{U}_m^+ C_m = 0, \qquad \sum_{m=1}^{4} (r_{23(m)}, r_{13(m)}, \Omega_{(m)}) \tilde{U}_m^+ S_m = 0, \tag{34}$$

and

$$\sum_{m=1}^{4} r_{33(m)} \tilde{U}_m^+ S_m = 0, \qquad \sum_{m=1}^{4} (r_{23(m)}, r_{13(m)}, \Omega_{(m)}) \tilde{U}_m^+ C_m = 0, \tag{35}$$

within which

$$C_m = \cos(\xi \alpha_m d), \;\; S_m = \sin(\xi \alpha_m d) \qquad \text{and} \qquad \tilde{U}_m^+ = V^{(2m-1)} + V^{(2m)}, \;\; \tilde{U}_m^- = V^{(2m-1)} - V^{(2m)}. \tag{36}$$

The condition that the system of Eqs. (34) and (35) admit a non-trivial solution give rise to the dispersion relations associated extensional with and flexural waves respectively.

6 ANTISYMMETRIC DISPERSION RELATION

The dispersion relation associated with anti symmetric waves is obtained by taking $V^{(2m-1)} = V^{(2m)}$, thus $\overline{u}_1$, $\overline{u}_2$, $\overline{u}_3$ and $\overline{T}$ have the form

$$\begin{aligned} &\overline{u}_1 = 2\sum_{m=1}^{4} V^{(2m)} C_m, \qquad \overline{u}_2 = 2\sum_{m=1}^{4} q_{2(m)} V^{(2m)} C_m, \qquad \overline{u}_3 = 2i\sum_{m=1}^{4} q_{3(m)} V^{(2m)} S_m, \\ &\overline{T} = 2\sum_{m=1}^{4} \Theta_m V^{(2m)} C_m. \end{aligned} \tag{37}$$

Therefore, system of Eqs. (34)-(35) admit a non-trivial solution provided the determinant of coefficients associated with these equations vanishes, which after a little and straight forward algebraic manipulation, may cast in the form

$$\sum_{j=1,3,5,7} (-1)^{\left(\frac{j+1}{2}\right)-1} r_{33(j)} G_k \tan(\gamma \alpha_j) = 0. \tag{38}$$

Corresponding to symmetric and anti symmetric thermoelastic modes, respectively, with

$$\begin{aligned} &G_1 = \det\begin{pmatrix} r_{13(3)} & r_{13(5)} & r_{13(7)} \\ r_{23(3)} & r_{23(5)} & r_{23(7)} \\ \Omega_3 & \Omega_5 & \Omega_7 \end{pmatrix}, \qquad G_3 = \det\begin{pmatrix} r_{13(1)} & r_{13(5)} & r_{13(7)} \\ r_{23(1)} & r_{23(5)} & r_{23(7)} \\ \Omega_1 & \Omega_5 & \Omega_7 \end{pmatrix}, \\ &G_5 = \det\begin{pmatrix} r_{13(1)} & r_{13(3)} & r_{13(7)} \\ r_{23(1)} & r_{23(3)} & r_{23(7)} \\ \Omega_1 & \Omega_3 & \Omega_7 \end{pmatrix}, \qquad G_7 = \det\begin{pmatrix} r_{13(1)} & r_{13(3)} & r_{13(5)} \\ r_{23(1)} & r_{23(3)} & r_{23(5)} \\ \Omega_1 & \Omega_3 & \Omega_5 \end{pmatrix}, \end{aligned} \tag{39}$$

$$\gamma = \frac{\xi d}{2} = \frac{\omega d}{2c}. \tag{40}$$

Thus far we have obtained characteristic Eq. (38) associated with flexural waves monoclinic materials in generalized thermoelasticity with two thermal relaxation times. These results are also valid for higher symmetry classes, such as orthotropic, transversely isotropic, cubic and isotropic.

7 SYMMETRIC DISPERSION RELATION

The dispersion relation associated with symmetric waves equation is obtained by taking $V^{(2m-1)} = -V^{(2m)}$, and determinant of coefficients of Eqs. (34)-(35) yields the dispersion relation associated with extensional waves, namely

$$\sum_{j=1,3,5,7} (-1)^{\left(\frac{j+1}{2}\right)-1} r_{33(j)} G_k \cot(\gamma\alpha_j) = 0. \tag{41}$$

Thus $\bar{u}_1, \bar{u}_3$ and $\bar{T}$ have the form

$$\begin{aligned} &\bar{u}_1 = -2i\sum_{m=1}^{4} V^{(2m)} S_m, \qquad \bar{u}_2 = -2i\sum_{m=1}^{4} q_{2(m)} V^{(2m)} S_m, \qquad \bar{u}_3 = -2\sum_{m=1}^{4} q_{3(m)} V^{(2m)} C_m, \\ &\bar{T} = -2i\sum_{m=1}^{4} \Theta_m V^{(2m)} S_m \end{aligned} \tag{42}$$

where G_1, G_3, G_5 and G_7, are defined in Eqs. (39). Thus so far we have obtained characteristic Eq. (41) associated with extensional waves monoclinic materials in generalized thermoelasticity with two thermal relaxation times. These results are also valid for higher symmetry classes, such as orthotropic, transversely isotropic, cubic and isotropic.

8 MATERIALS OF ADVANCED SYMMETRY CLASSES

Higher symmetry class's materials, such as orthotropic, transversely isotropic, cubic and isotropic, possess two orthogonal axes of symmetry in the plane of the plate, therefore taking advantage of simplifications in the definitions of the M_{ij} in Eq. (24), characteristic equation for higher symmetry classes are obtained as special cases. For off-principal axes propagation, one needs to assume that further appropriate restrictions on the number of non-zero thermoelastic constants of the monoclinic case. If x_1 and x_2 are chosen to coincide with the in-plane principal axes for orthotropic symmetry, then we have

$$c_{j6} = 0, \qquad c_{45} = 0, \qquad \alpha_{12} = 0. \qquad (j = 1,2,3) \tag{43}$$

Results for possessing transverse isotropy, whose x_1 axis is normal to the plane of isotropy, can be easily obtained by the additional conditions imposed by symmetry for cubic symmetry and finally, for the isotropic case.

Particularizing M_{ij} of Eq. (24) for orthotropic media in this case, inspection of the resulting entries leads to the conclusion that, for propagation along rotational symmetry axes, the matrix elements $c_{16,}$ $c_{26,}$ $c_{36,}$ c_{45} and α_{12} also vanish. This simplification of the thermoelastic constants has implications for the analysis commencing at Eq. (24). Of greatest importance is the fact that M_{12}, M_{23} and M_{42} in Eq. (24) Vanish. This means that SH wave motion decouple from the rest of the motion, as a consequence, Eq. (23), to reduces to

$$\alpha^6 + A_1'\alpha^4 + A_2'\alpha^2 + A_3' = 0, \tag{44}$$

and

$$c_3 + c_6\alpha^2 - \zeta^2 = 0. \tag{45}$$

Notice that roots of Eq. (33) correspond to the SH motion, gives a purely transverse wave, which is not affected by the temperature. This wave propagates without dispersion or damping. Eq. (45) corresponds to the sagittal plane waves, and has been studied in detail Verma and Hasebe [34].

The results of Eqs. (38) and (41) constitute the characteristic equations for anti symmetric and symmetric modes for waves propagating in thermoelastic plate in generalized thermoelasticity with two thermal relaxations time. The wave types uncouple since the wave vector is along the axis of symmetry. Furthermore, the relation Eqs. (38) and (41) implicitly includes corresponding results for higher materials, one only needs to exploit the appropriate restrictions on the thermoelastic properties .

9 SPECIAL CASES

Classical case: In this case the thermo-mechanical coupling constant ε_1 is identically zero, and therefore from Eq. (31) we have

$$M_{41} = M_{42} = M_{43} = 0. \tag{46}$$

The Eq. (31) with the help of Eq. (46) reduces to,

$$M_{44}(\Delta_E \alpha^6 + A_{E1}\alpha^4 + A_{E2}\alpha^2 + A_{E3}) = 0, \tag{47}$$

where Δ_E, A_{E1}, A_{E2} and A_{E3} are defined in the Appendix, and Eq.(47) gives us

$$M_{44} = n_1^2 - \tau\omega_1^* \zeta^2 + \bar{K}\alpha^2 = 0, \tag{48}$$

$$\Delta_E \alpha^6 + A_{E1}\alpha^4 + A_{E2}\alpha^2 + A_{3E} = 0. \tag{49}$$

Eq. (48) implies $n_1^2 - \tau\omega_1^* \zeta^2 + \bar{K}\alpha^2 = 0,$ which corresponds to the pure thermal wave, evidently it is influenced by the thermal relaxation time τ_0 only and not based on LS theory. Secular Eq. (49) corresponds to the purely anisotropic elastic material, which is obtained and discussed by [34]. In the case of orthotropic materials, Eq. (44) reduces to,

$$c_1 c_2 \alpha^4 + (c_2 F_{33} - c_1 F_{11} - F_{13}^2)\alpha^2 + F_{11}F_{33} = 0, \tag{50}$$

$$n_1^2 - \tau\omega_1^* \zeta^2 + \bar{K}\alpha^2 = 0, \tag{51}$$

which corresponds to the thermal wave and is the same as above. Clearly, this again is influenced by the thermal relaxation time τ_0 only of LS theory and not τ_1 concerned in the GL theory, and Eq. (50) is a secular equation corresponds to a purely orthotropic elastic material [35].

Coupled thermoelasticity: when $\tau_0 = \tau_1 = 0,$ and hence we have $\tau = \tau_g = i/\omega$. This is the case of coupled thermoelasticity, proceeding on the same lines; we again arrived at frequency equations of the form that is in agreement with the corresponding result [1, 36]. If $\tau_1 = \tau_0 \neq 0,$ become the frequency equations in the theory of generalized thermoelasticity with one thermal relaxation time [35] for anisotropic media.

10 NUMERICAL DISCUSSION

Numerical illustrations of the analytical characteristic equations are presented in the form of dispersion curves. Dispersion and Damping curves are plotted wave number Vs phase velocity. To find the numerical solutions of a characteristic equation, analytic function is solved by assuming physical reference [36] of the numerical constants:

$c_{11} = 1.628\times10^{11}$ N m^{-2}, $c_{12} = 0.362\times10^{11}$ N m^{-2}, $c_{13} = 0.508\times10^{11}$ N m^{-2},
$c_{33} = 0.627\times10^{11}$ N m^{-2}, $c_{44} = 0.385\times10^{11}$ N m^{-2}, $\rho = 7.14\times10^{3}$ kg m^{-3},
$\beta_1 = 5.75\times10^{6}$ N m^{-2} deg^{-1}, $\beta_3 = 5.07\times10^{6}$ N m^{-2} deg^{-1}, $C_e = 3.9\times10^{2}$ J kg m^{-1} deg,
$K_1 = 1.24\times10^{2}$ W m^{-1} deg^{-1}, $K_3 = 1.24\times10^{2}$ W m^{-1} deg^{-1}, $T_0 = 296K$, $\varepsilon_1 = 0.0221$.

Dispersion curves in the forms of variations phase velocity (dimensionless $\Omega = \omega d / \sqrt{c_{11}/\rho}$) versus frequency (dimensionless $\gamma = \xi d/2$) are constructed at different values of times relaxation time ratios $\tau_1 = 4.0\times10^{-7}$s, 10.0×10^{-7}s, 20.0×10^{-7}s and $\tau_0 = 2.10^{-7}$s, considering $\tau_1 \geq \tau_0 \geq 0$, for the first six lower modes of zinc plate in Figs. 1-12 in the generalized theories (Green and Lindasy (GL) & Lord and Shulman (LS)) of thermoelasticity and Figs. 13 and 14 in the classical theory (CT) of thermoelasticity (no relaxation time).

Dispersive character of quasi-longitudinal(QL), quasi-transverse(QL), and quasi-thermal(T-mode), wave modes are demonstrated in Figs. 1, 5 and 9 for anti symmetric modes, and in Figs. 3, 7 and 11 for symmetric modes in the Green and Lindasy GL) theory of thermoelasticity on considering the thermal relaxation time $\tau_1 = 4.0\times10^{-7}$s, 10.0×10^{-7}s, and 20.0×10^{-7}s and $\tau_0 = 2.10^{-7}$s, respectively. Likewise, in Figs. 2, 6, and 10 for anti symmetric, and in Figs. 4, 8 and 12 for symmetric modes, thermoelastic dispersive character of quasi-longitudinal, quasi-transverse and quasi-thermal waves are demonstrated on account of the Lord and Shulman (LS) theory of thermoelasticity. When τ_0 and τ_1 are set equal to zero (no relaxation time), Figs. 13 for anti symmetric thermoelastic and Fig. 14 for symmetric thermoelastic modes are demonstrated in the classical theory (CT) of thermoelasticity. From the Figs it is observed that three waves namely, quasi-longitudinal (QL), quasi-transverse (QT) and quasi-thermal (T-mode) of the medium are found coupled with each other due to the thermal and anisotropic effects. The wave-like behavior of the quasi-thermal modes is characterized in the thermoelasticity theory in both theories. Lower symmetric and anti symmetric modes are found more influenced by the thermal relaxation times at low values of wave number. The effect of thermal relaxation times is observed to be small as the inclusion of thermal relaxation times increases the amount of dissipation. It is also observed that both the anti symmetric and symmetric wave's mode have certain features in common with their equivalents and these modes tend to uncoupled as the lower order anti symmetric and symmetric modes for large wave number and approach to the Rayleigh velocity, while higher modes asymptote to shear velocity for large wave number.

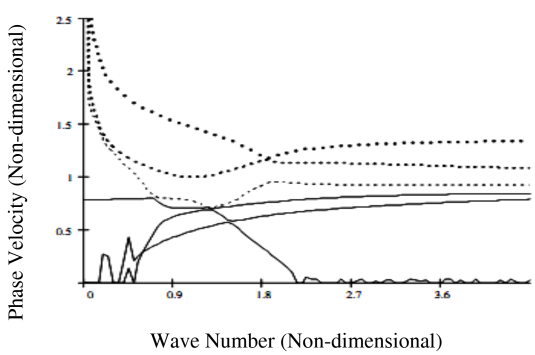

Fig. 1
Dispersion of anti symmetric wave modes in GL theory of generalized thermoelasticity when $\tau_0 = 2\times10^{-7}$s and $\tau_1 = 4\times10^{-7}$s.

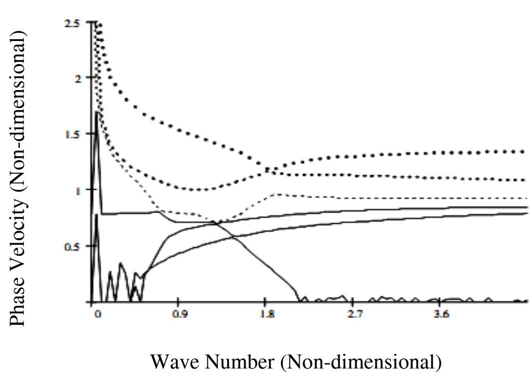

Fig. 2
Dispersion of anti symmetric wave modes in LS theory of generalized thermoelasticity when $\tau_0 = 2\times10^{-7}$s.

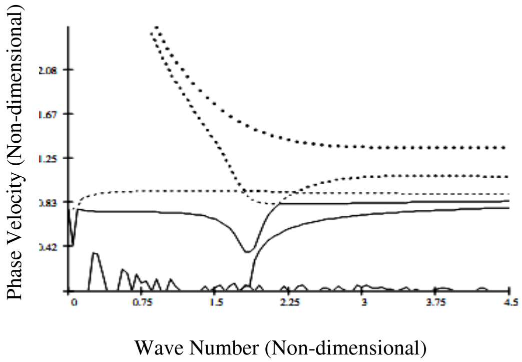

Fig. 3
Dispersion of symmetric wave modes in GL theory of generalized thermoelasticity when $\tau_0 = 2.10^{-7} s$ and $\tau_1 = 4.10^{-7} s$.

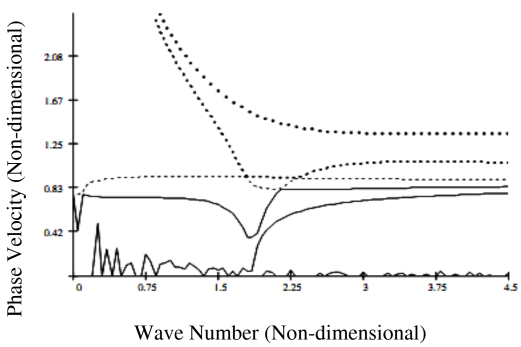

Fig. 4
Dispersion of symmetric wave modes in LS theory of generalized thermoelasticity when $\tau_0 = 2\times10^{-7}$s.

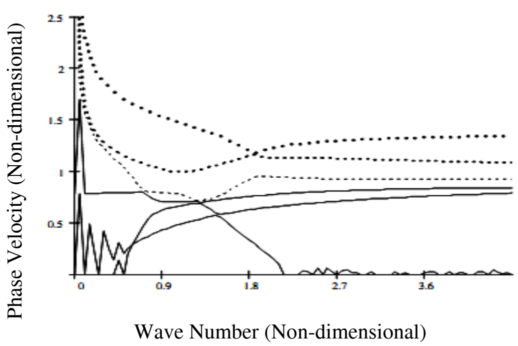

Fig. 5
Dispersion of anti symmetric wave modes in GL theory of generalized thermoelasticity when $\tau_0 = 2\times10^{-7}$s and $\tau_1 = 10\times10^{-7}$s.

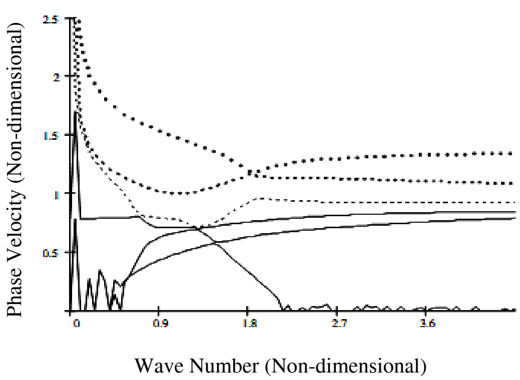

Fig. 6
Dispersion of anti symmetric wave modes in LS theory of generalized thermoelasticity when $\tau_0 = 4\times10^{-7}$s.

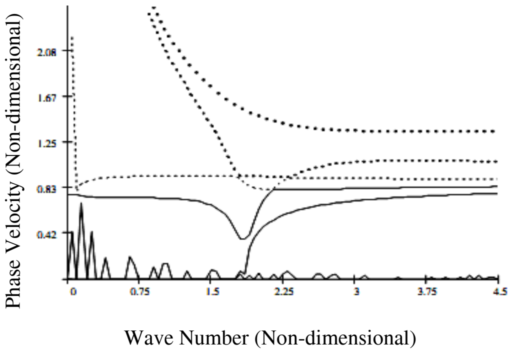

Fig. 7
Dispersion of symmetric wave modes in GL theory of generalized thermoelasticity when $\tau_0 = 2\times10^{-7}$s and $\tau_1 = 10\times10^{-7}$s..

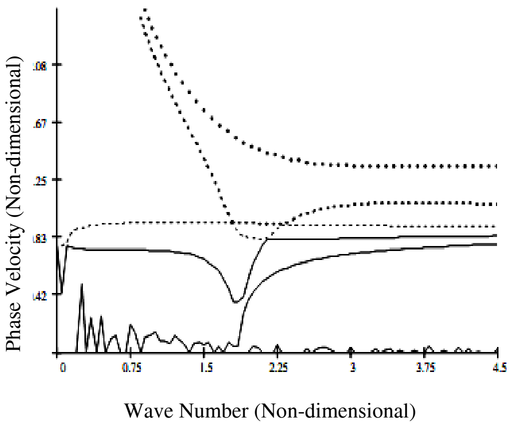

Fig. 8
Dispersion of symmetric wave modes in LS theory of generalized thermoelasticity when $\tau_0 = 4\times10^{-7}$s.

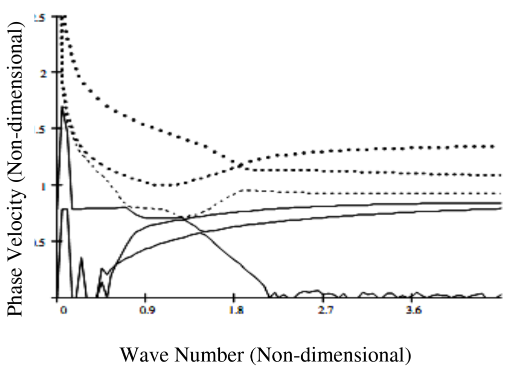

Fig. 9
Dispersion of anti symmetric wave modes in GL theory of generalized thermoelasticity when $\tau_0 = 2\times10^{-7}$s and $\tau_1 = 20\times10^{-7}$s.

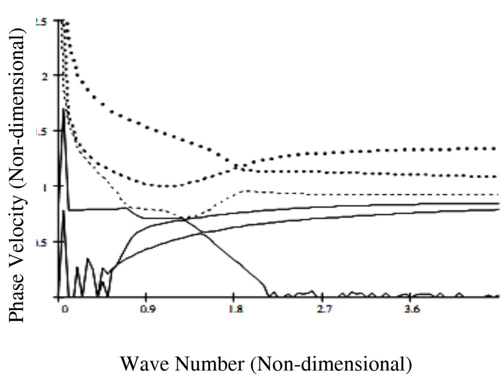

Fig. 10
Dispersion of anti symmetric wave modes in LS theory of generalized thermoelasticity when $\tau_0 = 10\times10^{-7}$s.

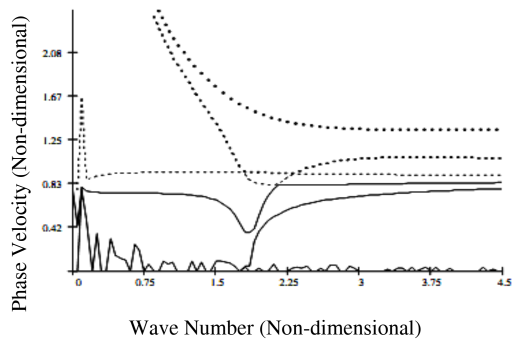

Fig. 11
Dispersion of anti symmetric wave modes in GL theory of generalized thermoelasticity when $\tau_0 = 2\times10^{-7}$s and $\tau_1 = 20\times10^{-7}$s.

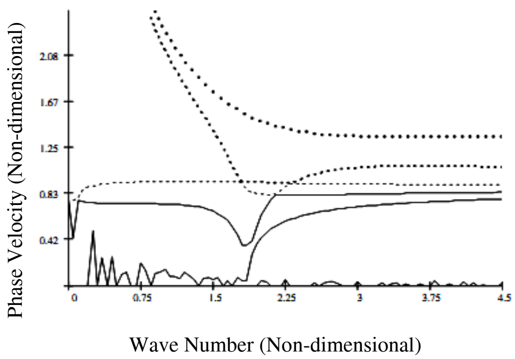

Fig. 12
Dispersion of symmetric wave modes in LS theory of generalized thermoelasticity when $\tau_0 = 10\times10^{-7}$s.

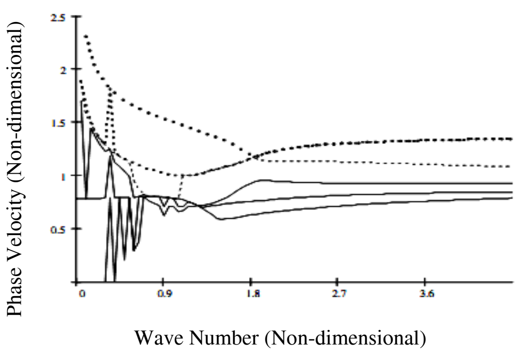

Fig. 13
Dispersion of anti symmetric wave modes in CT theory of thermoelasticity (no thermal relaxation time).

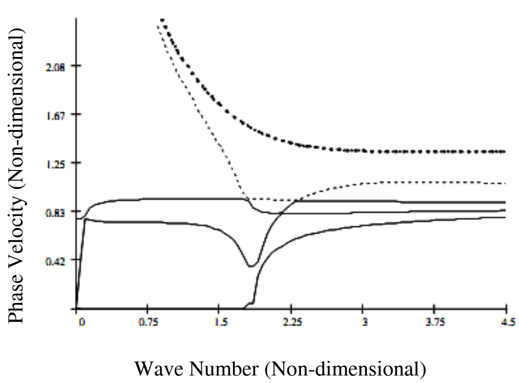

Fig. 14
Dispersion of symmetric wave modes in CT theory of thermoelasticity (no thermal relaxation time).

11 CONCLUSIONS

In this article, exact formal solution for the displacements, temperature and thermal stresses, temperature gradient in an infinite plate of arbitrary anisotropy of finite thickness are derived for the generalized theory of thermoelasticity with two thermal relaxation times. Dispersion relations are derived for thermoelastic waves for more specialized case of a monoclinic plate, dispersion relations for symmetric and anti symmetric are then derived in separate form. Results for thermoelastic plates of higher symmetry materials are implicitly contained in the analysis. The SH wave gets decoupled from the others motion and is not affected by thermal variations, if propagation occurs along an in-plane axis of symmetry and propagates without dispersion or damping. It is observed that GL, and LS theories, derived from distinctively different physical assumptions and physical laws, the spectral behaviors described by both theories is qualitatively similar. They resolve three waves, quasi-longitudinal (QL), quasi-transverse (QT) and quasi-thermal (T-mode).

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APPENDIX

$$A_{E1}=(c_1c_6F_{11}-c_6F_{13}^2+c_5F_{13}F_{23}-c_5^2F_{33}-c_2F_{23}^2-2c_1c_5F_{12}+c_5F_{21}F_{23}+c_2c_6F_{33}), \tag{A.1}$$

$$A_{E2}=(F_{11}F_{23}^2-F_{13}^2F_{22}+c_6F_{11}F_{33}+c_1F_{11}F_{22}-c_1F_{12}^2+2F_{12}F_{13}F_{23}+2c_5F_{12}F_{33}+c_2F_{22}F_{33}), \tag{A.2}$$

$$A_{E3}=(F_{11}F_{22}-F_{12}^2)F_{33}, \qquad \Delta_E=(c_2c_6-c_5^2)c_1, \tag{A.3}$$

$$F_{11}=n_1^2-\zeta^2,\ \ F_{12}=c_4n_1^2,\ \ F_{13}=c_7n_1,\ \ F_{22}=c_3n_1^2-\zeta^2,\ \ F_{23}=c_8n_1\alpha,\ \ F_{33}=c_2n_1^2-\zeta^2. \tag{A.4}$$

REFERENCES

[1] Chadwick P., 1960, *Progress in Solid Mechanic*s, edited by R. Hill and I. N. Sneddon **1**, North Holland Publishing Co., Amsterdam.

[2] Kraut E.A., 1963, Advances in the theory of anisotropic elastic wave propagation, *Reviews of Geophysics* **3**: 401-448.

[3] Fedorov F.I., 1968, *Theory of Elastic Waves in Crystals*, Plenun Press, New York.

[4] Achenbach J.D, 1973, *Wave Propagation in Elastic Solids*, North-Holland, Amsterdam.

[5] Auld B.A, 1973, *Acoustic Fields and Waves in Solids* **1**, John Wiley and Sons, New York.

[6] Rose J.L., 1999, *Ultrasonic Waves in Solid Media*, Cambridge, New York, Cambridge University Press.

[7] Brekhovskikh L.M., 1960, *Waves in Layered Media*, Academic Press, New York.

[8] Ewing, E. Jardetzky W.S., Press F., 1957, *Elastic Waves in Layered Media*, McGraw-Hill, New York.

[9] Kennett B.L.N, 1983, *Seismic Wave Propagation in Stratified Media*, Cambridge University Press, Cambridge.

[10] Thomson W.T., 1950, Transmission of elastic waves through a stratified solid medium, *Journal of Applied Physics* **21**: 89-93.

[11] Nayfeh A.H., 1995, Wave propagation in layered anisotropic media with application to composites, *Applied Mathematics and Mechanics*, North-Holland, Amsterdam.

[12] Biot, M.A., 1956, Thermoelasticity and irreversible thermodynamics, *Journal of Applied Physics* **27**:240-253.

[13] Nowacki, W., 1975, *Dynamic Problems of Thermoelasticity*, Noordhoff, Leyden, Netherlands.

[14] Nowacki, W. 1986, *Thermoelasticity*, 2nd edition, Pergamon Press, Oxford.

[15] Chandrasekharaiah D.S., 1986, Thermoelasticity with second sound-A Review, *Applied Mechanics Review* **39**: 355-376.

[16] Chandrasekharaiah D.S., 1998, Hyperbolic thermoelasticity.-A review of recent literature, *Applied Mechanics Review* **51**:705-729.

[17] Lord H.W., Shulman Y., 1967, A generalized dynamical theory of thermoelasticity, *Journal of Mechanics and Physics of Solids* **15**: 299-309.

[18] Green A.E., Lindsay K.A., 1972, Thermoelasticity, *Journal of Elasticity* **2**: 1-7.

[19] Banerjee D.K., Pao Y.K., 1974, Thermoelastic waves in anisotropy solids, *The Journal of the Acoustical Society of America* **56**: 1444-1454.

[20] Dhaliwal R.S., Sherief H.H., 1980, Generalized thermoelasticity for anisotropic media, *Applied Mathematics* **38**:1-8.

[21] Puri P., 1975, Plane waves in generalized thermoelasticity-errata, *International Journal of Engineering Science* **13**: 339-340.

[22] Agarwal Y.K., 1979, On surface waves in generalized thermoelasticity, *Journal of Elasticity* **8**: 171-177.

[23] Massalas C.V., Kalpakidis , V.K., 1987, Thermoelastic waves in a thin plate with mixed boundary conditions and thermal relaxation, *Archive of Applied Mechanics* **57**: 401-412.

[24] Verma K.L., Hasebe N., 1999, On the propagation of Generalized thermoelastic vibrations in plates. *Quarterly Journal of the Polish Academy of Science*, *Engineering Transactions* **47**: 299-319.

[25] Massalas C.V., Kalpakidis V.K., 1987, Thermoelastic waves in a waveguide, *International Journal of Engineering Science* **25**: 1207-1218.

[26] Nayfeh A.H., Hawwa M.A, 1996, Thermoelastic waves in laminated composites plate with a second sound effect, *Journal of Applied Physics* **80**: 2733-2738.

[27] Tao D.J., Prevost H., 1984, Relaxation effects on generalized thermoelastic waves, *Journal of Thermal Stresses* **7**:79-89.

[28] Nayfeh A.H., Hawwa M.A., 1995, The general problem of thermoelastic waves in anisotropic periodically laminated composites, *Composite Engineering* **5**:1499-1517.

[29] Verma K.L., Hasebe N., 2001, Wave propagation in plates of general anisotropic media in generalized thermoelasticity, *International Journal of Engineering Science* **39**: 1739-1763.

[30] Verma K. L., 2002, On the propagation of waves in layered anisotropic media in generalized thermoelasticity, *International Journal of Engineering Science* **40**: 2077-2096.

[31] Verma K.L., 2001, Thermoelastic vibrations of transversely isotropic plate with thermal relaxations, *International Journal of Solids and Structures* **38**: 8529-8546.

[32] Verma K.L., Hasebe N., 2002, Wave propagation in transversely isotropic plates in generalized thermoelasticity, *Archive of Applied Mechanics* **72**: 470-482.

[33] Jabbari M., Dehbani H., 2010, An exact solution for Lord-Shulman generalized coupled thermoporoelasticity in spherical coordinates, *Journal of Solid Mechanics* **2**(3): 214-230.

[34] Verma K.L., Hasebe N., 2004, On The flexural and extensional thermoelastic waves in orthotropic plates with thermal relaxation time, *Journal of Applied Mathematics* **1**: 69-83.

[35] Nayfeh A.H., Chementi D.E., 1989, Free wave propagation in plates of general anisotropic media, *Journal of Applied Mechanics* **56**: 881-886.

[36] Chadwick P., Seet L.T.C., 1970, Wave propagation in a transversely isotropic heat-conducting elastic material, *Mathematika* **17**: 255-274.