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Thermoelastic Analysis of a Rectangular Plate with Nonhomogeneous Material Properties and Internal Heat Source

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ABSTRACT

This article deals with the determination of temperature distribution, displacement and thermal stresses of a rectangular plate having nonhomogeneous material properties with internal heat generation. The plate is subjected to sectional heating. All the material properties except Poisson's ratio and density are assumed to be given by a simple power law along x direction. Solution of the two-dimensional heat conduction equation is obtained in the transient state. Integral transform method is used to solve the system of fundamental equation of heat conduction. The effects of inhomogeneity on temperature and thermal stress distributions are examined. For theoretical treatment, all the physical and mechanical quantities are taken as dimensional, whereas for numerical computations we have considered non-dimensional parameters. The transient state temperature field and its associated thermal stresses are discussed for a mixture of copper and zinc metals in the ratio 70:30 respectively. Numerical calculations are carried out for both homogeneous and nonhomogeneous cases and are represented graphically.

Keywords: Stresses; Inhomogeneity; Heat source; Shear modulus; Simple power law.

1 INTRODUCTION

FUNCTIONALLY graded materials (FGMs) refer to the composite materials where the compositions or the microstructures are locally varied so that a certain variation of the local material properties is achieved. The overall properties of FGM are unique and different from any of the individual material that forms it. As the field of functionally graded minerals has advanced, the study of nonhomogeneous solids has also gained revived importance. From the perspective of continuum mechanics, these materials can be regarded as nonhomogeneous solids which are modeled by variable elasticity moduli. Al-Hajri and Kalla [1] developed a new integral transform and its inversion involving combination of Bessel's function as a kernel and used it to solve mixed boundary value problems. Ding and Li [4] studied the thermoelastic analysis of nonhomogeneous structural materials. Gupta and Singhal [6] studied the thermal effect on vibration of non-homogeneous orthotropic visco-elastic rectangular plate of parabolically varying thickness having clamped boundary conditions on all the four edges. Gupta et al. [7] presented an analysis of the forced vibrations of non-homogeneous rectangular plate of variable thickness on the basis of

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classical plate theory by assuming the non-homogeneity of the plate material to arise due to the variation in density which is assumed to vary linearly. Hata [8] studied thermal stresses in a nonhomogeneous thick plate with surface radiation under steady state temperature distribution. Kassir [9] investigated thermal stress problems in a thick plate and a semi-infinite body in nonhomogeneous solids. Kawamura et al. [10] studied the thermoelastic deformation and stress analyses of an orthotropic nonhomogeneous rectangular plate. Kumar [11] studied the free transverse vibrations of thin simply supported nonhomogeneous isotropic rectangular plates of bilinearly varying thickness with elastically restrained edges against rotation. Lal and Kumar [12] analyzed the buckling and vibration behaviour of nonhomogeneous rectangular plates of uniform thickness on the basis of classical plate theory when the two opposite edges are simply supported and are subjected to linearly varying in-plane force by assuming young's modulus and density of the plate to vary exponentially along axial direction. Manthena et al. [13] studied the temperature distribution, displacement and thermal stresses in a rectangular plate with inhomogeneous material properties by taking the material properties to vary along y coordinate. Martynyak and Dmytriv [14] investigated the generalized plane stressed state of a rectangle of isotropic functionally gradient materials under the action of normal load using finite-element method. Morishita and Tanigawa [15] considered a nonhomogeneous semi-infinite body subject to an arbitrary shaped distributed load on its boundary surface as an analytical model, in which the fundamental equations system for the medium are given by three kinds of displacement functions. Muravskii [16] studied the action of surface vertical and horizontal forces applied to the half-space. Pandita and Kulkarni [17] studied the effect of variable thermal conductivity in thermal stress analysis of rectangular plate subjected to temperature variation. Sharma et al. [18] used Differential Quadrature Method (DQM) to analyse free vibration of non-homogeneous orthotropic rectangular plates of parabolically varying thickness resting on Winkler-type elastic foundation. Sugano [19] analyzed a plane thermoelastic problem in a nonhomogeneous doubly connected region under a transient temperature field by stress function method. Tanigawa et al. [20] presented thermal bending analysis of a laminated composite rectangular plate due to a partially distributed heat supply by introducing the methods of finite cosine transform and Laplace transform to the temperature field and adapted the classical plate theory based on Kirchhoff-Love's hypothesis to the thermoelastic field. Tanigawa [21] briefly discussed the method of analytical development of thermoelastic problems for nonhomogeneous materials where both the thermal and mechanical material constants are described by the function of the variable of coordinate system. Tanigawa et al. [22] established analytical method of development for the plane isothermal and thermoelastic problems by introducing two kinds of displacement functions. Tokovyy and Ma [23] presented a method for solving the plane elasticity and thermoelasticity problems for planes and half-planes which exhibit inhomogeneous material properties in one of the planar directions Wang and Wang [24] presented the exact solutions for the vibration problems of nonhomogeneous rectangular membranes with an exponential density distribution and with a linear density distribution. The behaviour of non-homogeneity has been assumed due to exponential variation in Young's moduli and density in one direction. Yang et al. [25] developed a general two-dimensional solution for a bilayer functionally graded cantilever beam with concentrated loads at the free end.

In this paper we have extended our own work [13]. Here we have considered a rectangular plate occupying the space $a \le x \le b,\ 0 \le y \le L$ subjected to sectional heating and internal heat generation. The material properties except Poisson's ratio and density are assumed to be nonhomogeneous given by a simple power law in x direction. The solutions are obtained in the transient state in the form of Bessel's and trigonometric functions. For theoretical treatment all physical and mechanical quantities are taken as dimensional, whereas for numerical computations we have considered non-dimensional parameters. Numerical computations are carried out by considering various values of the inhomogeneous parameter m.

2 STATEMENT OF THE PROBLEM

2.1 Heat conduction equation

We consider the transient heat conduction equation with initial and boundary conditions in a rectangular plate with heat source given by [13]

$$\frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda(x)\frac{\partial T}{\partial y}\right)+\Theta(x,y,t)=c(x)\ \rho\frac{\partial T}{\partial t} \tag{1}$$

$$
\begin{aligned}
&T = f_1(x,y), && at\ t=0 \\
&T = 0, && at\ x=a,\ \ 0 \le y \le L,\ t>0 \\
&T = 0, && at\ x=b,\ \ 0 \le y \le L,\ t>0 \\
&T = f_2(x,t), && at\ y=0,\ \ a \le x \le b,\ t>0 \\
&T = 0, && at\ y=L,\ \ a \le x \le b,\ t>0
\end{aligned} \tag{2}
$$

where $\lambda(x)$ and $c(x)$ are respectively, thermal conductivity and calorific capacity of the material in the inhomogeneous region, ρ is the density.

2.2 Thermoelastic equations

Let u_x and u_y be the displacement components in the in-plane directions of *x* and *y*. The strain-displacement components ε_{ij}, equilibrium equations of the forces and stress-strain components in *y* direction disregarding the body forces are given by [13]

$$
\varepsilon_{xx} = \frac{\partial u_x}{\partial x},\ \varepsilon_{yy} = \frac{\partial u_y}{\partial y},\ \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{3}
$$

$$
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0\ ,\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \tag{4}
$$

$$
\left.
\begin{aligned}
\sigma_{xx} &= \frac{2}{1-\upsilon} G(x)[\varepsilon_{xx} + \upsilon\,\varepsilon_{yy} - (1+\upsilon)\,\alpha(x) T] \\
\sigma_{yy} &= \frac{2}{1-\upsilon} G(x)[\upsilon\,\varepsilon_{xx} + \varepsilon_{yy} - (1+\upsilon)\,\alpha(x) T] \\
\sigma_{xy} &= 2G(x)\,\varepsilon_{xy}
\end{aligned}
\right\} \quad \text{for plane stress field} \tag{5}
$$

$$
\left.
\begin{aligned}
\sigma_{xx} &= \frac{2}{1-2\upsilon} G(x)[(1-\upsilon)\varepsilon_{xx} + \upsilon\,\varepsilon_{yy} - (1+\upsilon)\,\alpha(x) T] \\
\sigma_{yy} &= \frac{2}{1-2\upsilon} G(x)[\upsilon\varepsilon_{xx} + (1-\upsilon)\varepsilon_{yy} - (1+\upsilon)\,\alpha(x) T] \\
\sigma_{xy} &= 2G(x)\,\varepsilon_{xy} \\
\sigma_{zz} &= \upsilon(\sigma_{xx} + \sigma_{yy}) - 2(1+\upsilon)\,G(x)\alpha(x) T
\end{aligned}
\right\} \quad \text{for plane strain field} \tag{6}
$$

We assume that shear modulus of elasticity G and the coefficient of linear thermal expansion $\alpha(x)$ have an inhomogeneous material property in *x* direction and are changed arbitrarily in its direction, but Poisson's ratio υ is assumed to be constant. We consider $G(x)$ and $\alpha(x)$ given by simple power law [13]

$$
G(x) = G_0\,(x/L)^m,\ \ \alpha(x) = \alpha_0\,(x/L)^m \tag{7}
$$

Here G_0 and α_0 are the reference values of shear modulus of elasticity and coefficient of linear thermal expansion respectively. Also $m(\ge 0)$ is a constant which is related to Poisson's ratio ν by the relation $m\nu = 1-2\nu$.

2.3 Plane stress field

Using Eqs. (3) and (5) in (4) the displacement equations of equilibrium in x and y directions are obtained as:

$$\begin{aligned}
&\frac{2}{1-\upsilon}\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{1+\upsilon}{1-\upsilon}\frac{\partial^2 u_y}{\partial x\,\partial y}+\frac{1}{G}\frac{\partial G}{\partial y}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)-\frac{2(1+\upsilon)}{1-\upsilon}\alpha(x)\frac{\partial T}{\partial x}=0\\
&\frac{\partial^2 u_y}{\partial x^2}+\frac{2}{1-\upsilon}\frac{\partial^2 u_y}{\partial y^2}+\frac{1+\upsilon}{1-\upsilon}\frac{\partial^2 u_x}{\partial x\,\partial y}+\frac{2}{1-\upsilon}\frac{1}{G}\frac{\partial G}{\partial y}\left(\frac{\partial u_y}{\partial y}+\upsilon\frac{\partial u_x}{\partial x}\right)\\
&-\frac{2(1+\upsilon)}{1-\upsilon}\left[\frac{\partial}{\partial y}[\alpha(x)T]+\frac{1}{G}\frac{\partial G}{\partial y}[\alpha(x)T]\right]=0
\end{aligned} \tag{8}$$

2.4 Plane strain field

Similarly, the equilibrium equations in terms of displacement components are obtained by using Eqs. (3) and (6) into (4) as:

$$\begin{aligned}
&\frac{2(1-\upsilon)}{1-2\upsilon}\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{1}{1-2\upsilon}\frac{\partial^2 u_y}{\partial x\,\partial y}+\frac{1}{G}\frac{\partial G}{\partial y}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha(x)\frac{\partial T}{\partial x}=0\\
&\frac{\partial^2 u_y}{\partial x^2}+\frac{2(1-\upsilon)}{1-2\upsilon}\frac{\partial^2 u_y}{\partial y^2}+\frac{1}{1-2\upsilon}\frac{\partial^2 u_x}{\partial x\,\partial y}+\frac{2}{1-2\upsilon}\frac{1}{G}\frac{\partial G}{\partial y}\left((1-\nu)\frac{\partial u_y}{\partial y}+\upsilon\frac{\partial u_x}{\partial x}\right)\\
&-\frac{2(1+\upsilon)}{1-2\upsilon}\left[\frac{\partial}{\partial y}[\alpha(x)T]+\frac{1}{G}\frac{\partial G}{\partial y}[\alpha(x)T]\right]=0
\end{aligned} \tag{9}$$

The solution of Eqs. (8) and (9) without body forces can be expressed by the Goodier's thermoelastic displacement potential ϕ and the Boussinesq harmonic functions φ and ψ as:

$$\begin{aligned}
u_x&=\frac{\partial\varphi}{\partial x}+\frac{\partial\phi}{\partial x}+y\,\frac{\partial\psi}{\partial x}\\
u_y&=\frac{\partial\varphi}{\partial y}+\frac{\partial\phi}{\partial y}+y\,\frac{\partial\psi}{\partial y}-(3-4\nu)\psi
\end{aligned} \tag{10}$$

In which the three functions must satisfy the conditions

$$\nabla^2\phi=K\tau,\ \nabla^2\varphi=0 \quad \text{and} \quad \nabla^2\psi=0 \tag{11}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, $K(x)=\frac{(1+\nu)}{(1-\nu)}\alpha(x)$ is the restraint coefficient and $\tau=T-T_i$, T_i is the initial temperature.

Now by using Eqs. (7) and (10) in Eqs. (5) and (8) and then in Eqs. (6) and (9) the results for thermoelastic fields are obtained as

2.5 For plane stress field

The equations for the displacement functions are given by

$$\nabla^2\phi+\nabla^2\varphi+y\,\nabla^2\psi+(2\nu^2+(1-m)\nu+(1+m))\frac{\partial\psi}{\partial y}+\frac{m(1-\nu)}{y}\left(\frac{\partial\phi}{\partial y}+\frac{\partial\varphi}{\partial y}-(1-\nu)\psi\right)=(1+\nu)\alpha(x)T$$
$$\frac{\partial}{\partial y}(\nabla^2\phi+\nabla^2\varphi+y\,\nabla^2\psi)-\frac{2\nu^2-\nu(m+4)+1}{1-\nu}\frac{\partial^2\psi}{\partial x^2}+\frac{2+m}{1-\nu}\frac{\partial^2\psi}{\partial y^2}$$
$$+\frac{1}{1-\nu}\frac{m}{y}\left(\frac{\partial^2\phi}{\partial y^2}+\nu\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\nu\frac{\partial^2\varphi}{\partial x^2}+2\psi(2\nu-1)\right)=\frac{2(1+\nu)}{1-\nu}\left[\alpha(x)\frac{\partial T}{\partial y}+(y^2+1)\frac{m}{y}\alpha(x)T\right] \quad (12)$$

The corresponding stress functions are given by

$$\sigma_{xx}=\frac{2}{1-\upsilon}G(x)\left[\begin{array}{l}\left(\frac{\partial^2\phi}{\partial x^2}+\nu\frac{\partial^2\phi}{\partial y^2}\right)+\left(\frac{\partial^2\varphi}{\partial x^2}+\nu\frac{\partial^2\varphi}{\partial y^2}\right)+y\left(\frac{\partial^2\psi}{\partial x^2}+\nu\frac{\partial^2\psi}{\partial y^2}\right)\\+2\nu(2\nu-1)\frac{\partial\psi}{\partial y}-(1+\upsilon)\alpha(x)T\end{array}\right],$$
$$\sigma_{yy}=\frac{2}{1-\upsilon}G(x)\left[\begin{array}{l}\left(\nu\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\left(\nu\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)+y\left(\nu\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\\+(4\nu-2)\frac{\partial\psi}{\partial y}-(1+\upsilon)\alpha(x)T\end{array}\right], \quad (13)$$
$$\sigma_{xy}=2G(x)\left[\frac{\partial^2\phi}{\partial x\,\partial y}+\frac{\partial^2\varphi}{\partial x\,\partial y}+y\,\frac{\partial^2\psi}{\partial x\,\partial y}-(1-\nu)\frac{\partial\psi}{\partial x}\right]$$

2.6 For plane strain field

The fundamental equations for the displacement function are given by

$$\nabla^2\phi+\nabla^2\varphi+y\,\nabla^2\psi-\frac{4\nu^2-4\nu-m(1-2\nu)+1}{1-\nu}\frac{\partial\psi}{\partial y}+\frac{m(1-2\nu)}{y(1-\nu)}\left(\frac{\partial\phi}{\partial y}+\frac{\partial\varphi}{\partial y}-(1-\nu)\psi\right)=\frac{(1+\nu)}{(1-\nu)}\alpha(x)T$$
$$\frac{\partial}{\partial y}(\nabla^2\phi+\nabla^2\varphi+y\,\nabla^2\psi)-\frac{2\nu^2-\nu(m+4)+1}{1-2\nu}\frac{\partial^2\psi}{\partial x^2}+\frac{2+m}{1-2\nu}\frac{\partial^2\psi}{\partial y^2}$$
$$+\frac{1}{1-2\nu}\frac{m}{y}\left(\frac{\partial^2\phi}{\partial y^2}+\nu\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\nu\frac{\partial^2\varphi}{\partial x^2}+2\psi(2\nu-1)\right)=\frac{2(1+\nu)}{1-2\nu}\left[\alpha(x)\frac{\partial\theta}{\partial y}+(y^2+1)\frac{m}{y}\alpha(x)T\right] \quad (14)$$

The corresponding stress functions are given by

$$\sigma_{xx}=\frac{2}{1-2\upsilon}G(x)\left[\begin{array}{l}\left((1-\nu)\frac{\partial^2\phi}{\partial x^2}+\nu\frac{\partial^2\phi}{\partial y^2}\right)+\left((1-\nu)\frac{\partial^2\varphi}{\partial x^2}+\nu\frac{\partial^2\varphi}{\partial y^2}\right)+y\left((1-\nu)\frac{\partial^2\psi}{\partial x^2}+\nu\frac{\partial^2\psi}{\partial y^2}\right)\\-2\nu(1-2\nu)\frac{\partial\psi}{\partial y}-(1+\upsilon)\alpha(x)T\end{array}\right],$$
$$\sigma_{yy}=\frac{2}{1-2\upsilon}G(x)\left[\begin{array}{l}\left(\nu\frac{\partial^2\phi}{\partial x^2}+(1-\nu)\frac{\partial^2\phi}{\partial y^2}\right)+\left(\nu\frac{\partial^2\varphi}{\partial x^2}+(1-\nu)\frac{\partial^2\varphi}{\partial y^2}\right)+y\left(\nu\frac{\partial^2\psi}{\partial x^2}+(1-\nu)\frac{\partial^2\psi}{\partial y^2}\right)\\+2(2\nu^2+3\nu-1)\frac{\partial\psi}{\partial y}-(1+\upsilon)\alpha(x)T\end{array}\right], \quad (15)$$
$$\sigma_{xy}=2G(x)\left[\frac{\partial^2\phi}{\partial x\,\partial y}+\frac{\partial^2\varphi}{\partial x\,\partial y}+y\,\frac{\partial^2\psi}{\partial x\,\partial y}-(1-\nu)\frac{\partial\psi}{\partial x}\right],$$
$$\sigma_{zz}=\nu(\sigma_{xx}+\sigma_{yy})-2(1+\nu)G(x)\alpha(x)T$$

The boundary condition on the traction free surface stress functions are

$$\sigma_{xx}\big|_{y=0}=\sigma_{xy}\big|_{y=0}=\sigma_{yy}\big|_{y=0}=0 \quad (16)$$

Eqs. (1) to (16) constitute the mathematical formulation of the problem.

3 SOLUTION OF THE PROBLEM

3.1 Heat conduction equation

The heat conduction equation is given by

$$\lambda(x)\frac{\partial^2 T}{\partial x^2}+\lambda'(x)\frac{\partial T}{\partial x}+\lambda(x)\frac{\partial^2 T}{\partial y^2}+\Theta(x,y,t)=c(x)\,\rho\frac{\partial T}{\partial t} \tag{17}$$

For the sake of brevity, we consider

$$\begin{aligned}&\lambda(x)=\lambda_0(x/L)^m,\ c(x)=c_0(x/L)^m,\ \rho=\rho_0,\ f_1(x,y)=Q_0\delta(x-x_0)\,\delta(y-y_0)\\&f_2(x,t)=Q_1\delta(x-x_0)\sinh(\omega t),\\&\Theta(x,y,t)=(\lambda_0/L^m)\,x^{((1+m)/2)}\Theta_1(y,t),\quad \Theta_1(y,t)=\delta(y-y_0)\delta(t-t_0)\end{aligned} \tag{18}$$

Here λ_0, c_0 and ρ_0 are the reference values of thermal conductivity, calorific capacity and density, respectively. Using Eq. (18) in (17), we obtain

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{m}{x}\frac{\partial T}{\partial x}\right)+\left(\frac{\partial^2 T}{\partial y^2}\right)+x^{((1-m)/2)}\Theta_1(y,t)=\frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{19}$$

where $\kappa=\dfrac{\lambda_0}{c_0\,\rho_0}$ and

$$\begin{aligned}&T=Q_0\delta(x-x_0)\,\delta(y-y_0), && at\ t=0\\&T=0, && at\ x=a,\ 0\le y\le L,\ t>0\\&T=0, && at\ x=b,\ 0\le y\le L,\ t>0\\&T=Q_1\delta(x-x_0)\sinh(\omega t), && at\ y=0,\ a\le x\le b,\ t>0\\&T=0, && at\ y=L,\ a\le x\le b,\ t>0\end{aligned} \tag{20}$$

To remove m from the numerator of Eq. (19), we use the variable transformation $T=x^{((1-m)/2)}\theta$ and obtain

$$\left(\frac{\partial^2\theta}{\partial x^2}+\frac{1}{x}\frac{\partial\theta}{\partial x}+\frac{\gamma^2}{x^2}\theta\right)+\left(\frac{\partial^2\theta}{\partial y^2}\right)+\Theta_1(y,t)=\frac{1}{\kappa}\frac{\partial\theta}{\partial t} \tag{21}$$

where $\gamma^2=((m-1)/2)$

$$\begin{aligned}&\theta=Q_0x^{((m-1)/2)}\delta(x-x_0)\,\delta(y-y_0), && at\ t=0\\&\theta=0, && at\ x=a,\ 0\le y\le L,\ t>0\\&\theta=0, && at\ x=b,\ 0\le y\le L,\ t>0\\&\theta=Q_1x^{((m-1)/2)}\delta(x-x_0)\sinh(\omega t), && at\ y=0,\ a\le x\le b,\ t>0\\&\theta=0, && at\ y=L,\ a\le x\le b,\ t>0\end{aligned} \tag{22}$$

To solve the differential Eq. (21) using integral transform technique, we introduce the extended integral transform [1] of order i over the variable x as given below (Refer Appendix A).

$$T[f(x),a,b;\gamma_i]=\bar{f}(\gamma_i)=\int_a^b x f(x)S(\gamma_i x)dx \tag{23}$$

Here $S(\gamma_i x)$ is the kernel of the transform given by

$$S(\gamma_i x)=Z_i\cos(\gamma_i\log x)-W_i\sin(\gamma_i\log x) \tag{24}$$

where $Z_i=\sin(\gamma_i\log a)+\sin(\gamma_i\log b)$, $W_i=\cos(\gamma_i\log a)+\cos(\gamma_i\log b)$ and $\gamma_i\,(i=1,2,3,........)$ are the real and positive roots of the transcendental equation

$$\sin(\gamma\log a)\cos(\gamma\log b)-\sin(\gamma\log b)\cos(\gamma\log a)=0 \tag{25}$$

The inversion formula is

$$f(x)=\sum_{i=1}^{\infty}\frac{\bar{f}(\gamma_i)}{S(\gamma_i)}S(\gamma_i x) \tag{26}$$

where

$$\int_a^b x\,S(\gamma_i x)S(\gamma_j x)dx=\begin{cases}S(\gamma_i); & i=j\\ 0; & i\neq j\end{cases} \tag{27}$$

Hence Eq. (21) becomes

$$-\gamma_i^2\bar{\theta}+\left(\frac{\partial^2\bar{\theta}}{\partial y^2}\right)+\bar{\Theta}_1(y,t)=\frac{1}{\kappa}\frac{\partial\bar{\theta}}{\partial t} \tag{28}$$

$$\begin{aligned}&\bar{\theta}=Q_0 g_0\,\delta(y-y_0), && at\ t=0\\ &\bar{\theta}=Q_1 g_0\sinh(\omega t), && at\ y=0,\ a\le x\le b,\ t>0\\ &\bar{\theta}=0, && at\ y=L,\ a\le x\le b,\ t>0\end{aligned} \tag{29}$$

where $\bar{\Theta}_1(y,t)=\delta(y-y_0)\delta(t-t_0)$, $g_0=\int_a^b x^{((1+m)/2)}\delta(x-x_0)S(\gamma_i x)dx$

Applying finite Fourier sine transform to Eq. (28) and using the boundary conditions given in Eq. (29), we obtain

$$\frac{\partial\bar{\bar{\theta}}}{\partial t}+A_1\bar{\bar{\theta}}=A_2\sinh(\omega t)+A_3\,\delta(t-t_0) \tag{30}$$

where

$A_1 = \kappa(\gamma_i^2 + \alpha_n^2)$
$A_2 = \kappa \alpha_n Q_1 g_0$
$A_3 = \kappa y_0 \sin(n\pi y_0 / L)$

$\alpha_n = (n\pi / L)$, n is the transform parameter and

$$\bar{\bar{\theta}} = Q_0 g_0 y_0 \sin(n\pi y_0 / L), \qquad at \;\; t = 0 \tag{31}$$

The kernel of the transform is $\sin(n\pi y / L)$. Applying Laplace transform and its inverse on Eq. (30) by using the initial condition given in Eq. (31), we obtain

$$\bar{\bar{\theta}}(n,t) = \left[\frac{A_2 \omega}{A_1^2 - \omega^2} + A_4\right]\exp(-A_1 t) + \frac{A_2}{2\omega - 2A_1}\exp(-\omega t) + \frac{A_2}{2\omega + 2A_1}\exp(\omega t) + A_3 \exp(-A_1(t - t_0))\theta^*(t - t_0) \tag{32}$$

where $A_4 = Q_0 g_0 y_0 \sin(n\pi y_0 / L$. Here $\theta^*(t - t_0)$ is the Heaviside Theta function given by

$$\theta^*(t - t_0) = \begin{cases} 0; & t < t_0 \\ 1; & t > t_0 \end{cases}$$

Applying inverse Fourier sine transform on Eq. (32) and following [3], we obtain

$$\bar{\theta}(y,t) = \frac{2}{L}\sum_{n=1}^{\infty} \{Q_1 g_0 \sinh(\omega t)[(L - y)/L] + \bar{\bar{\theta}}(n,t)\sin(n\pi y / L)\} \tag{33}$$

Applying inverse transform defined in Eq. (26) on the above equation, we obtain

$$\theta(x,y,t) = \frac{2}{L}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \frac{Q_1 g_0 \sinh(\omega t)[(L - y)/L] + \bar{\bar{\theta}}(n,t)\sin(n\pi y / L)}{S(\gamma_i)} \times S(\gamma_i x), \; x > 0 \tag{34}$$

Using Eq. (34) in the equation $T = x^{((1-m)/2)}\theta$, we obtain

$$T(x,y,t) = \frac{2}{L}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \xi_1 \times [Q_1 g_0 \sinh(\omega t)[(L - y)/L] + \bar{\bar{\theta}}(n,t)\sin(n\pi y / L)] \times g_1(x) \tag{35}$$

where

$$\xi_1 = 1/S(\gamma_i), \;\; g_1(x) = x^{((1-m)/2)} \times [Z_i \cos(\gamma_i \log x) - W_i \sin(\gamma_i \log x)], \quad x > 0$$

3.2 Thermoelastic equations

Using the solution of heat conduction Eq. (17) given by Eq. (35), the solutions for Goodier's thermoelastic displacement potential ϕ from Eq. (11) is obtained as:

$$\phi = \frac{2}{L}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \frac{K(x)[T^2(x,y,t) - T(x,y,t)T_i]}{g_2(x,y,t)} \tag{36}$$

where

$$g_2(x,y,t)=g_3(x,y,t)+g_4(x,y,t)$$
$$g_3(x,y,t)=\xi_1\times[Q_1 g_0\sinh(\omega t)[(L-y)/L]+\bar{\bar{\theta}}(n,t)\sin(n\pi y/L)]\times g_1''(x)$$
$$g_4(x,y,t)=\xi_1\times[\bar{\bar{\theta}}(n,t)\alpha_n^2\sin(n\pi y/L)]\times g_1(x)$$

For the sake of brevity, to avoid complexity, we assume the Boussinesq harmonic functions φ and ψ so as to satisfy Eq. (11) as:

$$\varphi=\psi=\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\{\sinh(pt)[A_n\cos(\gamma_i\log x)+B_n\sin(\gamma_i\log x)]\sin(y)\} \tag{37}$$

where A_n, B_n are constants.

Now, to obtain the displacement components, we substitute the values of ϕ, φ, ψ from Eqs. (36) and (37) in Eq. (10) and obtain

$$u_x=\sum_{n=1}^{\infty}\{\sinh(pt)\}\{\varphi_{,x}+(1+y)[-A_n(\gamma_i/x)\sin(\gamma_i\log x)+B_n(\gamma_i/x)\cos(\gamma_i\log x)]\sin(y)\}$$
$$u_y=\sum_{n=1}^{\infty}\{\sinh(pt)\}\{\varphi_{,y}+(1+y)[A_n\cos(\gamma_i\log x)+B_n\sin(\gamma_i\log x)]\cos(y)\}$$
$$-(3-4\nu)[A_n\cos(\gamma_i\log x)+B_n\sin(\gamma_i\log x)]\sin(y)\} \tag{38}$$

where a comma denotes differentiation with respect to the following variable.

By substituting the values of displacement components given by Eq. (38) in Eqs. (13) and (15), the resulting components of stresses in plane stress field and plane strain field can be obtained. By using the traction free conditions given by Eq. (16) in the equation of stresses (13) and (15), the constants A_n and B_n can be obtained.

Since the equations of stresses and constants A_n and B_n obtained so are very large, we have not mentioned them here. However numerical calculations are carried out by using Mathematica software.

4 NUMERICAL RESULTS AND DISCUSSION

The numerical computations have been carried out for a mixture of Copper and Zinc metals [5, 8] in the ratio 70:30 respectively, with non-dimensional variables as given below.

$$\theta=\frac{T}{T_R},\quad \bar{x}=\frac{x}{L},\quad \bar{y}=\frac{y}{L},\quad \tau=\frac{\kappa t}{L^2},\quad (\bar{u}_x,\bar{u}_y)=\frac{(u_x,u_y)}{K_0\theta_R L},$$
$$(\bar{\sigma}_{xx},\bar{\sigma}_{yy},\bar{\sigma}_{xy})=\frac{(\sigma_{xx},\sigma_{yy},\sigma_{xy})}{EG_0\theta_R},\quad K_0=\frac{1+\nu}{1-\nu}\alpha_0$$

with parameters $L=2cm$, $a=1\,cm$, $b=3cm$, $t=2$ sec, Reference Temperature $\theta_R=32^o C$, Thermal expansion coefficient $\alpha_0=17\times10^{-6}/^o C$, Thermal diffusivity $\kappa=1.11\,cm^2/\text{sec}$. The Young's modulus E is given by the following equation [5, 8]

$$E(x)=(0.14-0.09x-0.93x^2+5.72x^4-10.28x^5)\times9.8\times10^7 N/cm^2$$

Here x : weight of zinc $\div$ 100 , $0 \le x \le 0.3$. For $x = 0.3$, $E = 4.963 \times 10^6\ N/cm^2$

For different values of parameter m , the Poisson's ratio ν and Shear modulus G_0 are calculated by using the formula $m\nu = 1 - 2\nu$, $G_0 = [E/(2(1+\nu))]$

For $m = 0$, Poisson's ratio $\nu = 0.5$, Shear modulus $G_0 = 1.654 \times 10^6\ N/cm^2$

For $m = 1$, Poisson's ratio $\nu = 0.33$, Shear modulus $G_0 = 1.866 \times 10^6\ N/cm^2$

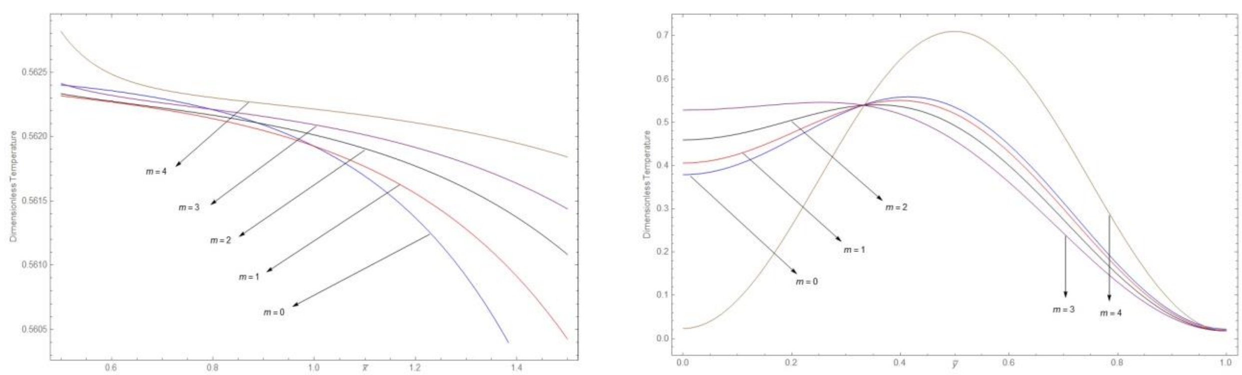
For $m = 2$, Poisson's ratio $\nu = 0.25$, Shear modulus $G_0 = 1.985 \times 10^6\ N/cm^2$

For $m = 3$, Poisson's ratio $\nu = 0.2$, Shear modulus $G_0 = 2.0679 \times 10^6\ N/cm^2$

For $m = 4$, Poisson's ratio $\nu = 0.1667$, Shear modulus $G_0 = 2.127 \times 10^6\ N/cm^2$

Fig.1 (a) shows the variation of dimensionless temperature along *x*–axis for different values of parameter $m = 0, 1, 2, 3, 4$. From the graph it is seen that the nature is exponential. Due to internal heat source, the temperature has a finite value at the outer part of the plate. Because of the sectional heating at the outer part, the absolute value of temperature is slowly and steadily increasing towards the inner region of the plate. The magnitude of temperature is increasing with increase in the parameter *m*. The magnitude of temperature is low in the homogeneous region $m = 0$ and is peak in the nonhomogeneous region $m = 4$.

Fig. 1(b) shows the variation of dimensionless temperature along *y*–axis for different values of parameter *m*. From the graph it is seen that the nature is sinusoidal. The temperature is increasing in the region $0 < \bar{y} < 0.35$ and then suddenly decreasing towards the end in the region $0.35 < \bar{y} < 1$ and gradually approaching zero at the upper part of the plate. Thermal energy is accumulated in the middle region at $\bar{y} = 0.35$ causing material deformation. Also the magnitude of temperature is peak in the central part of the plate at $\bar{y} = 0.5$ in the nonhomogeneous region, whereas it is peak at $\bar{y} = 0.4$ for the remaining values of *m*.

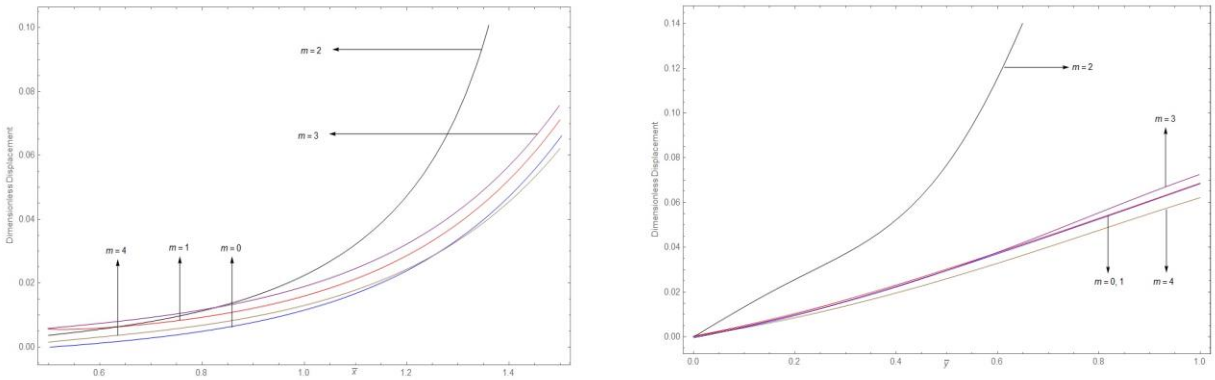


(a) (b)

Fig.1
a)Variation of dimensionless temperature along *x*–axis. b)Variation of dimensionless temperature along *y*–axis.

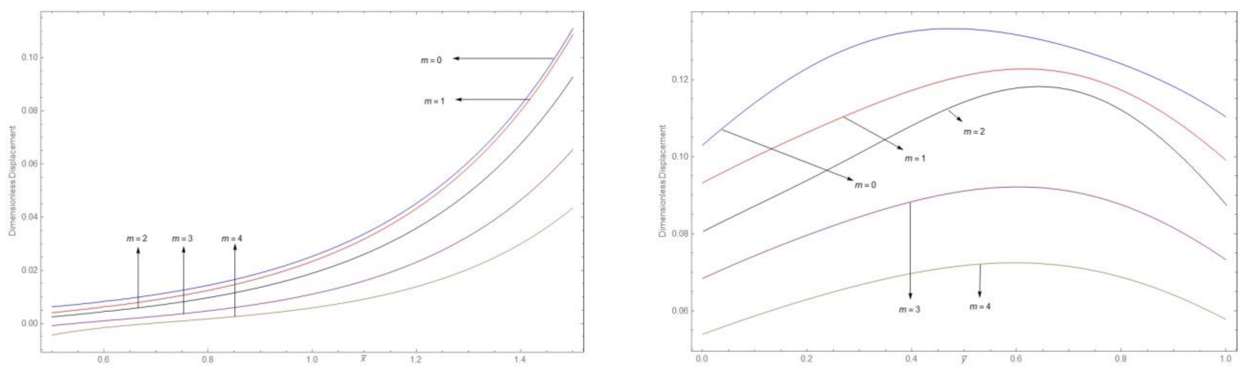
Figs. 2(a) and 2(b) shows the variation of dimensionless displacement $\bar{u}_x$ along *x*–axis and *y*-axis respectively for different values of parameter *m*. Along *x*-axis, it is seen that the absolute value of displacement is increasing and is peak at the rightmost end at $\bar{x} = 1.5$, where the heat is applied. Also the magnitude of displacement is high in the nonhomogeneous region as compared to that of homogeneous region. In the nonhomogeneous region the magnitude is high for $m = 2$ and is found to be decreased with increase in the parameter *m*. Along *y*-axis, it is seen that the no displacement is happening at the lower end. Also it is increasing and towards the upper part of the plate.

Figs. 3(a) and 3(b) shows the variation of dimensionless displacement $\bar{u}_y$ along *x*–axis and *y*-axis respectively for different values of parameter *m*. Along *x*-axis, due to the application of heat source at the upper part of *x*–axis $\bar{x} = 1.5$, it is observed that the absolute value of displacement is exponentially increasing and is peak at the upper part. The displacement has high magnitude in the homogeneous region $m = 0$ as compared to that of nonhomogeneous region and is slowly decreasing with the increase in the parameter *m*. Along *y*-axis, it is seen that the displacement is increasing in $0 < \bar{y} < 0.45$ for the homogeneous region $m = 0$ and then decreasing towards the end. Whereas in the nonhomogeneous region, the absolute value of displacement is increasing for $0 < \bar{y} < 0.65$ and then decreasing towards the end.



(a) (b)

Fig.2
a) Variation of dimensionless displacement $\bar{u}_x$ along *x*–axis. b) Variation of dimensionless displacement $\bar{u}_x$ along *y*–axis.



(a) (b)

Fig.3
a) Variation of dimensionless displacement $\bar{u}_y$ along *x*–axis. b) Variation of dimensionless displacement $\bar{u}_y$ along *y*–axis.

Fig. 4(a) shows the variation of dimensionless stresses in the plane stress field along *x*–axis for different values of parameter *m*. It is seen that the stresses $\bar{\sigma}_{xx}, \bar{\sigma}_{xy}$ are exponentially increasing along *x*–axis in both homogeneous and nonhomogeneous regions. The absolute value of the stresses $\bar{\sigma}_{xx}, \bar{\sigma}_{xy}$ is more in the homogeneous region as compared to nonhomogeneous region, whereas for the stress $\bar{\sigma}_{yy}$ it is more in the nonhomogeneous region as compared to homogeneous region. The stresses $\bar{\sigma}_{xx}, \bar{\sigma}_{xy}$ are tensile while the stress $\bar{\sigma}_{yy}$ is compressive in both homogeneous and nonhomogeneous regions.

Fig. 4(b) shows the variation of dimensionless stresses in the plane stress field along *y*–axis for different values of parameter *m*. It is seen that all the stress $\bar{\sigma}_{xx}$ has a peak value at the upper part of the plate and is tensile and is gradually decreasing towards the lower part of the plate and becoming zero at the lower part in both homogeneous and nonhomogeneous regions. The absolute value of the stress $\bar{\sigma}_{xy}$ is peak at the lower part of the plate and is decreasing towards the upper part. The stress $\bar{\sigma}_{yy}$ is compressive and its magnitude is a bit high in the homogeneous region as compared to that of the nonhomogeneous region. We also observe that the stresses $\bar{\sigma}_{xx}, \bar{\sigma}_{yy}$ are zero at the beginning (irrespective of inhomogeneity parameter *m*), which agrees with the prescribed traction free boundary conditions given in Eq. (16).

Fig. 5(a) shows the variation of dimensionless stresses in the plane strain field along *x*–axis for different values of parameter *m*. It is observed that the absolute value of the stresses $\bar{\sigma}_{xx}, \bar{\sigma}_{zz}$ is exponentially increasing along *x*–

axis in both homogeneous and nonhomogeneous regions. The stresses are tensile and the magnitude is low in the nonhomogeneous region as compared to homogeneous region. The stress $\bar{\sigma}_{xy}$ is tensile in the region $0 < \bar{x} < 1.457$ and is compressive towards the end in the nonhomogeneous region, whereas it is compressive throughout, in the homogeneous region.

Fig. 5(b) shows the variation of dimensionless stresses in the plane strain field along *y*–axis for different values of parameter *m*. It is seen that all the stresses are linearly increasing from the lower part of the plate towards the upper part. The stresses $\bar{\sigma}_{xx}, \bar{\sigma}_{zz}$ are tensile and have a peak value at the upper part of the plate in both homogeneous and nonhomogeneous regions. The magnitude is high in the nonhomogeneous region as compared to that of homogeneous region.

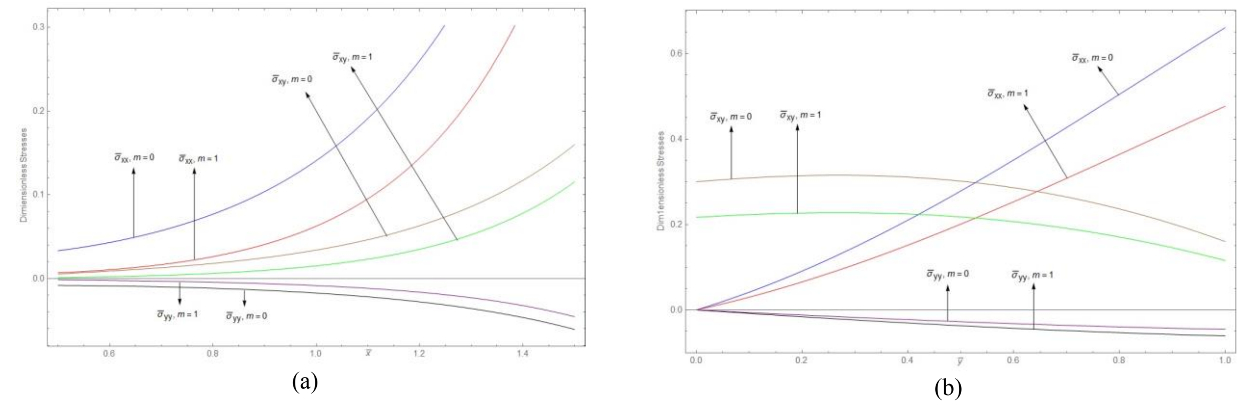


Fig.4
a) Variation of dimensionless stresses (plane stress field) along *x*–axis. b) Variation of dimensionless stresses (plane stress field) along *y*–axis.

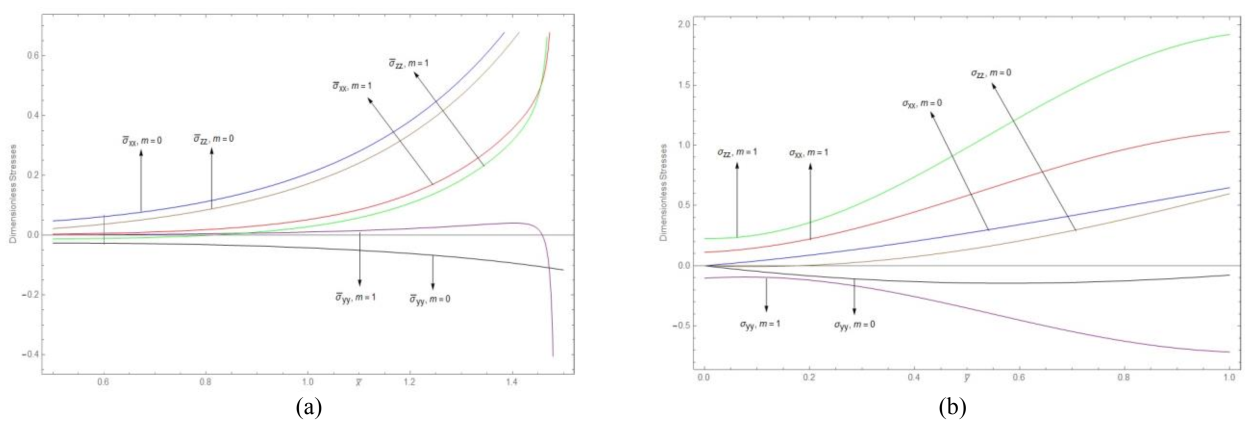


Fig.5
a) Variation of dimensionless stresses (plane strain field) along *x*–axis. b) Variation of dimensionless stresses (plane strain field) along *y*–axis.

5 CONCLUSIONS

In this paper, we have investigated temperature and thermal stresses in a rectangular plate subjected to sectional heating. The material properties except Poisson's ratio and density are considered to vary by simple power law along *x* direction. Solution for the transient two-dimensional heat conduction equation with internal heat generation and its associated thermal stresses for a rectangular plate with inhomogeneous material properties is obtained in the form of Bessel's and trigonometric series. Numerical computations are carried out for a mixture of Copper and Zinc metals in the ratio 70:30 respectively and the transient state temperature field and thermal stresses are examined.

The influence of inhomogeneity grading is investigated by changing parameter m. We have obtained the following results during our investigation.

The nature of temperature distribution and displacement is exponential when plotted along x–direction and sinusoidal when plotted along y–direction for different values of parameter m.

The nature of stresses in the plane stress field is exponential along x-direction and linear along y-direction for both homogeneous and nonhomogeneous regions. Whereas in the plane strain field it is exponential along both x and y-directions.

With increase in the parameter m the magnitude of temperature is found to be increased along both x and y-directions. Due to internal heat generation, sudden change in temperature is observed for $\bar{X} > 1$ and $\bar{Y} > 0.5$.

By choosing some different material for numerical computations, particular cases of special interest can be studied. Also by assigning suitable values to the material parameters in the equations of temperature and thermal stresses special case study can also be done.

APPENDIX A

Consider the differential equation

$$x^2\theta'' + x\theta' + \gamma^2\theta = 0, \quad x \in [a,b], a > 0, b > 0 \tag{A.1}$$

with boundary conditions

$$\begin{aligned} \theta &= 0, \qquad at\ x = a \\ \theta &= 0, \qquad at\ x = b \end{aligned} \tag{A.2}$$

The general solution of (A.1) is given by

$$\theta(x) = C_1\cos(\gamma\log x) + C_2\sin(\gamma\log x),\ x > 0 \tag{A.3}$$

where C_1 and C_2 are arbitrary constants.

To obtain the solution of (A.1) that satisfies conditions (A.2), we have

$$C_1\cos(\gamma\log a) + C_2\sin(\gamma\log a) = 0 \tag{A.4}$$

$$C_1\cos(\gamma\log b) + C_2\sin(\gamma\log b) = 0 \tag{A.5}$$

From (4) and (5), we get

$$\begin{aligned} \frac{C_1}{C_2} &= -\tan(\gamma\log a) \\ \frac{C_1}{C_2} &= -\tan(\gamma\log b) \end{aligned} \tag{A.6}$$

Then the function given by (A.3) is a solution of (A.1) subject to conditions (A.2), if γ is a root of the transcendental equation

$$\sin(\gamma\log a)\cos(\gamma\log b) - \sin(\gamma\log b)\cos(\gamma\log a) = 0 \tag{A.7}$$

Hence we take $\gamma_i\ (i = 1,2,3,........)$ to be the real and positive roots of Eq. (A.7).

From Eq. (A.4) and (A.5), we have

$$\theta_i(x) = \frac{C_1}{\sin(\gamma_i \log a)}\left[\cos(\gamma_i \log x)\sin(\gamma_i \log a) - \sin(\gamma_i \log x)\cos(\gamma_i \log a)\right] \tag{A.8}$$

$$\theta_i(x) = \frac{C_1}{\sin(\gamma_i \log b)}\left[\cos(\gamma_i \log x)\sin(\gamma_i \log b) - \sin(\gamma_i \log x)\cos(\gamma_i \log b)\right] \tag{A.9}$$

We define

$$Z_i = \sin(\gamma_i \log a) + \sin(\gamma_i \log b)$$
$$W_i = \cos(\gamma_i \log a) + \cos(\gamma_i \log b)$$

Then

$$S(\gamma_i x) = Z_i \cos(\gamma_i \log x) - W_i \sin(\gamma_i \log x) \tag{A.10}$$

Is taken to be the solution of (A.1) - (A.2). By Sturm-Liouville theory [2], the functions of the system (A.10) are orthogonal on the interval [*a, b*] with weight function *x* that is

$$\int_a^b x\, S(\gamma_i x) S(\gamma_j x)\, dx = \begin{cases} S(\gamma_i); & i = j \\ 0; & i \neq j \end{cases} \tag{A.11}$$

where $S(\gamma_i) = \left\| \sqrt{x}\, S(\gamma_i x) \right\|_2^2$ is the weighted L^2 norm. If a function *f(x)* and its first derivative are piecewise continuous on the interval $[a, b]$, then the relation

$$T[f(x), a, b; \gamma_i] = \bar{f}(\gamma_i) = \int_a^b x\, f(x) S(\gamma_i x)\, dx \tag{A.12}$$

Defines a linear integral transform. To derive the inversion formula for this transform, let

$$f(x) = \sum_{i=1}^{\infty} [a_i\, S(\gamma_i x)] \tag{A.13}$$

On multiplying Eq. (A.13) by $x\, S(\gamma_i x)$ and integrating both sides with respect to *x*, we obtain the coefficients as:

$$a_i = \frac{1}{S(\gamma_i)} \int_a^b x\, f(x) S(\gamma_i x)\, dx = \frac{\bar{f}(\gamma_i)}{S(\gamma_i)}; \quad i = 1, 2, 3, ... \tag{A.14}$$

Hence the inversion formula becomes

$$f(x) = \sum_{i=1}^{\infty} \frac{\bar{f}(\gamma_i)}{S(\gamma_i)} S(\gamma_i x) \tag{A.15}$$

We derive the transform of the following operator

$$Df(x) = \frac{d^2}{dx^2}f(x) + \frac{1}{x}\frac{d}{dx}f(x) + \frac{\gamma^2}{x^2}f(x);\ x \in [a,b] \tag{A.16}$$

Let *I* be the transform of first two terms of *D*, that is

$$I = \int_a^b x\,[f''(x) + (1/x)f'(x)]S(\gamma_i x)\,dx$$

On solving *I*, we get

$$I = x[f'(x)S(\gamma_i x) - \gamma_i f(x)S'(\gamma_i x)]\Big|_a^b + \int_a^b x^{-1}[\gamma_i^2 x^2 S''(\gamma_i x) + \gamma_i x S'(\gamma_i x)]f(x)\,dx \tag{A.17}$$

Since *S* satisfies Eq. (A.1), we have

$$\gamma_i^2 x^2 S''(\gamma_i x) + \gamma_i x S'(\gamma_i x) = -\gamma_i^2 S(\gamma_i x)$$

and

$$\int_a^b x^{-1}[\gamma_i^2 x^2 S''(\gamma_i x) + \gamma_i x S'(\gamma_i x)]f(x)\,dx = \int_a^b x\,[-\gamma^2 / x^2]f(x)S(\gamma_i x)\,dx$$

Also from (A.2), we get

$$S(\gamma_i a) = S(\gamma_i b) == 0$$

Hence

$$I = b\,S(\gamma_i b)f(b) - a\,S(\gamma_i a)f(a) - \gamma_i^2 \bar{f}(\gamma_i) + T[-\gamma^2 / x^2]f(x)$$

Therefore

$$T[Df(x)] = b\,S(\gamma_i b)f(b) - a\,S(\gamma_i a)f(a) - \gamma_i^2 \bar{f}(\gamma_i) \tag{A.18}$$

REFERENCES

[1] Al-Hajri M., Kalla S.L., 2004, On an integral transform involving Bessel functions, *Proceedings of the International Conference on Mathematics and its Applications*.

[2] Birkoff G., Rota G. C., 1989, *Ordinary Differential Equations*, Wiley, New York.

[3] Churchill R.V., 1972, *Operational Mathematics*, Mc-Graw Hill.

[4] Ding S.H., Li X., 2015, Thermoelastic analysis of nonhomogeneous structural materials with an interface crack under uniform heat flow, *Applied Mathematics and Computation Archive* **271**: 22-33.

[5] Edited by the Japan Society of Mechanical Engineers, 1980, *Elastic Coefficient of Metallic Materials*, Japan Society of Mechanical Engineers.

[6] Gupta A.K., Singhal P., 2010, Thermal effect on free vibration of non-homogeneous orthotropic visco-elastic rectangular plate of parabolically varying thickness, *Applied Mathematics* **1**: 456-463.

[7] Gupta A.K., Saini M., Singh S., Kumar R., 2014, Forced vibrations of non-homogeneous rectangular plate of linearly varying thickness, *Journal of Vibration Control* **20**: 876-884.

[8] Hata T., 1983, Thermal stresses in a nonhomogeneous thick plate with surface radiation under steady distribution of temperature, *The Japan Society of Mechanical Engineers* **49**: 1515-1521.

[9] Kassir M.K., 1972, Boussinesq problems for nonhomogeneous solid, *Proceedings of the American Society of Civil Engineers*, *Journal of the Engineering Mechanics Division* **98**: 457-470.

[10] Kawamura R., Huang D., Tanigawa Y., 2001, Thermoelastic deformation and stress analyses of an orthotropic nonhomogeneous rectangular plate, *Proceedings of the Fourth International Congress on Thermal Stresses*.

[11] Kumar Y., 2012, Free vibrations of simply supported nonhomogeneous isotropic rectangular plates of bilinearly varying thickness and elastically restrained edges against rotation using Rayleigh-Ritz method, *Earthquake Engineering and Engineering Vibration* **11**: 273-280.

[12] Lal R., Kumar Y., 2013, Transverse vibrations of nonhomogeneous rectangular plates with variable thickness, *Mechanics of Advanced Materials and Structures* **20**: 264-275.

[13] Manthena V.R., Lamba N.K., Kedar G.D., 2016, Transient thermoelastic problem of a nonhomogeneous rectangular plate, *Journal of Thermal Stresses* **40**: 627-640.

[14] Martynyak R.M., Dmytriv M.I., 2010, Finite-element investigation of the stress-strain state of an inhomogeneous rectangular plate, *Journal of Mathematical Sciences* **168**: 633-642.

[15] Morishita H., Tanigawa Y., 1998, Formulation of three dimensional elastic problem for nonhomogeneous medium and its analytical development for semi-infinite body, *The Japan Society of Mechanical Engineers* **97**: 97-104.

[16] Muravskii G.B., 2008, Response of an elastic half-space with power-law nonhomogeneity to static loads, *Archive of Applied Mechanics* **78**: 965.

[17] Pandita B.B., Kulkarni V.S., 2015, Finite difference approach for nonhomogeneous problem of thermal stresses in cartesian domain, *International Journal of Advances in Applied Mathematics and Mechanics* **3**: 100-112.

[18] Sharma S., Gupta U.S., Singhal P., 2012, Vibration analysis of nonhomogeneous orthotropic rectangular plates of variable thickness resting on winkler foundation, *Journal of Applied Science and Engineering* **15**: 291-300.

[19] Sugano Y., 1987, Transient thermal stresses in a nonhomogeneous doubly connected region, *The Japan Society of Mechanical Engineers* **53**: 941-946.

[20] Tanigawa Y., Ootao Y., Kawamura R., 1991, Thermal bending of laminated composite rectangular plates and nonhomogeneous plates due to partial heating, *Journal of Thermal Stresses* **14**: 285-308.

[21] Tanigawa Y., 1995, Some basic thermoelastic problems for nonhomogeneous structural materials, *Applied Mechanics Reviews* **48**: 287-300.

[22] Tanigawa Y., Kawamura R., Ishida S., 2002, Derivation of fundamental equation systems of plane isothermal and thermoelastic problems for in-homogeneous solids and its applications to semi-infinite body and slab, *Theoretical and Applied Mechanics* **51**: 267-279.

[23] Tokovyy Y., Ma C-C., 2009, Analytical solutions to the 2D elasticity and thermoelasticity problems for inhomogeneous planes and half-planes, *Archive of Applied Mechanics* **79**: 441-456.

[24] Wang C.Y., Wang C.M., 2011, Exact solutions for vibrating nonhomogeneous rectangular membranes with exponential density distribution, *The IES Journal Part A: Civil & Structural Engineering* **4**: 37-40.

[25] Yang Q., Zheng B., Zhang K., Zhu J., 2013, Analytical solution of a bilayer functionally graded cantilever beam with concentrated loads, *Archive of Applied Mechanics* **83**: 455-466.