

Mathematical Modeling of Thermoelastic State of a Thick Hollow Cylinder With Nonhomogeneous Material Properties

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ABSTRACT

The object of the present paper is to study heat conduction and thermal stresses in a hollow cylinder with nonhomogeneous material properties. The cylinder is subjected to sectional heating at the curved surface. All the material properties except for Poisson's ratio and density are assumed to be given by a simple power law in the axial direction. A solution of the two-dimensional heat conduction equation is obtained in the transient state. The solutions are obtained in the form of Bessel's and trigonometric functions. For theoretical treatment, all the physical and mechanical quantities are taken as dimensional, whereas we have considered non-dimensional parameters, for numerical analysis. The influence of inhomogeneity on the thermal and mechanical behaviour is examined. The transient state temperature field and its associated thermal stresses are discussed for a mixture of copper and tin metals in the ratio 70:30 respectively. Numerical calculations are carried out for both homogeneous and nonhomogeneous cylinders and are represented graphically.

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1 INTRODUCTION

THE advancement in efficient modeling and methodology for thermoelastic analysis of structure members requires a variety of the material characteristics to be taken into consideration. Due to the critical importance of such analysis for adequate determination of operational performance of structures, it presents a great deal of interest for scientists in both academia and industry. However, the assumption that the material properties depend on spatial coordinates (material inhomogeneity) presents a major challenge for analytical treatment of relevant heat conduction and thermoelasticity problems. The main difficulty lies in the need to solve the governing equations in the differential form with variable coefficients which are not pre given for arbitrary dependence of thermophysical and thermoelastic material properties on the coordinate. Particularly, for functionally graded materials, whose properties vary continuously from one surface to another, it is impossible, except for few particular cases, to solve the mentioned problems analytically (Tanigawa, [21]). The analytical, semi analytical, and numerical methods for solving the heat conduction and thermoelasticity problems in inhomogeneous solids attract considerable attention in recent years. Many of the existing analytical methods are developed for particular cases of inhomogeneity (e.g. in the form of power or exponential functions of a coordinate, etc.). The methods applicable for wider ranges of

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material properties are oriented mostly on representation of the inhomogeneous solid as a composite consisting of tailored homogeneous layers. However, such approaches are inconvenient for applying to inhomogeneous materials with large gradients of inhomogeneity due to the weak convergence of the solution with increasing the number of layers.

Al-Hajri et al. [1] used an extended integral transform to solve heat conduction equation of a hollow cylinder. Arefi et al. [2] studied the thermoelastic analysis of an arbitrary structure (functionally graded piezoelectric) using energy method. Cho et al. [4] investigated an elastodynamic solution for the thermal shock stress in an orthotropic cylindrical shell using finite Hankel transform and Laplace transform. Deshmukh et al. [5] studied the problem of dynamic thermal stresses in a circular plate by a moving heat source. Deshmukh et al. [6] determined quasi-static thermal stresses due to an instantaneous point heat source. Ghasemi et al. [8] presented the analytical and numerical investigation of a cylindrical shell made of functionally graded materials (FGMs) reinforced by laminated composite subjected to internal pressure. Ghorbanpour et al. [9, 10] studied thermopiezoelectric behaviour of a thick walled cylinder with functionally graded material and stress analysis of carbon nano-tube reinforced composite cylinder made of poly-vinylidene fluoride. Hata [11] studied thermal stresses in a nonhomogeneous medium whose shear modulus and coefficient of thermal expansion are assumed to vary axially. Hosseini et al. [12] obtained an analytical solution in transient thermoelasticity of functionally graded thick hollow cylinders. Jabbari et al. [13] developed the general solution of steady-state one dimensional asymmetric thermal stresses and electrical and mechanical displacements of a hollow cylinder made of functionally graded material and piezoelectric layers. Kassir [14] investigated thermal stress problems in a thick plate and a semi-infinite body in non-homogeneous solids. Kaur et al. [15] applied Seth's transition theory to the problems of thickness variation parameter in a thin rotating disc by finite deformation. Kim et al. [16] adopted a Green's function approach based on the laminate theory for solving the two-dimensional unsteady temperature field (r, z) and the associated thermal stresses in an infinite hollow circular cylinder. Morishita et al. [17] considered a nonhomogeneous semi-infinite body subject to an arbitrary shaped distributed load on its boundary surface as an analytical model, in which the fundamental equations system for the medium are given by three kinds of displacement functions. Noda et al. [19] studied the theoretical treatment of a transient thermoelastic problem involving a functionally graded solid circular disk with piecewise power law due to uniform heat supply from an outer surface. Ootao et al. [20, 21] developed a theoretical analysis of a three-dimensional transient thermal stress problem for a nonhomogeneous hollow sphere with respect to rotating heat source and a hollow circular cylinder due to a moving heat source by assuming nonhomogeneous thermal and mechanical material properties.

Ootao [22, 23, and 24] used piecewise power law nonhomogeneity to study transient thermoelastic analysis for a multilayered hollow cylinder and a functionally graded hollow circular disk. Rezaei et al. [25] presented thermal buckling analysis of rectangular functionally graded plates (FG plates) with an eccentrically located elliptic cutout. Sugano [26] analyzed a plane thermoelastic problem in a non-homogeneous doubly connected region under a transient temperature field by stress function method. Sugano [27] presented an expression for nonzero thermal stress in a nonhomogeneous flat plate with arbitrary variation in mechanical properties under a transient temperature distribution. Sugano [28] formulated a plane thermoelastic problem in a nonhomogeneous doubly connected region under a transient temperature field. Sugano et al. [29] studied transient plane thermal stress problem in a non-homogeneous hollow circular plate by expressing Young's modulus and thermal conductivity in different power laws of radial coordinate. Tanigawa et al. [30] studied elastic behavior for a medium with Kasser's nonhomogeneous material property. They theoretically developed axisymmetric problem for a semi-infinite body subjected to an arbitrarily shaped distributed load and a concentrated load on its boundary surface using the fundamental equations for such a system. Tanigawa et al. [31] established analytical method of development for the plane isothermal and thermoelastic problems by introducing two kinds of displacement functions. Vimal et al. [32] presented a simple formulation for studying the free vibration of shear-deformable functionally graded plates of different shapes with different cutouts using the finite element method. Wang [33] presented analytical method of solving the elastodynamic problem of a solid sphere by decomposing the problem into a quasi-static solution satisfying the inhomogeneous boundary conditions and a dynamic solution satisfying the homogeneous boundary conditions. Zamani et al. [34] used a semi-analytical iterative method for the elastic analysis of thick-walled spherical pressure vessels made of functionally graded materials subjected to internal pressure.

In the present article, we have considered a two-dimensional transient thermoelastic problem of a thick circular disc occupying the space $a \leq r \leq b$, $h_1 \leq z \leq h_2$, subjected to sectional heating at the curved surface. The material properties except Poisson's ratio and density are assumed to be nonhomogeneous given by a simple power law in axial direction.

2 STATEMENT OF THE PROBLEM

2.1 Heat conduction equation

We consider the transient heat conduction equation with initial and boundary conditions in a hollow cylinder given by [29]

$$\frac{1}{r} \frac{\partial}{\partial r} \left(\bar{\lambda}(z) r \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left(\bar{\lambda}(z) \frac{\partial T}{\partial z} \right) = c(z) \rho \frac{\partial T}{\partial t} \quad (1)$$

$$\left. \begin{aligned} T &= f_1(r, z), & \text{at } t = 0 \\ e_1 T + k_1 \frac{\partial T}{\partial r} &= 0, & \text{at } r = a, \quad h_1 \leq z \leq h_2, \quad t > 0 \\ e_2 T + k_2 \frac{\partial T}{\partial r} &= f_2(z, t), & \text{at } r = b, \quad h_1 \leq z \leq h_2, \quad t > 0 \\ T &= 0, & \text{at } z = h_1, \quad a \leq r \leq b, \quad t > 0 \\ T &= 0, & \text{at } z = h_2, \quad a \leq r \leq b, \quad t > 0 \end{aligned} \right\} \quad (2)$$

where $\bar{\lambda}(z)$ and $c(z)$ are the thermal conductivity and calorific capacity of the material respectively in the inhomogeneous region, ρ is the constant density, e_1, e_2, k_1, k_2 are constants.

2.2 Thermoelastic equations

The strain displacement relations, stress-strain relations and equilibrium condition are given by [11]

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \quad e_{rz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \quad (3)$$

$$\left. \begin{aligned} \sigma_{rr} &= 2\mu(z)e_{rr} + \lambda(z)e - (3\lambda(z) + 2\mu(z))\alpha_r(z)T \\ \sigma_{\theta\theta} &= 2\mu(z)e_{\theta\theta} + \lambda(z)e - (3\lambda(z) + 2\mu(z))\alpha_r(z)T \\ \sigma_{zz} &= 2\mu(z)e_{zz} + \lambda(z)e - (3\lambda(z) + 2\mu(z))\alpha_r(z)T \\ \sigma_{rz} &= 2\mu(z)e_{rz} \end{aligned} \right\} \quad (4)$$

$$\left. \begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0 \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} &= 0 \end{aligned} \right\} \quad (5)$$

where $e_{rr}, e_{\theta\theta}, e_{zz}$ are the strain components ($e = e_{rr} + e_{\theta\theta} + e_{zz}$), $\lambda(z)$ and $\mu(z)$ are Lamé constants, $\alpha_r(z)$ is the coefficient of thermal expansion.

Following [11], we assume that the shear modulus $\mu(z)$ and the coefficient of thermal expansion $\alpha_r(z)$ vary in the axial direction given by $\mu(z) = \mu_0 z^\alpha$, $\alpha_r(z) = \alpha_0 z^\alpha$, where μ_0 and $\alpha \geq 0$ are constants, provided that α is related to Poisson's ratio ν by the relation $\alpha\nu = 1 - 2\nu$, where ν is constant.

Using Eqs. (3) and (4) in (5), the displacement equations of equilibrium are obtained as:

$$\left. \begin{aligned} \nabla^2 u - \frac{u}{r^2} + \frac{\nu}{1-2\nu} \frac{\partial e}{\partial r} + \frac{e_{rz}}{\mu(z)} \frac{\partial \mu(z)}{\partial z} + 2 \frac{\partial e_{rz}}{\partial z} - \frac{1+\nu}{1-2\nu} \alpha_T(z) \frac{\partial T}{\partial r} &= 0 \\ \nabla^2 w + \frac{\nu}{1-2\nu} \frac{\partial e}{\partial z} + \frac{\partial e_{rz}}{\partial z} + \frac{e_{zz}}{\mu(z)} \frac{\partial \mu(z)}{\partial z} + \frac{e}{2\mu(z)} \frac{\partial \lambda(z)}{\partial z} - \frac{1+\nu}{1-2\nu} \left[\alpha_T(z) \frac{\partial T}{\partial z} + T \frac{\partial \alpha_T(z)}{\partial z} \right] & \\ - \frac{\alpha_T(z)}{2\mu(z)} T \left[3 \frac{\partial \lambda(z)}{\partial z} + 2 \frac{\partial \mu(z)}{\partial z} \right] + \frac{1}{r} \frac{\partial u}{\partial z} &= 0 \end{aligned} \right\} \quad (6)$$

where ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (7)$$

The solution of Eq. (6) without body forces can be expressed by the Goodier's thermoelastic displacement potential ϕ and the Boussinesq harmonic functions φ and ψ as: [18]

$$\left. \begin{aligned} u &= \frac{\partial \varphi}{\partial r} + \frac{\partial \phi}{\partial r} + z \frac{\partial \psi}{\partial r} \\ w &= \frac{\partial \varphi}{\partial z} + \frac{\partial \phi}{\partial z} + z \frac{\partial \psi}{\partial z} - (3-4\nu)\psi \end{aligned} \right\} \quad (8)$$

In which the three functions must satisfy the conditions

$$\nabla^2 \phi = K \tau, \quad \nabla^2 \varphi = 0 \quad \text{and} \quad \nabla^2 \psi = 0 \quad (9)$$

where $K(z) = \frac{(1+\nu)}{(1-\nu)} \alpha_T(z)$ is the restraint coefficient and $\tau = T - T_i$, in which T_i is the surrounding temperature. If we take

$$-\int (\varphi + z \psi) dz = M \quad (10)$$

In the above Eq. (8), Michell's function M may be used instead of Boussinesq harmonic functions φ and ψ . Hence Eq. (8) reduces to

$$\left. \begin{aligned} u &= \frac{\partial \varphi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \\ w &= \frac{\partial \varphi}{\partial z} + 2(1-\nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \end{aligned} \right\} \quad (11)$$

In which Michell's function M must satisfy the condition

$$\nabla^2 \nabla^2 M = 0 \quad (12)$$

Now by using Eq. (11) in Eqs. (4) and (6), the results for thermoelastic fields are obtained as:

$$\nabla^2 \phi - \frac{\partial}{\partial z} (\nabla^2 M) - \frac{1}{r^2} (\phi - \frac{\partial M}{\partial z}) - \frac{1+\nu}{1-2\nu} \alpha_T(z) T = 0 \quad (13)$$

$$\begin{aligned} & \nabla^2 \phi + 2(1-\nu) \left[\nabla^2 M + \frac{\partial}{\partial z} (\nabla^2 M) + \frac{\partial}{\partial r} (\nabla^2 M) \right] + 2(1-\nu) \left[\frac{2-r}{r^3} \frac{\partial M}{\partial r} - \frac{2}{r^2} \frac{\partial^2 M}{\partial r^2} \right] \\ & - \frac{1+\nu}{1-2\nu} \alpha_T(z) T - (3\lambda + 2\mu) \frac{\alpha_T(z)}{2\mu(z)} T + \frac{1}{r} \left[\frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \right] = 0 \end{aligned} \quad (14)$$

The corresponding stresses are given by

$$\begin{aligned} \sigma_{rr} &= 2\mu(z) \frac{\partial}{\partial r} \left(\frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \right) + \lambda(z) \left[\nabla^2 \phi + (1-2\nu) \frac{\partial}{\partial z} (\nabla^2 M) \right] - (3\lambda(z) + 2\mu(z)) \alpha_T(z) T \\ \sigma_{\theta\theta} &= 2\mu(z) \frac{1}{r} \left(\frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \right) + \lambda(z) \left[\nabla^2 \phi + (1-2\nu) \frac{\partial}{\partial z} (\nabla^2 M) \right] - (3\lambda(z) + 2\mu(z)) \alpha_T(z) T \\ \sigma_{zz} &= 2\mu(z) \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial z} + 2(1-\nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) + \lambda(z) \left[\nabla^2 \phi + (1-2\nu) \frac{\partial}{\partial z} (\nabla^2 M) \right] - (3\lambda(z) + 2\mu(z)) \alpha_T(z) T \\ \sigma_{rz} &= \mu(z) \left[\frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \right) + \frac{\partial}{\partial r} \left(\frac{\partial \phi}{\partial z} + 2(1-\nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right] \end{aligned} \quad (15)$$

The boundary conditions on the traction free surface stress functions are

$$\sigma_{rr} \Big|_{r=a} = \sigma_{rr} \Big|_{r=b} = 0 \quad (16)$$

Eqs. (1) to (16) constitute the mathematical formulation of the problem.

3 SOLUTION OF THE PROBLEM

3.1 Heat conduction equation

From Eq. (1) we have

$$\bar{\lambda}(z) \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \bar{\lambda}(z) \frac{\partial^2 T}{\partial z^2} + \frac{\partial T}{\partial z} \frac{\partial \bar{\lambda}(z)}{\partial z} = c(z) \rho \frac{\partial T}{\partial t} \quad (17)$$

The initial and boundary conditions are

$$\left. \begin{aligned} T &= f_1(r, z), & \text{at } t = 0 \\ e_1 T + k_1 \frac{\partial T}{\partial r} &= 0, & \text{at } r = a, \quad h_1 \leq z \leq h_2, \quad t > 0 \\ e_2 T + k_2 \frac{\partial T}{\partial r} &= f_2(z, t), & \text{at } r = b, \quad h_1 \leq z \leq h_2, \quad t > 0 \\ T &= 0, & \text{at } z = h_1, \quad a \leq r \leq b, \quad t > 0 \\ T &= 0, & \text{at } z = h_2, \quad a \leq r \leq b, \quad t > 0 \end{aligned} \right\} \quad (18)$$

For the sake of brevity, we consider

$$\bar{\lambda}(z) = \bar{\lambda}_0 z^\alpha, \quad c(z) = c_0 z^\alpha, \quad \rho = \rho_0, \quad f_1(r, z) = Q_0 \delta(r - r_0) \delta(z - z_0), \quad f_2(z, t) = Q_1 \delta(z - z_0) \sinh(\omega t) \quad (19)$$

Using Eq. (19) in (17), we obtain

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r}\right) + \left(\frac{\partial^2 T}{\partial z^2} + \frac{\alpha}{z} \frac{\partial T}{\partial z}\right) = \frac{1}{\kappa} \frac{\partial T}{\partial t} \tag{20}$$

where $\kappa = (\bar{\lambda}_0 / c_0 \rho_0)$ and

$$\left. \begin{aligned} T &= Q_0 \delta(r - r_0) \delta(z - z_0), & \text{at } t = 0 \\ e_1 T + k_1 \frac{\partial T}{\partial r} &= 0, & \text{at } r = a, \quad h_1 \leq z \leq h_2, \quad t > 0 \\ e_2 T + k_2 \frac{\partial T}{\partial r} &= Q_1 \delta(z - z_0) \sinh(\omega t), & \text{at } r = b, \quad h_1 \leq z \leq h_2, \quad t > 0 \\ T &= 0, & \text{at } z = h_1, \quad a \leq r \leq b, \quad t > 0 \\ T &= 0, & \text{at } z = h_2, \quad a \leq r \leq b, \quad t > 0 \end{aligned} \right\} \tag{21}$$

To remove α from the numerator of Eq. (20), we use the variable transformation $T = z^{(1-\alpha)/2} \Theta(r, z, t)$ Hence Eq. (20) reduces to

$$\left(\frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r} \frac{\partial \Theta}{\partial r}\right) + \left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{1}{z} \frac{\partial \Theta}{\partial z} + \frac{\gamma^2}{z^2} \Theta\right) = \frac{1}{\kappa} \frac{\partial \Theta}{\partial t} \tag{22}$$

where $\gamma^2 = ((\alpha - 1) / 2)$ and

$$\left. \begin{aligned} \Theta &= Q_0 z^{((\alpha-1)/2)} \delta(r - r_0) \delta(z - z_0), & \text{at } t = 0 \\ e_1 \Theta + k_1 \frac{\partial \Theta}{\partial r} &= 0, & \text{at } r = a, \quad h_1 \leq z \leq h_2, \quad t > 0 \\ e_2 \Theta + k_2 \frac{\partial \Theta}{\partial r} &= Q_1 z^{((\alpha-1)/2)} \delta(z - z_0) \sinh(\omega t), & \text{at } r = b, \quad h_1 \leq z \leq h_2, \quad t > 0 \\ \Theta &= 0, & \text{at } z = h_1, \quad a \leq r \leq b, \quad t > 0 \\ \Theta &= 0, & \text{at } z = h_2, \quad a \leq r \leq b, \quad t > 0 \end{aligned} \right\} \tag{23}$$

To solve the differential Eq. (22) using integral transform technique, we introduce the extended integral transform [1] of order i over the variable z as given below (Refer Appendix A).

$$T[f(z), a, b; \gamma_i] = \bar{f}(\gamma_i) = \int_{h_1}^{h_2} z f(z) S(\gamma_i z) dz \tag{24}$$

Here $S(\gamma_i z)$ is the kernel of the transform given by

$$S(\gamma_i z) = Z_i \cos(\gamma_i \log z) - W_i \sin(\gamma_i \log z), \quad z > 0 \tag{25}$$

where $Z_i = \sin(\gamma_i \log h_1) + \sin(\gamma_i \log h_2)$, $W_i = \cos(\gamma_i \log h_1) + \cos(\gamma_i \log h_2)$ and $\gamma_i (i = 1, 2, 3, \dots)$ are the real and positive roots of the transcendental equation

$$\sin(\gamma \log h_1) \cos(\gamma \log h_2) - \sin(\gamma \log h_2) \cos(\gamma \log h_1) = 0 \tag{26}$$

The inversion formula is

$$f(z) = \sum_{i=1}^{\infty} \frac{\bar{f}(\gamma_i)}{S(\gamma_i)} S(\gamma_i z) \quad (27)$$

where

$$\int_{h_1}^{h_2} z S(\gamma_i z) S(\gamma_j z) dz = \begin{cases} S(\gamma_i); & i = j \\ 0; & i \neq j \end{cases} \quad (28)$$

Hence Eq. (22) becomes

$$\left(\frac{\partial^2 \bar{\Theta}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\Theta}}{\partial r} \right) - \gamma_i^2 \bar{\Theta} = \frac{1}{\kappa} \frac{\partial \bar{\Theta}}{\partial t} \quad (29)$$

$$\left. \begin{aligned} \bar{\Theta} &= Q_0 g_0 \delta(r - r_0), & \text{at } t = 0 \\ e_1 \bar{\Theta} + k_1 \frac{\partial \bar{\Theta}}{\partial r} &= 0, & \text{at } r = a, \quad h_1 \leq z \leq h_2, \quad t > 0 \\ e_2 \bar{\Theta} + k_2 \frac{\partial \bar{\Theta}}{\partial r} &= Q_1 g_0 \sinh(\omega t), & \text{at } r = b, \quad h_1 \leq z \leq h_2, \quad t > 0 \end{aligned} \right\} \quad (30)$$

where $g_0 = \int_{h_1}^{h_2} z^{((\alpha+1)/2)} \delta(z - z_0) S(\gamma_i z) dz$

We use the transform given in [1] to solve Eq. (29) and use the boundary conditions given by Eq. (30), and obtain

$$\frac{\partial \bar{\bar{\Theta}}}{\partial t} + A_1 \bar{\bar{\Theta}} = A_2 \sinh(\omega t) \quad (31)$$

where $A_1 = \kappa(q_n^2 + \gamma_i^2)$, $A_2 = \frac{\kappa h_1}{k_2} M(q_n h_1) Q_1 g_0$

$$\bar{\bar{\Theta}} = Q_0 g_0 r_0 M(q_n r_0), \quad \text{at } t = 0 \quad (32)$$

Here $M(q_n r)$ is the kernel of the transformation given by

$$M(q_n r) = [B(q_n a, e_1, k_1) + B(q_n b, e_2, k_2)] J_0(q_n r) - [A(q_n a, e_1, k_1) + A(q_n b, e_2, k_2)] Y_0(q_n r)$$

In which

$$\begin{aligned} A(q_n r, e_n, k_n) &= e_n J_0(q_n r) + k_n q_n J_0'(q_n r); \quad n=1,2; \quad r=a,b \\ B(q_n r, e_n, k_n) &= e_n Y_0(q_n r) + k_n q_n Y_0'(q_n r); \quad n=1,2; \quad r=a,b \end{aligned}$$

Here J_0 and Y_0 are Bessel's function of first kind and second kind, respectively and q_n are the positive roots of the transcendental equation $B(q_n a, e_1, k_1) \times A(q_n b, e_2, k_2) - A(q_n a, e_1, k_1) \times B(q_n b, e_2, k_2) = 0$

Applying Laplace transform and its inverse on Eq. (31) by using the initial condition given in Eq. (32), we obtain

$$\bar{\bar{\Theta}}(n, t) = E_1 \exp(-A_1 t) + E_2 \exp(-\omega t) + E_3 \exp(\omega t) \quad (33)$$

where

$$E_1 = \frac{A_2 \omega}{A_1^2 - \omega^2} + A_3, \quad E_2 = \frac{A_2}{2\omega - 2A_1}, \quad E_3 = \frac{A_2}{2\omega + 2A_1}, \quad A_3 = Q_0 g_0 r_0 M(q_n r_0)$$

Applying inverse transform on Eq. (33), we obtain [1]

$$\bar{\Theta}(r, t) = \sum_{n=1}^{\infty} \frac{E_1 \exp(-A_1 t) + E_2 \exp(-\omega t) + E_3 \exp(\omega t)}{M(q_n)} M(q_n r) \tag{34}$$

Applying inverse transform defined in Eq. (27) on the above Eq. (34), we obtain

$$\Theta(r, z, t) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{\bar{\Theta}(r, t)}{S(\gamma_i)} \times S(\gamma_i z), \quad z > 0 \tag{35}$$

Using Eq. (35) in the equation $T = z^{((1-\alpha)/2)} \Theta(r, z, t)$, we obtain

$$T(r, z, t) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \{ \xi_1 \times \xi_2(t) \times [\xi_3 J_0(q_n r) - \xi_4 Y_0(q_n r)] \times g_1(z) \} \tag{36}$$

where

$$\begin{aligned} \xi_1 &= 1/[M(q_n) \times S(\gamma_i)], \quad \xi_2(t) = E_1 \exp(-A_1 t) + E_2 \exp(-\omega t) + E_3 \exp(\omega t), \\ \xi_3 &= [B(q_n h_1, e_1, k_1) + B(q_n h_2, e_2, k_2)], \quad \xi_4 = [A(q_n h_1, e_1, k_1) + A(q_n h_2, e_2, k_2)] \\ g_1(z) &= z^{((1-\alpha)/2)} \times S(\gamma_i z), \quad z > 0 \end{aligned}$$

3.2 Thermoelastic equations

Referring to the heat conduction Eq. (17) and its solution given by Eq. (36), the solution for the Goodier’s thermoelastic displacement potential ϕ governed by Eq. (9) is obtained as:

$$\phi = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{K(z)(T^2(r, z, t) - T_i(r, z))}{\xi_1 \times \xi_2(t) \times [\xi_3 J_0(q_n r) - \xi_4 Y_0(q_n r)] \times [-q_n^2 g_1(z) + g_1''(z)]} \tag{37}$$

Similarly, the solution for Michell’s function M assumed so as to satisfy the governed condition of Eq.(12) is obtained as:

$$M = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \{ z^{((1-\alpha)/2)} \times [\cos(\log z) - \sin(\log z)] \exp(\omega t) [C_m J_0(q_n r) + D_m r Y_0(q_n r)] \} \tag{38}$$

where C_m and D_m are constants.

Now, in order to obtain the displacement components, we substitute the values of ϕ and M in Eq. (11), and obtain

$$u = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \{ \phi_{,r} - g_2'(z) \exp(\omega t) [-C_m q_n J_1(q_n r) + D_m [-r q_n Y_1(q_n r) + Y_0(q_n r)]] \} \tag{39}$$

$$w = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \{ \phi_{,z} + (2-2\nu)[g_2(z) \exp(\omega t)[-C_m q_n^2 J_0(q_n r) + D_m[-3q_n Y_1(q_n r) + (1/r)Y_0(q_n r) - q_n r Y_0(q_n r) + (1/r)Y_1(q_n r)]]] + (1-2\nu)[g_2''(z) \exp(\omega t)[C_m J_0(q_n r) + D_m r Y_0(q_n r)]] \} \quad (40)$$

where $g_2(z) = z^{(1-\alpha)/2} \times [\cos(\log z) - \sin(\log z)]$. Using the displacement components given by Eqs. (39) and (40) in Eq. (15), the components of stresses can be obtained. Also by using the traction free conditions given by Eq. (16) the constants C_m and D_m can be determined. Since the equations of stresses and constants C_m and D_m obtained so are very large, hence we have not mentioned them here. However numerical calculations are carried out by using Mathematica software.

Based on composite material law, final product properties will be sum of primary materials properties multiplied by their volume fractions. In functionally graded materials where each material ratio is variable through thickness, as shown in Fig. 1, final properties are also varied through thickness.

4 NUMERICAL RESULTS AND DISCUSSION

The numerical computations have been carried out for a mixture of Copper and Tin metals [7] in the ratio 70:30 respectively, with non-dimensional variables as given below.

$$\theta = \frac{T}{T_R}, \quad \rho = \frac{r}{a}, \quad \zeta = \frac{z - h_1}{a}, \quad \tau = \frac{\kappa t}{a^2}, \quad \bar{h} = \frac{h}{a}, \quad (\bar{\sigma}_{rr}, \bar{\sigma}_{\theta\theta}, \bar{\sigma}_{zz}, \bar{\sigma}_{rz}) = \frac{(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz})}{E \alpha_0 \theta_R}$$

with parameters $a = 1 \text{ cm}$, $b = 3 \text{ cm}$, $h_1 = 2 \text{ cm}$, $h_2 = 4 \text{ cm}$, Reference Temperature $T_R = 32^\circ \text{C}$, Thermal expansion coefficient $\alpha_0 = 17 \times 10^{-6} / ^\circ \text{C}$, $\kappa = 1.11 \text{ cm}^2 / \text{sec}$. Here 1.3, 2.4, 4.1, 5.5, 7.2, 8.7, 10.3, 11.8, 13.4, 14.9, 16.5, 18.1, 21.2, 24.4, 25.9 are the real and positive roots of the transcendental equation $B(q_n a, e_1, k_1) \times A(q_n b, e_2, k_2) - A(q_n a, e_1, k_1) \times B(q_n b, e_2, k_2) = 0$. Also 2.2, 3.5, 4.9, 6.5, 8.1, 10.2, 12.6, 14.8, 16.3, 18.8, 20.5, 22.3, 25.2, 28.1, 32.2 are the real and positive roots of the transcendental equation $\sin(\gamma \log h_1) \cos(\gamma \log h_2) - \sin(\gamma \log h_2) \cos(\gamma \log h_1) = 0$.

The Young's modulus E is given by the following equation [7]

$$E(z) = (0.1174 - 0.2246z - 1.347z^2 - 5.814z^3 + 8.495z^4) \times 9.8 \times 10^7 \text{ N/cm}^2$$

Here z : weight of tin % $\div 100$, $0 \leq z \leq 0.3$, Let $z = 0.3$, then $E = 4.41 \times 10^7 \text{ N/cm}^2$.

For different values of parameter α , the Poisson's ratio ν and Shear modulus μ_0 are calculated by using the formula $\alpha\nu = 1 - 2\nu$, $\mu_0 = \frac{E}{2(1+\nu)}$

For Homogeneous Cylinder: $\alpha = 0$, Poisson's ratio $\nu = 0.5$. Shear modulus $\mu_0 = 1.47 \times 10^7 \text{ N/cm}^2$

For Nonhomogeneous Cylinder: $\alpha = 1.5$, Poisson's ratio $\nu = 0.286$. Shear modulus $\mu_0 = 1.715 \times 10^7 \text{ N/cm}^2$

The Figs. (1 to 5) on the left are of homogeneous hollow cylinder and on the right are those of non-homogeneous hollow cylinder.

Fig. 1 shows the variation of dimensionless temperature in radial line (ρ - direction) for different values of dimensionless thickness $\zeta = 0.25, 0.5, 0.75, 1, 1.5$. From the graph it is seen that the nature is sinusoidal. The temperature has a finite value at the curved surface due to surrounding temperature. Because of the sectional heating at the curved surface, thermal energy is accumulated at the central region causing deformation and hence thermal expansion. The temperature is slowly decreasing towards the middle of the cylinder and increasing towards the inner radius in the range $1 \leq \rho \leq 1.8$. The absolute value of temperature is less at the lower and upper surfaces whereas high in the middle portion of the cylinder. Also the magnitude of temperature is low for homogeneous cylinder as compared to non-homogeneous cylinder.

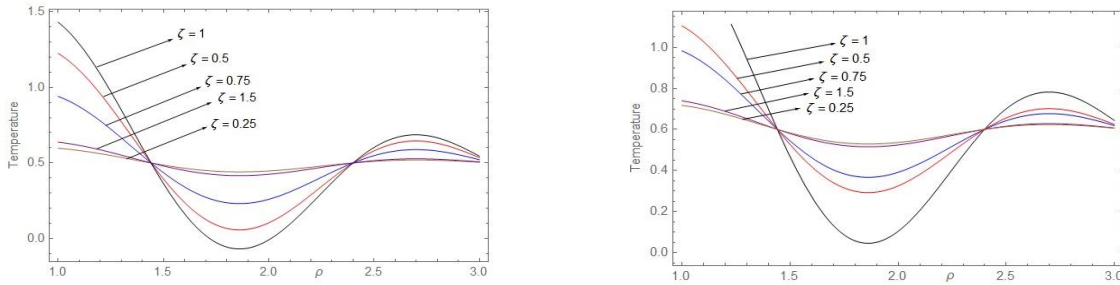


Fig.1
Variation of dimensionless temperature with ρ and ζ .

Fig. 2 shows the variation of dimensionless radial stress in radial direction for different values of dimensionless thickness ζ . It is seen that tensile stress occurs in both homogeneous as well as non-homogeneous cylinders. The radial stress is peak at $\rho = 2.1$. Its magnitude is decreasing from the upper surface towards the lower surface. Also the radial stress is zero at both the radial ends (for both homogeneous and non-homogeneous cylinders), which agrees with the prescribed traction free boundary conditions.

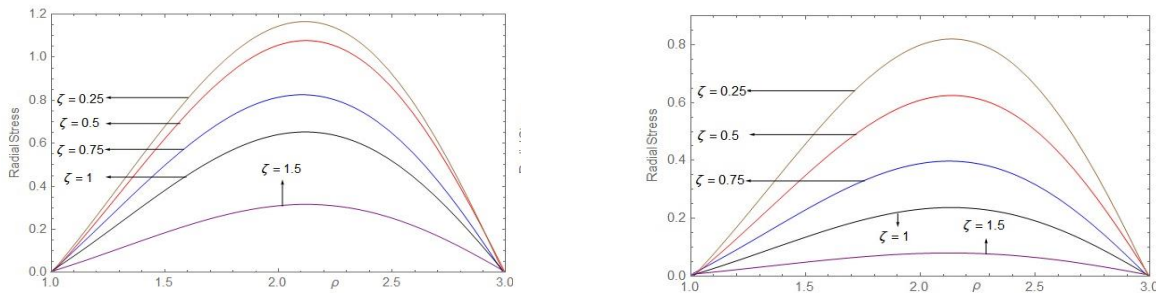


Fig.2
Variation of dimensionless radial stress with ρ and ζ .

Fig. 3 shows the variation of dimensionless tangential stress in radial direction for different values of dimensionless thickness ζ . The tangential stress is tensile in the range $2.5 \leq \rho \leq 3$ and $1.3 \leq \rho \leq 1.5$, whereas compressive for the remaining range. Its magnitude is high at the curved surface and is slowly decreasing towards the inner radius. For homogeneous cylinder the magnitude of tangential stress is high at both the lower and upper surfaces, while low for in the middle portion. For non-homogeneous cylinder it is high at $\zeta = 0.25, 0.5$ and low for remaining values of ζ .

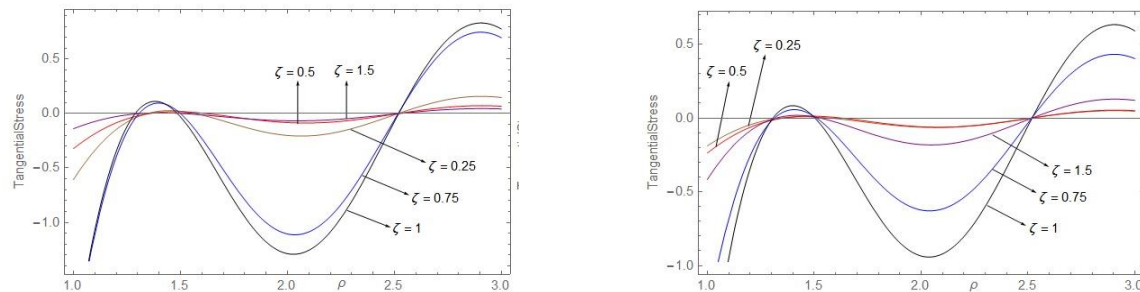


Fig.3
Variation of dimensionless tangential stress with ρ and ζ .

Fig. 4 shows the variation of dimensionless axial stress in radial direction for different values of dimensionless thickness ζ . The axial stress is tensile in the range $2.35 \leq \rho \leq 3$ and $1 \leq \rho \leq 1.5$, whereas compressive for the

remaining range. It is converging to zero at the central region for both homogeneous and non-homogeneous cylinders. The magnitude of crest and trough is high in the outer and inner radius due to the accumulation of thermal energy, whereas low in the middle region. Also the magnitude is decreasing from the upper towards the lower surface in the range $1.5 \leq \rho \leq 2.35$, while increasing in the remaining regions.

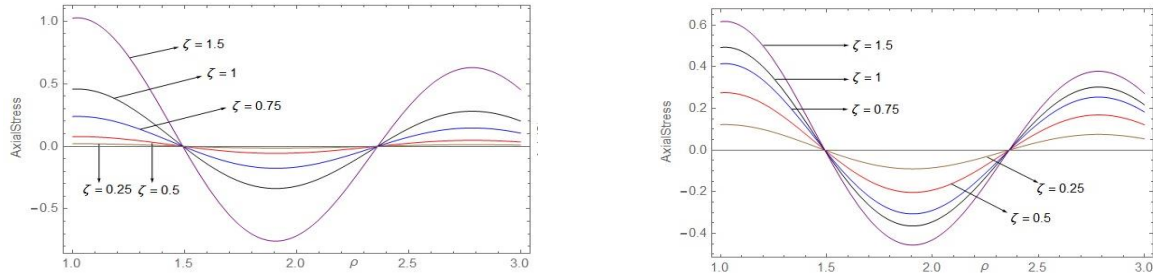


Fig.4
Variation of dimensionless axial stress with ρ and ζ .

Fig. 5 shows the variation of dimensionless shear stress in radial direction for different values of dimensionless thickness ζ . It is observed that the graphs of both homogeneous and non-homogeneous cylinders are sinusoidal in nature. The shear stress is tensile in the range $2.4 \leq \rho \leq 2.9$ and $1.35 \leq \rho \leq 1.9$, whereas compressive for the remaining range. In the tensile region the magnitude is decreasing from the upper towards the lower surface of the cylinder, whereas in the compressive regions it is increasing from the upper towards the lower surface. The magnitude of homogeneous cylinder is high as compared to that of non-homogeneous cylinder.

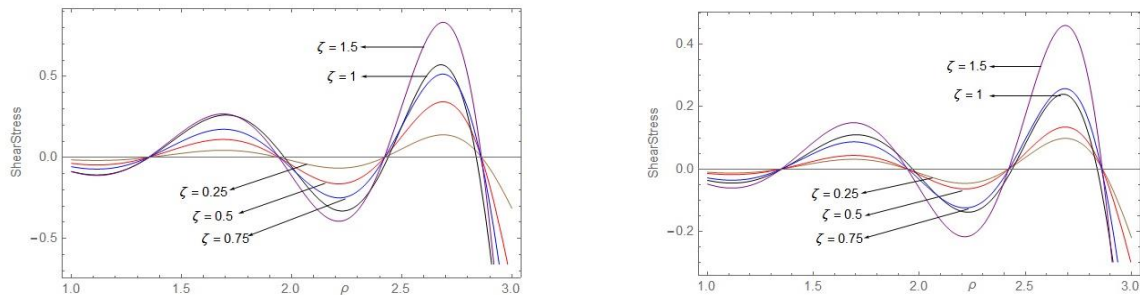


Fig.5
Variation of dimensionless shear stress with ρ and ζ .

5 CONCLUSIONS

In the present paper, we have investigated temperature and thermal stresses in a thick hollow cylinder subjected to sectional heating. The material properties except for Poisson's ratio and density are considered to vary by simple power law along the axial direction. We obtained the solution for transient two-dimensional conductivity equation and its associated thermal stresses for a thick hollow cylinder with inhomogeneous material properties. The solutions are obtained in the form of Bessel's and trigonometric functions. Numerical analysis are carried out for a mixture of copper and tin metals in the ratio 70:30 respectively and the transient state temperature field and thermal stresses are examined. Furthermore, the influence of inhomogeneity grading is investigated by changing parameter α .

During our investigation, the following results were obtained.

- (1) The nature of the figures of temperature and all stresses are sinusoidal when plotted for radial direction.
- (2) By increasing the parameter α , it was observed that the absolute values of temperature and thermal stresses were decreased for non-homogeneous cylinder.

- (3) By decreasing the parameter α , it was observed that the absolute values of temperature and thermal stresses were increased for non-homogeneous cylinder.
- (4) Particular cases of special interest can be studied by assigning suitable values to the material parameters in the equations of temperature and thermal stresses as well as by taking some different material for numerical analysis.
- (5) This type of theoretical analysis may be used under high temperature conditions in non-homogeneous and functionally graded materials.

APPENDIX A

Consider the differential equation

$$z^2 \theta'' + z \theta' + \gamma^2 \theta = 0, \quad z \in [h_1, h_2], \quad h_1 > 0, \quad h_2 > 0 \quad (\text{A.1})$$

with boundary conditions

$$\begin{aligned} \theta &= 0, & \text{at } z &= h_1 \\ \theta &= 0, & \text{at } z &= h_2 \end{aligned} \quad (\text{A.2})$$

The general solution of (A.1) is given by

$$\theta(z) = C_1 \cos(\gamma \log z) + C_2 \sin(\gamma \log z), \quad z > 0 \quad (\text{A.3})$$

where C_1 and C_2 are arbitrary constants.

To obtain the solution of (A.1) that satisfies conditions (A.2), we have

$$C_1 \cos(\gamma \log h_1) + C_2 \sin(\gamma \log h_1) = 0 \quad (\text{A.4})$$

$$C_1 \cos(\gamma \log h_2) + C_2 \sin(\gamma \log h_2) = 0 \quad (\text{A.5})$$

From (4) and (5), we get

$$\begin{aligned} \frac{C_1}{C_2} &= -\tan(\gamma \log h_1) \\ \frac{C_1}{C_2} &= -\tan(\gamma \log h_2) \end{aligned} \quad (\text{A.6})$$

Then the function given by (A.3) is a solution of (A.1) subject to conditions (A.2), if γ is a root of the transcendental equation

$$\sin(\gamma \log h_1) \cos(\gamma \log h_2) - \sin(\gamma \log h_2) \cos(\gamma \log h_1) = 0 \quad (\text{A.7})$$

Hence we take γ_i ($i = 1, 2, 3, \dots$) to be the real and positive roots of Eq. (A.7).

From Eq. (A.4) and (A.5), we have

$$\theta_i(z) = \frac{C_1}{\sin(\gamma_i \log h_1)} [\cos(\gamma_i \log z) \sin(\gamma_i \log h_1) - \sin(\gamma_i \log z) \cos(\gamma_i \log h_1)] \quad (\text{A.8})$$

$$\theta_i(z) = \frac{C_1}{\sin(\gamma_i \log h_2)} [\cos(\gamma_i \log z) \sin(\gamma_i \log h_2) - \sin(\gamma_i \log z) \cos(\gamma_i \log h_2)] \quad (\text{A.9})$$

We define

$$\begin{aligned} Z_i &= \sin(\gamma_i \log h_1) + \sin(\gamma_i \log h_2) \\ W_i &= \cos(\gamma_i \log h_1) + \cos(\gamma_i \log h_2) \end{aligned}$$

Then

$$S(\gamma_i z) = Z_i \cos(\gamma_i \log z) - W_i \sin(\gamma_i \log z) \quad (\text{A.10})$$

Is taken to be the solution of (A.1) - (A.2). By Sturm-Liouville theory [3], the functions of the system (A.10) are orthogonal on the interval $[a, b]$ with weight function x , that is

$$\int_{h_1}^{h_2} z S(\gamma_i z) S(\gamma_j z) dz = \begin{cases} S(\gamma_i); & i = j \\ 0; & i \neq j \end{cases} \quad (\text{A.11})$$

where $S(\gamma_i) = \left\| \sqrt{z} S(\gamma_i z) \right\|_2^2$ is the weighted L^2 norm. If a function $f(z)$ and its first derivative are piecewise continuous on the interval $[h_1, h_2]$, then the relation

$$T[f(z), a, b; \gamma_i] = \bar{f}(\gamma_i) = \int_{h_1}^{h_2} z f(z) S(\gamma_i z) dz \quad (\text{A.12})$$

Defines a linear integral transform. To derive the inversion formula for this transform, let

$$f(z) = \sum_{i=1}^{\infty} [a_i S(\gamma_i z)] \quad (\text{A.13})$$

On multiplying Eq. (A.13) by and integrating both sides with respect to z , we obtain the coefficients as:

$$a_i = \frac{1}{S(\gamma_i)} \int_{h_1}^{h_2} z f(z) S(\gamma_i z) dz = \frac{\bar{f}(\gamma_i)}{S(\gamma_i)}; \quad i = 1, 2, 3, \dots \quad (\text{A.14})$$

Hence the inversion formula becomes

$$f(z) = \sum_{i=1}^{\infty} \frac{\bar{f}(\gamma_i)}{S(\gamma_i)} S(\gamma_i z) \quad (\text{A.15})$$

We derive the transform of the following operator

$$Df(z) = \frac{d^2}{dz^2} f(z) + \frac{1}{z} \frac{d}{dz} f(z) + \frac{\gamma^2}{z^2} f(z); \quad z \in [h_1, h_2] \quad (\text{A.16})$$

Let I be the transform of first two terms of D , that is

$$I = \int_{h_1}^{h_2} z [f''(z) + (1/z)f'(z)] S(\gamma_i z) dz$$

On solving I , we get

$$I = z [f'(z)S(\gamma_i z) - \gamma_i f(z)S'(\gamma_i z)] \Big|_{h_1}^{h_2} + \int_{h_1}^{h_2} z^{-1} [\gamma_i^2 z^2 S''(\gamma_i z) + \gamma_i z S'(\gamma_i z)] f(z) dz \quad (\text{A.17})$$

Since S satisfies Eq. (A.1), we have

$$\gamma_i^2 z^2 S''(\gamma_i z) + \gamma_i z S'(\gamma_i z) = -\gamma_i^2 S(\gamma_i z)$$

and

$$\int_{h_1}^{h_2} z^{-1} [\gamma_i^2 z^2 S''(\gamma_i z) + \gamma_i z S'(\gamma_i z)] f(z) dz = \int_{h_1}^{h_2} z [-\gamma^2 / z^2] f(z) S(\gamma_i z) dz$$

Also from (A.2), we get

$$S(\gamma_i h_1) = S(\gamma_i h_2) = 0$$

Hence

$$I = h_2 S(\gamma_i h_2) f(h_2) - h_1 S(\gamma_i h_1) f(h_1) - \gamma_i^2 \bar{f}(\gamma_i) + T[-\gamma^2 / z^2] f(z)$$

Therefore

$$T[Df(z)] = h_2 S(\gamma_i h_2) f(h_2) - h_1 S(\gamma_i h_1) f(h_1) - \gamma_i^2 \bar{f}(\gamma_i) \quad (\text{A.18})$$

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