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A New Finite Element Formulation for Buckling and Free Vibration Analysis of Timoshenko Beams on Variable Elastic Foundation

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ABSTRACT

In this study, the buckling and free vibration of Timoshenko beams resting on variable elastic foundation analyzed by means of a new finite element formulation. The Winkler model has been applied for elastic foundation. A two-node element with four degrees of freedom is suggested for finite element formulation. Displacement and rotational fields are approximated by cubic and quadratic polynomial interpolation functions, respectively. The length of the element is assumed to be so small, so that linear variation could be considered for elastic foundation through the length of the element. By these assumptions and using energy method, stiffness matrix, mass matrix and geometric stiffness matrix of the proposed beam element are obtained and applied to buckling and free vibration analysis. Accuracy of obtained formulation is approved by comparison with the special cases of present problem in other studies. Present formulation shows faster convergence in comparison with conventional finite element formulation. The effects of different parameters on the stability and free vibration of Timoshenko beams investigated and results are completely new.

Keywords : Buckling; Vibration; Timoshenko beam; Variable elastic foundation; Finite element formulation.

1 INTRODUCTION

BEAMS are fundamental component in engineering and have wide applications in structures and machines design and fabrication. They are also used as simple and accurate model for analysis of complex engineering structures. Three well-known theories have been developed for beams analysis. In the primary and widely used theory, the beams considered as thin or Euler-Bernoulli beam which means the length of beams at least 10 times larger than the height. For this model, the rotation of cross-section and distortion due to shear is neglected compared to the translation and the bending deformation, respectively [1]. The inertia due to the axial displacement of the beam or rotary inertia effect, is considered in Rayleigh's theory. The third theories, evaluates the effects of rotary inertia and shear deformation and called Timoshenko theory. The Euler-Bernoulli model has simple mathematical model for handling and closed form solution could be obtain by this model, but results only valid for thin beams and significant discrepancy observed for short or thick beams in this model. The Timoshenko model has complex mathematical model rather than other theories, while results of this model are very accurate for short and thick beams. There are other models developed for beams which includes warping of the cross-section and allow variation

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in the longitudinal direction like Levinson [2] and Bickford [3] theories. Many studies have been presented for dynamical analysis of beams by using Timoshenko theory. Rossi and Laura [4] presented analytical solution of the free vibration of Timoshenko beams carrying elastically mounted masses. The exact solution of the vibration and the stability analysis for a non-uniform Timoshenko beam subjected to axial and distributed tangential loads has been presented by Esmailzadeh and Ohadi [5]. Lee and Schultz [6] employed the Chebyshev pseudospectral method to study of the free vibration of Timoshenko beams and axisymmetric Mindlin plates. Finite element formulation has been derived by Moallemi-Oreh and Karkon [7] for stability and free vibration analysis of Timoshenko beams. They used simple two-node elements and assumed that shear strain of the element has the constant value. Constant shear strain assumption let them to consider the polynomial interpolation functions with un-known coefficients while in previous work by Yokoyama [8], the known value introduced in interpolation function for bending rotation. Lee and Park [9] developed a thick beam element by using isogeometrical approach for the free vibration analysis of Timoshenko beams. Using the non-local elasticity theory, Timoshenko beam model is developed by Mohammadimehr et al. [10] to study the elastic buckling of double-walled carbon nanotubes (DWCNTs) embedded in an elastic medium under axial compression and so on [11-13].

It is obvious that the correct analysis and design of structures required an understanding of soil-structure interaction. The surrounding soil increase resistance of buried structures such as pipelines and significantly change dynamical behavior of structures. Many practical cases in engineering related to soil-structure interaction can be modeled by means of a beam on elastic foundation. The well-known models for elastic foundations are Winkler and Pasternak. The Winkler model of elastic foundation is the most preliminary in which the vertical displacement is assumed to be proportional to the contact pressure at an arbitrary point [14], in another words, the foundation modeled as a series of closely spaced and mutually independent linear elastic springs. Pasternak model or two-parameter foundation adds shearing layer to Winkler model, where shearing layer play a role same as axial load in equation of motion. Mentioned model has been used for different problems of beams and plates that resting on elastic foundation [15-23]. Usually, researchers assumed that the foundation has constant value through the length of the beam length and only limited studies exist for dynamic analysis of beams on variables foundations. Eisenberger and Clastornik [24] studied free vibration and buckling of the Euler-Bernoulli beams on variables Winkler foundation, also, they studied free vibration and buckling of the Euler-Bernoulli beams on variables Pasternak foundation [25]. Zhou [26] by considering the reaction force of the foundation on the beam as the external force acting on the beam derived a general solution to vibrations of the Euler-Bernoulli beams on variables Winkler foundation. Differential quadrature method applied by Pradhan and Murmu [27] to thermo-mechanical vibration analysis of sandwich beam resting on variable Winkler foundation. Kacar et al. [28] studied free vibration of the Euler-Bernoulli beams on variables Winkler foundation by means of semi-analytical approach which called differential transform method (DTM). Teodoru and Musat [29] derived mass, stiffness and geometrical matrices for the Euler-Bernoulli beam on linear variables Pasternak foundation by Galerkin based finite element.

According to literature survey, the stability and free vibration analysis of Timoshenko beam resting on variable elastic foundation has not been studied before and for the first time is studied in this paper. At first, a new finite element (FE) formulation derive for Timoshenko beams by two-node elements with the constant shear value and linear variation for elastic foundation through the length of the element assumptions. Then, comparisons are made with studies in open literature in which special cases of present problem have been studied and very good agreement observed. Finally, some new and more useful results extracted from present formulation.

2 FINITE ELEMENT FORMULATION

Consider a beam under axial load and resting on variable Winkler foundation as shown in Fig. 1. The beam has length L, rectangular cross section with height of h and width of w. The beam made from homogenous and isotropic material with E as modulus of elasticity, G as the shear modulus and ρ as mass per unit volume. A beam element with length of l is depicted in Fig. 2. The beam element has two nodes, in which two degrees of freedom associated with the degree of freedom of transverse displacement and bending rotation considered for each node. Length of the element is assumed to be so small, so that we are able to consider a linear variation for elastic foundation. Following the work by Moallemi-Oreh and Karkon [7], it is assumed that shearing strain has the constant value. A cubic polynomial interpolation function considered for displacement field as follow:

$$y^e = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{1}$$

In which a_0 to a_3 are unknown constant. In Timoshenko beam theory, the shear deformation is equal to $\frac{\partial y}{\partial x}-\theta$, where θ is the bending rotation. It is clear displacement derivation with respect to x in the shear strain formula, reduced order of polynomial in Eq. (1); therefore a polynomial with same power should be consider for the bending rotation as follow:

$$\theta^e = b_0 + b_1 x + b_2 x^2 \tag{2}$$

In which b_0 to b_2 are unknown constant. The constant value assumption for the shear strain considered as follow:

$$\frac{\partial y}{\partial x} - \theta = \phi_0 \tag{3}$$

where ϕ_0 is a constant value. The unknown constants in Eq. (1) and Eq. (2) determined from nodal variables at two ends of the element and using Eq. (3) as follows:

$$a_0 = y_i \tag{4}$$

$$a_1 = \theta_i + \phi_0 \tag{5}$$

$$a_2 = -\frac{3y_i + 2l\theta_i + 3l\phi_0 - 3y_j + l\theta_j}{l^2} \tag{6}$$

$$a_3 = \frac{2y_i + l\theta_i + 2l\phi_0 - 2y_j + l\theta_j}{l^3} \tag{7}$$

$$b_0 = \theta_i \tag{8}$$

$$b_1 = -\frac{2(3y_i + 2l\theta_i + 3l\phi_0 - 3y_j + l\theta_j)}{l^2} \tag{9}$$

$$b_2 = \frac{3(2y_i + l\theta_i + 2l\phi_0 - 2y_j + l\theta_j)}{l^3} \tag{10}$$

Now, only one constant remain unknown in formulation which is ϕ_0. By using the condition of minimum strain energy in element, the value of ϕ_0 could be determined. The strain energy in Timoshenko beam element without foundation is the sum of bending and shear strain energies and calculated as following:

$$\Pi^e_{Strian} = \frac{1}{2}\int_0^l EI\left(\frac{\partial \theta^e}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^l k'GA(\frac{\partial y^e}{\partial x} - \theta^e)^2\, dx \tag{11}$$

where A is the area of cross-section, I is the cross-sectional moment of inertia and k' is shear correction factor. The strain energy of element obtained from substituting Eqs. (2-10) into Eq. (5), then, for minimizing strain energy, the following stationary condition is applied:

$$\frac{\partial \Pi^e_{Strain}}{\partial \phi_0} = 0 \tag{12}$$

which yields the following expression for the constant value of shear strain:

$$\phi_0 = \Lambda \frac{[2(y_j - y_i) - l(\theta_i + \theta_j)]}{l}, \Lambda = \frac{6EI}{12EI + k'GAl^2} \tag{13}$$

Substituting Eqs. (4-10) and Eq. (13) into Eq. (1) and Eq. (2) yields the shape functions for Timoshenko beam element as following:

$$y^e = \begin{bmatrix} N_y^1 & N_y^2 & N_y^3 & N_y^4 \end{bmatrix} \{D\}^e = [N_y]\{D^e\} \tag{14}$$

$$\theta^e = \begin{bmatrix} N_\theta^1 & N_\theta^2 & N_\theta^3 & N_\theta^4 \end{bmatrix} \{D\}^e = [N_\theta]\{D^e\} \tag{15}$$

$$\{D^e\} = \{y_i \quad \theta_i \quad y_j \quad \theta_j\}^T \tag{16}$$

$$N_y^1 = \frac{(x-l)(2x^2 + 2\Lambda xl - 4\Lambda x^2 - xl - l^2)}{l^3} \tag{17}$$

$$N_y^2 = \frac{x(x-l)(\Lambda l - 2\Lambda x + x - l)}{l^2} \tag{18}$$

$$N_y^3 = \frac{x(3xl + 4\Lambda x^2 + 2\Lambda l^2 - 2x^2 - 6\Lambda xl)}{l^3} \tag{19}$$

$$N_y^4 = \frac{x(x-l)(\Lambda l - 2\Lambda x + x)}{l^2} \tag{20}$$

$$N_\theta^1 = \frac{6x(l-x)(2\Lambda - 1)}{l^3} \tag{21}$$

$$N_\theta^2 = \frac{(l-x)(l + 6\Lambda x - 3x)}{l^2} \tag{22}$$

$$N_\theta^3 = -\frac{6x(l-x)(2\Lambda - 1)}{l^3} \tag{23}$$

$$N_\theta^4 = \frac{x(3x - 6\Lambda x + 6\Lambda l - 2l)}{l^2} \tag{24}$$

As mentioned before, a linear variation considered for elastic foundation through the length of the element, if the stiffness of foundation be k_1 at the left node ($x = 0$) and be k_2 at the right node ($x = l$), then, the following function considered for variation of foundation through the length of the element:

$$k(x)^e = k_1 + (k_2 - k_1)\frac{x}{l} \tag{25}$$

The strain energy of the Timoshenko beam element with elastic foundation effect written as following:

$$\Pi^e = \frac{1}{2}\int_0^l EI\left(\frac{\partial\theta^e}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^l k'GA(\frac{\partial y^e}{\partial x} - \theta^e)^2 dx + \frac{1}{2}\int_0^l k(x)(y^e)^2 dx \tag{26}$$

The kinetic energy of the Timoshenko beam element with inclusion of the rotary inertia effect is given by

$$T^e = \frac{1}{2}\int_0^l \rho A(\frac{\partial y^e}{\partial t})^2 dx + \frac{1}{2}\int_0^l \rho I(\frac{\partial\theta^e}{\partial t})^2 dx \tag{27}$$

The external work done by a compressive axial load (positive for tension) can be written as:

$$W^e = -\frac{p}{2}\int_0^l (\frac{\partial y^e}{\partial x})^2 dx \tag{28}$$

Obtained expressions for the strain energy, the kinetic energy and the external work, re-write in terms of the element displacement vector ($\{D^e\}$) as:

$$\Pi^e = \frac{1}{2}\{D^e\}^T[K_b]^e\{D^e\} + \frac{1}{2}\{D^e\}^T[K_s]^e\{D^e\} + \frac{1}{2}\{D^e\}^T[K_f]^e\{D^e\} \tag{29}$$

$$T^e = \frac{1}{2}\{\dot{D}^e\}^T[M_t]^e\{\dot{D}^e\} + \frac{1}{2}\{\dot{D}^e\}^T[M_r]^e\{\dot{D}^e\} \tag{30}$$

$$W^e = -\frac{1}{2}\{D^e\}^T[K_g]^e\{D^e\} \tag{31}$$

In Eqs.(29-31), $[K_b]^e, [K_s]^e, [K_f]^e, [M_t]^e, [M_r]^e$ and $[K_g]^e$ are bending stiffness matrix, shear stiffness matrix, stiffness matrix due to the elastic foundation, translational mass matrix, rotary inertia mass matrix and geometric stiffness matrix, respectively. These matrices defined as follows:

$$[K_b]^e = \int_0^l [B_b]^T EI[B_b]dx, \quad [B_b] = \frac{\partial}{\partial x}[N_\theta] \tag{32}$$

$$[K_s]^e = \int_0^l [B_s]^T k'GA[B_s]dx, \quad [B_s] = \frac{\partial}{\partial x}[N_y] - [N_\theta] \tag{33}$$

$$[K_f]^e = \int_0^l [N_y]^T k(x)^e [N_y]dx \tag{34}$$

$$[M_t]^e = \int_0^l [N_y]^T \rho A[N_y]dx \tag{35}$$

$$[M_r]^e = \int_0^l [N_\theta]^T \rho I[N_\theta]dx \tag{36}$$

$$[K_g]^e = \int_0^l [B_v]^T p[B_v]dx, \quad [B_v] = \frac{\partial}{\partial x}[N_y] \tag{37}$$

The explicit expressions for matrices in Eq.(32)-Eq. (37) are listed in Appendix.
To derive equation of motion, Lagrangian function is defined as follows:

$$L = \sum (\Pi^e - T^e + W^e) \tag{38}$$

Inserting Lagrangian function in Eq. (38) into Hamilton's principle [8] and by using Eqs. (29-31) leads to the governing equation of motion in matrix form as follows:

$$[M]\{\ddot{D}^e\} + [K]\{D^e\} = 0 \tag{39}$$

where $[M]$ is global consistent mass matrix, in the following form:

$$[M] = \sum ([M_t]^e + [M_r]^e) \tag{40}$$

and $[K]$ is global stiffness matrix, in the following form:

$$[K] = \sum ([K_b]^e + [K_s]^e + [K_f]^e - [K_g]^e) \tag{41}$$

By harmonic motion assumption with circular frequency ω, equation of motion in (39) is changed to:

$$([K] - \omega^2 [M])\{D^e\} = 0 \tag{42}$$

Eq. (42) is an eigenvalue problem, in which for the non-trivial solution, it is necessary that the determinant of the coefficient matrix is set equal to zero. Obtained eigenvalues are corresponding to natural frequencies of vibration. For determination of the critical buckling load, the following eigenvalue will be achieved:

$$\sum ([K_b]^e + [K_s]^e + [K_f]^e - p_{Cr}.[K_g]^e) = 0 \tag{43}$$

The lowest positive eigenvalue of Eq. (43) is the critical buckling load.

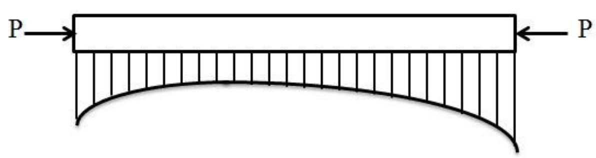


Fig.1
A Timoshenko beam resting on variable elastic foundation.

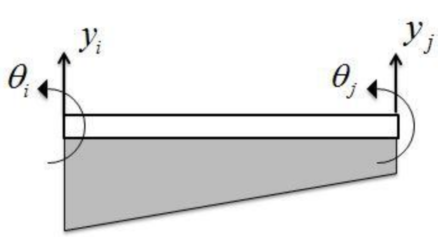


Fig.2
A two-node beam element, resting on variable elastic foundation.

3 NUMERICAL RESULTS

A computer code has been developed in Matlab software to calculate numerical results. As same as other studies, some dimensionless parameters defined to better representation of the numerical results. A general form for variable Winkler foundation considered as follows:

$$k(x)=k_w f(\frac{x}{L}),\ k_w=\frac{K_w EI}{L^4} \tag{44}$$

In Eq. (44), $f(\frac{x}{L})$ is a dimensionless function which shows variation of elastic foundation through the length of the beam, also, K_w is a dimensionless parameter which known as dimensionless moduli of Winkler foundation. When the beam discretized over its length by two node elements, the effective value of elastic foundation at each node calculated from Eq. (44) and then stiffness matrix obtained from Eq. (34) for each element. Dimensionless axial load, defined as follows:

$$P=\frac{pL^2}{EI} \tag{45}$$

After calculation natural frequencies from eigenvalue problem in Eq. (42), dimensionless frequencies obtained as follows:

$$\beta^4=\frac{\rho AL^4\omega^2}{EI} \tag{46}$$

To validate the obtained finite element formulation, the first three dimensionless frequencies of the Euler-Bernoulli beam resting on variable Winkler foundation with linear and parabolic distribution under different boundary conditions studied by the differential transform method [28] are re-examined and results are presented in Table 1. Very good agreement between results can be observed, which confirmed accuracy of proposed element for the beam on variable Winkler foundation. It should be noted, obtained FE formulation is applicable for the Euler-Bernoulli beams when Λ in Eq. (13) is set equal to zero and $[M_r]^e$ is omitted.

Table 1
Dimensionless frequencies of the Euler-Bernoulli beam resting on variable Winkler foundation.

B.C.		$f(\frac{x}{L})=1-0.2(\frac{x}{L}),K_w=100$			$f(\frac{x}{L})=1-0.2(\frac{x}{L})^2,K_w=100$		
		β_1	β_2	β_3	β_1	β_2	β_3
H-H	Present	3.699	6.372	9.452	3.721	6.375	9.453
	Kacar *et al.* [28]	3.699	6.372	9.452	3.721	6.375	9.453
C-C	Present	4.930	7.899	11.013	4.939	7.901	11.013
	Kacar *et al.* [28]	4.930	7.899	11.013	4.939	7.901	11.013

For all the subsequent results, the Poisson's ratio is $\nu=0.3$ and shear correction factor is taken $5/6$. The first three dimensionless frequencies of Timoshenko beams with hinged-hinged and clamped-clamped boundary conditions calculated by presented FE formulation and compared with other well-known studies in Table 2. The exact solution of beam critical buckling load with shear deformation effect is obtained as succeeding form [30]:

$$P_{cr}=\frac{\pi^2 EI}{L_{eff}^2}(\frac{k'GAL_{eff}^2}{k'GAL_{eff}^2+\pi^2 EI}) \tag{47}$$

where L_{eff} is the effective beam length in which $L_{eff}=L, L_{eff}=2L$ and $L_{eff}=0.5L$ are used for hinged-hinged boundary condition, clamp-free boundary condition and clamped-clamped boundary condition, respectively. Critical buckling load for different ratios of height to length calculated by presented FE formulation and analytical solution in Eq. (47) and convert to dimensionless form by using Eq. (45). Obtained results for dimensionless critical buckling load are presented in Table 3. Very good accuracy can be seen for presented FE formulation for buckling and free vibration analysis of Timoshenko beams, as shown in Table 2. and Table 3.

Table 2
Dimensionless frequencies of the Timoshenko beam without elastic foundation

B.C.		$L=5h$			$L=10h$		
		β_1	β_2	β_3	β_1	β_2	β_3
H-H	Present	3.045	5.672	7.840	3.116	6.091	8.841
	Lee and Schultz [6]	3.045	5.672	7.839	3.116	6.091	8.841
	Attar *et al*. [22]	3.045	5.672	7.840	-	-	-
C-C	Present	4.242	6.418	8.287	4.580	7.331	9.857
	Lee and Schultz [6]	4.242	6.418	8.285	4.580	7.331	9.856

Table 3
Dimensionless critical buckling load of the Timoshenko beam without elastic foundation

B.C.	C-F		H-H		C-C	
	Present	Analytical	Present	Analytical	Present	Analytical
$h=0.1L$	2.45167	2.45167	9.6227	9.6227	35.8044	35.8034
$h=0.2L$	2.40567	2.40567	8.9508	8.9508	27.9894	27.9875
$h=0.3L$	2.33272	2.33272	8.0179	8.0179	20.5228	20.5211

In order to show efficiency of present FE formulation to conventional FE formulation, a comparison between convergence rates of two methods has been made for first two dimensionless frequencies of hinged-clamped Timoshenko beam resting on variable elastic foundation with following distribution:

$$f\left(\frac{x}{L}\right)=1+\left(\frac{x}{L}\right)^2,\quad K_w=100 \tag{48}$$

Results are depicted in Fig. 3 where faster convergence rates of present FE formulation are observed. Seven different types of distribution considered for foundation, where one type is constant distribution and other six types of distribution are plotted in Fig. 4. For all types of foundation distribution, the average value of the variable Winkler foundation is same and equal to 2, also, the dimensionless moduli of Winkler foundation is same for all distribution types. The average value of the variable Winkler foundation has been defined as follows:

$$\frac{1}{L}\int_0^L f\left(\frac{x}{L}\right)dx=2 \tag{49}$$

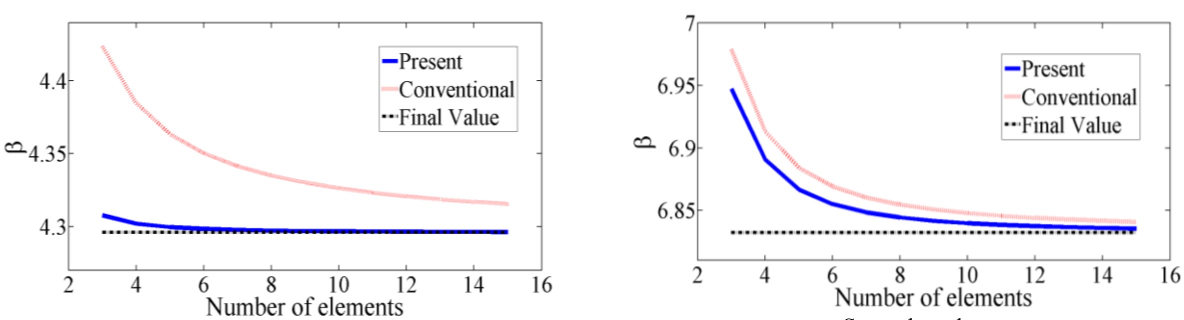


Fig.3
Comparison between convergence rates of present and conventional FE formulation ($L=10h$).

The effects of the Winkler foundation distribution and ratio of height to length of beams on first four dimensionless frequencies investigated in Tables 4-7. Results obtained for four different kinds of boundary conditions i.e. C-F, H-H, C-C and H-C. It is obvious, for all kinds of boundary conditions; the first dimensionless frequency is strongly depend on Winkler foundation distribution, despite the fact that the average value of the variable Winkler foundation and the dimensionless moduli of Winkler foundation are same for all types of distribution. For C-F boundary condition (Table 4), Winkler foundation distribution like those presented for Case 1 and Case 2 in Fig. 4 yield the maximum value for fundamental frequency for both the Euler-Bernoulli and Timoshenko beam theories. For other kinds of boundary conditions which investigated in Tables 5-7., Winkler foundation distribution like those presented for Case 4 and Case 5 in Fig. 4 yields the maximum value for fundamental frequency for both the Euler-Bernoulli and Timoshenko beam theories. It seems the maximum fundamental frequency occurred when Winkler foundation distribution is closed to fundamental mode shape of the beam; which depend on boundary conditions of the beam. Also, it is clearly obtained from Tables 4-7. that third and fourth frequencies are not sensitive to Winkler foundation distribution.

Table 4
Dimensionless frequencies of beams resting on variable elastic foundation under C-F boundary conditions ($K_w = 100$).

Theory	Mode	Constant	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Euler-Bernoulli	1st	3.8174	4.0592	4.1759	3.9329	3.5197	3.3389	3.6698
	2nd	5.1169	5.1548	5.1780	5.1117	5.1782	5.2444	5.1591
	3rd	7.9560	7.9593	7.9604	7.9528	7.9602	7.9606	7.9561
	4th	11.033	11.034	11.034	11.032	11.034	11.034	11.034
Timoshenko $L = 15h$	1st	3.8155	4.0569	4.1723	3.92874	3.5158	3.3336	3.6672
	2nd	5.0694	5.1066	5.1298	5.0639	5.1338	5.2014	5.1132
	3rd	7.7486	7.7512	7.7519	7.7453	7.7530	7.7531	7.7486
	4th	10.520	10.520	10.520	10.519	10.521	10.521	10.521
Timoshenko $L = 7.5h$	1st	3.8099	4.0503	4.1618	3.9166	3.5043	3.3177	3.6594
	2nd	4.9453	4.9803	5.0037	4.9389	5.0187	5.0900	4.9938
	3rd	7.2780	7.2786	7.2783	7.2743	7.2829	7.2819	7.2777
	4th	9.5267	9.5258	9.5252	9.5261	9.5274	9.5267	9.5274

Table 5
Dimensionless frequencies of beams resting on variable elastic foundation under H-H Boundary conditions ($K_w = 100$).

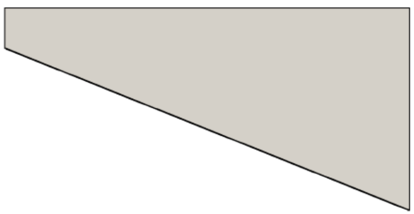
Theory	Mode	Constant	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Euler-Bernoulli	1st	4.1528	4.1497	4.0915	3.9212	4.3668	4.4609	4.2869
	2nd	6.4757	6.4763	6.4736	6.4616	6.4878	6.4753	6.4757
	3rd	9.4839	9.4840	9.4835	9.4820	9.4857	9.4843	9.4853
	4th	12.591	12.591	12.591	12.591	12.592	12.592	12.591
Timoshenko $L = 15h$	1st	4.1452	4.1420	4.0835	3.9131	4.3595	4.4534	4.2795
	2nd	6.3910	6.3916	6.3888	6.3765	6.4033	6.3905	6.3909
	3rd	9.1998	9.1998	9.1993	9.1977	9.2016	9.2001	9.2012
	4th	11.957	11.957	11.957	11.957	11.958	11.957	11.957
Timoshenko $L = 7.5h$	1st	4.1241	4.1204	4.0609	3.8901	4.3390	4.4325	4.2588
	2nd	6.1781	6.1788	6.1758	6.1624	6.1912	6.1773	6.1779
	3rd	8.5838	8.5839	8.5834	8.5815	8.5861	8.5844	8.5856
	4th	10.784	10.784	10.784	10.784	10.785	10.784	10.784

Table 6
Dimensionless frequencies of beams resting on variable elastic foundation under C-C Boundary conditions ($K_w = 100$).

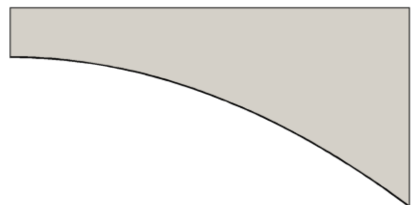
Theory	Mode	Constant	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Euler-Bernoulli	1st	5.1447	5.1442	5.1095	5.0037	5.2883	5.3641	5.2371
	2nd	7.9545	7.9545	7.9501	7.9364	7.9727	7.9731	7.9613
	3rd	11.033	11.033	11.032	11.029	11.037	11.036	11.035
	4th	14.155	14.155	14.155	14.154	14.156	14.156	14.155
Timoshenko $L = 15h$	1st	5.0891	5.0885	5.0528	4.9447	5.2355	5.3122	5.1832
	2nd	7.7074	7.7075	7.7030	7.6888	7.7260	7.7257	7.7141
	3rd	10.447	10.447	10.447	10.444	10.451	10.449	10.449
	4th	13.067	13.067	13.067	13.067	13.068	13.067	13.068
Timoshenko $L = 7.5h$	1st	4.9489	4.9480	4.9098	4.7951	5.1032	5.1823	5.0478
	2nd	7.1641	7.1642	7.1594	7.1442	7.1836	7.1811	7.1704
	3rd	9.3532	9.3532	9.3525	9.3504	9.3555	9.3526	9.3541
	4th	11.335	11.335	11.335	11.335	11.335	11.334	11.335

Table 7
Dimensionless frequencies of beams resting on variable elastic foundation under H-C Boundary conditions ($K_w = 100$).

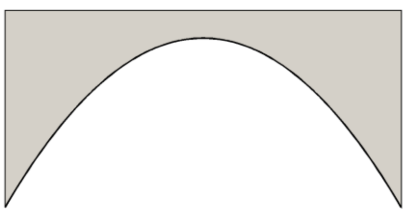
Theory	Mode	Constant	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Euler-Bernoulli	1st	4.5740	4.5365	4.4738	4.3991	4.7438	4.8222	4.6804
	2nd	7.2061	7.2036	7.1978	7.1877	7.2245	7.2232	7.2127
	3rd	10.257	10.256	10.255	10.254	10.260	10.259	10.258
	4th	13.373	13.373	13.372	13.372	13.374	13.373	13.373
Timoshenko $L=15h$	1st	4.5504	4.5132	4.4501	4.3731	4.7219	4.8009	4.6579
	2nd	7.0517	7.0498	7.0441	7.0328	7.0705	7.0687	7.0583
	3rd	9.8346	9.8345	9.8337	9.8314	9.8377	9.8364	9.8362
	4th	12.525	12.525	12.524	12.524	12.525	12.525	12.525
Timoshenko $L=7.5h$	1st	4.4873	4.4511	4.3867	4.3034	4.6637	4.7442	4.5980
	2nd	6.6885	6.6882	6.6829	6.6682	6.7084	6.7048	6.6948
	3rd	8.9855	8.9862	8.9858	8.9826	8.9881	8.9858	8.9868
	4th	11.071	11.072	11.072	11.071	11.072	11.071	11.071



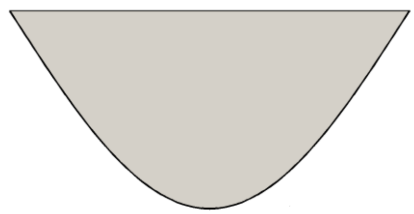
Case 1: $f(\frac{x}{L}) = 1 + 2(\frac{x}{L}).$



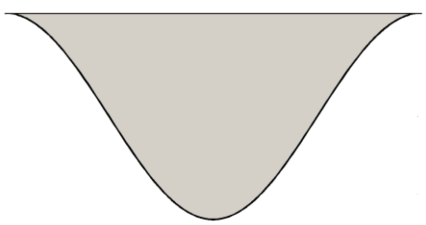
Case 2: $f(\frac{x}{L}) = 1 + 3(\frac{x}{L})^2.$



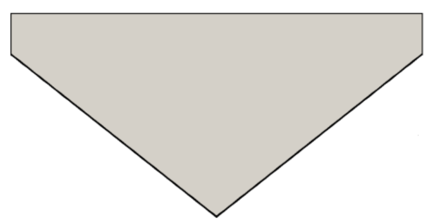
Case 3: $f(\frac{x}{L}) = 4 - 12(\frac{x}{L}) + 12(\frac{x}{L})^2.$



Case 4: $f(\frac{x}{L}) = \pi \sin(\pi(\frac{x}{L})).$



Case 5: $f(\frac{x}{L}) = 2 - 2\cos(2\pi(\frac{x}{L})).$



Case 6: $f(\frac{x}{L}) = \begin{cases} 1 + 4(\frac{x}{L}) & 0 < \frac{x}{L} < \frac{1}{2} \\ 5 - 4(\frac{x}{L}) & \frac{1}{2} < \frac{x}{L} < 1 \end{cases}.$

Fig.4
Six different types of elastic foundation distribution through the length of the beam.

Table 8. is given to study the influence of Winkler foundation distribution on dimensionless critical buckling load of Timoshenko beams with different height to length ratios and different boundary conditions. As same as behavior observed in free vibration analysis, critical buckling load is affected by Winkler foundation distribution and maximum critical buckling load obtained when Winkler foundation distribution is closed to fundamental mode shape of beam.

Effect of the dimensionless moduli of Winkler foundation on first and second dimensionless frequency (D.F.) of Timoshenko beams with C-F and H-C boundary conditions are plotted in Fig. 5 and Fig. 6, respectively. Two types of distribution considered for elastic foundation i.e. Case 1 and Case 4 (see Fig. 4). It is clear, by increasing dimensionless moduli of Winkler foundation frequency increased in both first and second mode. The type of foundation distribution has little influence on second frequency and second frequencies are almost equal for both types of distribution and almost change linearly with dimensionless moduli of Winkler foundation, while the first frequency greatly affected by the type of foundation distribution (special for C-F) and this issue is more obvious when the value of K_w has been increased.

In order to deduce effect of distribution type of elastic foundation on first and second normalized mode shapes of Timoshenko beam, Fig. 7-10 are presented. Great effect of distribution type on first normalized mode shape is seen in Fig. 7 and Fig. 9.

Table 8
Dimensionless critical buckling load of the Timoshenko beams resting on variable Winkler foundation under different boundary conditions ($K_w = 100$).

B.C.		Constant	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
C-F	$L=15h$	15.4216	17.2984	18.4323	17.5928	11.2936	9.48781	13.3806
	$L=7.5h$	14.8072	16.5764	17.6389	16.8106	10.8534	9.11652	12.872
H-H	$L=15h$	30.0226	29.7672	27.9746	23.8257	36.7034	39.884	34.0905
	$L=7.5h$	29.7033	29.3374	27.475	23.4897	36.3421	38.4248	33.7502
C-C	$L=15h$	52.5455	52.5002	51.0558	46.9558	58.6911	62.2904	56.5235
	$L=7.5h$	47.7559	47.6833	46.2714	42.3762	53.5525	56.9423	51.5519
H-C	$L=15h$	35.5042	33.8322	31.9387	30.9348	39.841	41.4181	38.1682
	$L=7.5h$	33.8830	32.1552	30.3115	29.506	37.6986	38.8084	36.2437

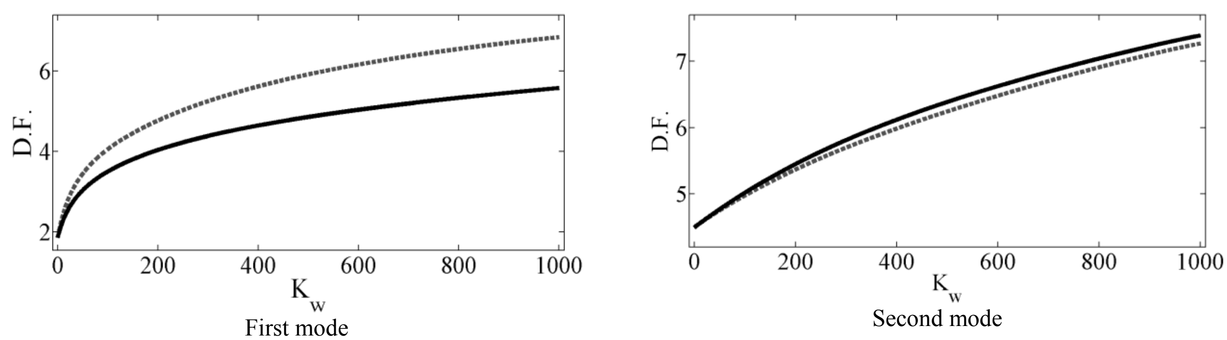


Fig.5
Effect of the dimensionless moduli of Winkler foundation on dimensionless frequencies of Timoshenko beams with C-F boundary conditions. (Dashed line: Case 1; Solid line: Case 4).

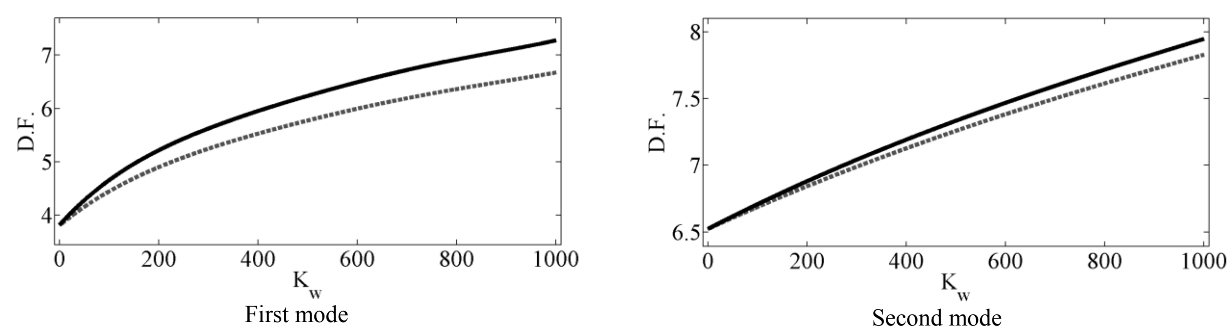


Fig.6
Effect of the dimensionless moduli of Winkler foundation on dimensionless frequencies of Timoshenko beams with H-C boundary conditions. (Dashed line: Case 1; Solid line: Case 4).

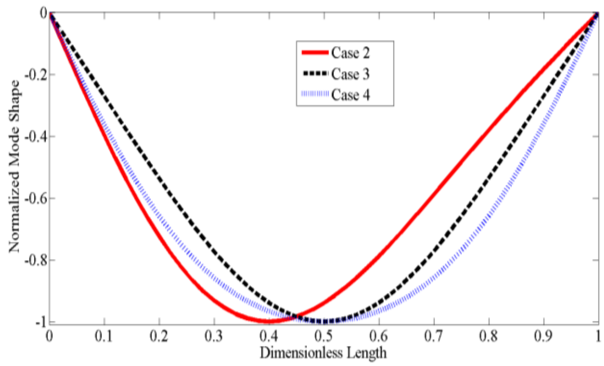


Fig.7
The first normalized mode shape of H-H beams with different distribution of elastic foundation. ($L=10h, K_w=500$)

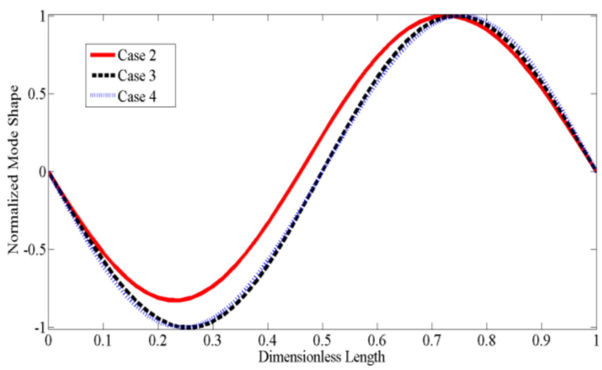


Fig.8
The second normalized mode shape of H-H beams with different distribution of elastic foundation. ($L=10h, K_w=500$)

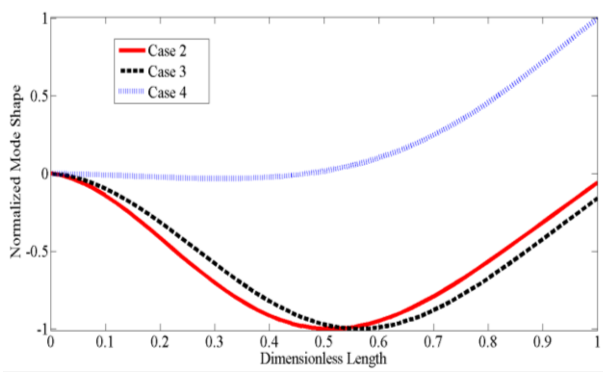


Fig.9
The first normalized mode shape of C-F beams with different distribution of elastic foundation. ($L=10h, K_w=500$)

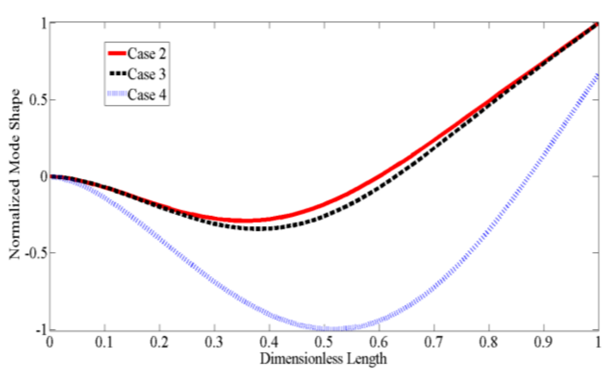


Fig.10
The second normalized mode shape of C-F beams with different distribution of elastic foundation. ($L=10h, K_w=500$)

APPENDIX

$$[K_b]^e = \frac{EI}{l^3}\begin{bmatrix} 12\alpha^2 & 6l\alpha^2 & -12\alpha^2 & 6l\alpha^2 \\ 6l\alpha^2 & 4l^2(3\beta+1) & -6l\alpha^2 & 2l^2(6\beta+1) \\ -12\alpha^2 & -6l\alpha^2 & 12\alpha^2 & -6l\alpha^2 \\ 6l\alpha^2 & 2l^2(6\beta+1) & -6l\alpha^2 & 4l^2(3\beta+1) \end{bmatrix}, \alpha = 2\Lambda - 1, \beta = \Lambda^2 - \Lambda$$

$$[K_s]^e = \frac{k'GA\Lambda^2}{l}\begin{bmatrix} 4 & 2l & -4 & 2l \\ 2l & l^2 & -2l & l^2 \\ -4 & -2l & 4 & -2l \\ 2l & l^2 & -2l & l^2 \end{bmatrix},$$

$$[K_g]^e = \frac{p}{30l}\begin{bmatrix} 12(2\beta+3) & 3l\alpha^2 & -12(2\beta+3) & 3l\alpha^2 \\ 3l\alpha^2 & 2l^2(3\beta+2) & -3l\alpha^2 & l^2(6\beta-1) \\ -12(2\beta+3) & -3l\alpha^2 & 12(2\beta+3) & -3l\alpha^2 \\ 3l\alpha^2 & l^2(6\beta-1) & -3l\alpha^2 & 2l^2(3\beta+2) \end{bmatrix},$$

$$[M_t]^e = \frac{\rho A l}{420}\begin{bmatrix} m_t^1 & m_t^2 & m_t^3 & m_t^4 \\ m_t^2 & m_t^5 & -m_t^4 & m_t^6 \\ m_t^3 & -m_t^4 & m_t^1 & -m_t^2 \\ m_t^4 & m_t^6 & -m_t^2 & m_t^5 \end{bmatrix},$$

$$m_t^1 = 4(2\Lambda^2 - 9\Lambda + 39), m_t^2 = l(4\Lambda^2 - 11\Lambda + 22), m_t^3 = 2(-4\Lambda^2 + 18\Lambda + 27)$$
$$m_t^4 = l(4\Lambda^2 - 11\Lambda - 13), m_t^5 = 2l^2(\Lambda^2 - \Lambda + 2), m_t^6 = l^2(2\Lambda^2 - 2\Lambda - 3)$$

$$[M_r]^e = \frac{\rho I}{30l}\begin{bmatrix} 36\alpha^2 & 3l\alpha(6\alpha+5) & -36\alpha^2 & 3l\alpha(6\alpha+5) \\ 3l\alpha(6\alpha+5) & l^2(15\alpha+36\beta+19) & -3l\alpha(6\alpha+5) & l^2(15\alpha+36\beta+14) \\ -36\alpha^2 & -3l\alpha(6\alpha+5) & 36\alpha^2 & -3l\alpha(6\alpha+5) \\ 3l\alpha(6\alpha+5) & l^2(15\alpha+36\beta+14) & -3l\alpha(6\alpha+5) & l^2(15\alpha+36\beta+19) \end{bmatrix},$$

$$[K_w]^e = \frac{l}{840}\begin{bmatrix} k_w^1 & k_w^2 & k_w^3 & k_w^4 \\ k_w^2 & k_w^5 & k_w^6 & k_w^7 \\ k_w^3 & k_w^6 & k_w^8 & k_w^9 \\ k_w^4 & k_w^7 & k_w^9 & k_w^{10} \end{bmatrix},$$

$$k_w^1 = 8[30k_1 + 9k_2 - \Lambda(8k_1 + k_2) + \Lambda^2(k_1 + k_2)], k_w^2 = 2l[15k_1 + 7k_2 - \Lambda(10k_1 + k_2) + 2\Lambda^2(k_1 + k_2)]$$
$$k_w^3 = 2(k_1 + k_2)(-4\Lambda^2 + 18\Lambda + 27), k_w^4 = -2l[7k_1 + 6k_2 + \Lambda(8k_1 + 3k_2) - 2\Lambda^2(k_1 + k_2)]$$
$$k_w^5 = l^2[5k_1 + 3k_2 - 4\Lambda k_1 + 2\Lambda^2(k_1 + k_2)], k_w^6 = 2l[6k_1 + 7k_2 + \Lambda(3k_1 + 8k_2) - 2\Lambda^2(k_1 + k_2)]$$
$$k_w^7 = l^2(k_1 + k_2)(2\Lambda^2 - 2\Lambda - 3), k_w^8 = 8[9k_1 + 30k_2 - \Lambda(k_1 + 8k_2) + \Lambda^2(k_1 + k_2)]$$
$$k_w^9 = -2l[7k_1 + 15k_2 - \Lambda(k_1 + 10k_2) + 2\Lambda^2(k_1 + k_2)], k_w^{10} = l^2[3k_1 + 5k_2 - 4\Lambda k_2 + 2\Lambda^2(k_1 + k_2)]$$

4 CONCLUSIONS

In this study, a new finite element formulation has been developed for the buckling and free vibration analysis of Timoshenko beams resting on variable Winkler type elastic foundation. A two-node element with sufficiently small length and constant shear strain suggested for FE formulation. Small length of element permits us to consider linear variation for elastic foundation through the length of the element. The stiffness matrix, mass matrix and geometric stiffness matrix of the proposed beam element derived using energy method. Comparison between results obtained from presented FE formulation with those obtained from other well-known methods shows very good accuracy. Proposed FE formulation applied for the static and dynamic analysis of beams resting on variable Winkler foundation and some results presented for the first time. Results shows fundamental frequency and critical buckling load are sensitive to foundation distribution through the length of the beam while the higher mode shapes are not. Proposed finite element formulation shows faster convergence in comparison with conventional finite element formulation and is capable for analyzing the beams resting on variable elastic foundation with any arbitrary distribution of elastic foundation.

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