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Vibration and Stability Analysis of a Pasternak Bonded Double-GNR-System Based on Different Nonlocal Theories

A. Ghorbanpour Arani[1,2*], R. Kolahchi[1], H. Vossough[1], M. Abdollahian[1]

[1]*Faculty of Mechanical Engineering, University of Kashan, Kashan, Islamic Republic of Iran*
[2]*Institute of Nanoscience & Nanotechnology, University of Kashan, Kashan, Islamic Republic of Iran*

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ABSTRACT

This study deals with the vibration and stability analysis of double-graphene nanoribbon-system (DGNRS) based on different nonlocal elasticity theories such as Eringen's nonlocal, strain gradient, and modified couple stress within the framework of Rayleigh beam theory. In this system, two graphene nanoribbons (GNRs) are bonded by Pasternak medium which characterized by Winkler modulus and shear modulus. An analytical approach is utilized to determine the frequency and critical buckling load of the coupled system. The three vibrational states including out-of-phase vibration, in-phase vibration and one GNR being stationary are discussed. A detailed parametric study is conducted to elucidate the influences of the small scale coefficients, stiffness of the internal elastic medium, mode number and axial load on the vibration of the DGNRS. The results reveal that the dimensionless frequency and critical buckling load obtained by the strain gradient theory is higher than the Eringen's and modified couple stress theories. Moreover, the small scale effect in the case of in-phase vibration is higher than that in the other cases. This study might be useful for the design of nano-devices in which GNRs act as basic elements.

Keywords: GNR; Strain gradient theory; Rayleigh beam theory; Coupled system; Modified couple stress theory

1 INTRODUCTION

IN recent years, nano-structural carbon materials including graphene sheet (GS), carbon nanotube (CNT), buckyball, and GNR have attracted considerable interest of scientific communities due to their exceptional and remarkable properties. GS, which can be described as a monolayer of carbon atoms tightly packed into a two-dimensional honeycomb lattice in which carbon atoms bond covalently with their neighbors and was first produced in 2004 [1-3]. GNRs are obtained by patterning GS into narrow strips, which can be observed as quasi-one-dimensional nanomaterials. The electronic structure of GNRs depends on not only the shapes of the edges (zigzag and armchair) but also the ribbon width [4-5]. GNRs possess extraordinary mechanical, thermal, electronic transport and spin transport properties, and a large aspect ratio, which have made them as desirable materials for a wide range of device applications, such as sensors [6].

In order to the mechanical modeling of micro- and nano-structures, the higher-order continuum theories, such as Eringen's nonlocal elasticity, strain gradient elasticity, and modified couple stress have been recently employed. These theories are capable of account and statement of the size-dependent behavior while the classical (local) theory can not explain size-dependent manner and it assumes that the stress at a defined point depends uniquely on the

* Corresponding author. Tel.: +98 9131626594; Fax: +98 361 5912424.
E-mail address: aghorban@kashanu.ac.ir (A.Ghorbanpour Arani).

strain at the same point. Among the size-dependent theories, Eringen's nonlocal theory expresses the stress at a defined point as a function of the strain at all the points of the continuum [7-9]. There are many works in the literature that have used this theory. For instance, Ghorbanpoure Arani et al. [10] investigated the Pasternak foundation effect on the axial and torsional waves propagation in embedded double-walled CNTs using nonlocal elasticity cylindrical shell theory. They deduced that with increasing the value of nonlocal parameter, the wave propagation frequencies are decreased. The strain gradient is more complete theory which was originated by Mindlin [11]. For more details, researchers can refer to Lam et al. [12] and Kong et al. [13]. Based on the strain gradient theory, the strain energy density is considered as a function of the symmetric strain tensor, the dilatation gradient vector, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor. Yin et al. [14] employed strain gradient beam model for dynamics of microscale pipes conveying fluid. Their results showed that the microscale pipe displays remarkable size effect when its outside diameter becomes comparable to the material length scale parameter while the size effect is almost diminishing as the diameter is far greater than the material length scale parameter. Among the aforementioned nonlocal theories, the modified couple stress theory was first introduced by Yang et al. [15] in which the strain energy has been demonstrated to be a quadratic function of the strain tensor and symmetric part of the curvature tensor. For example, Şimşek [16] utilized the modified couple stress theory to study dynamic analysis of an embedded microbeam carrying a moving microparticle. He concluded that the deflection of the microbeam under action of a microparticle predicted by the classical beam theory are always larger than those by the modified couple stress theory. This study aims to analyze vibration and critical buckling load of DGNRS based on aforementioned different nonlocal theories.

Understanding mechanical behaviors of GNRs is a key step for designing many nano-devices. Particularly, vibration and stability responses of GNRs as nano-devices component have great importance. Herein, few researchers analyzed mechanical behaviors of GNRs based on nonlocal elasticity theory which was initiated in the papers of Eringen [7-9]. For instance, nonlocal vibration of embedded double-layer graphene nanoribbons (DLGNRs) in in-phase and anti-phase modes was studied by Shi et al. [17]. They concluded that the vibrational properties of DLGNRs show different behaviors in in-phase and anti-phase modes. Moreover, they showed that the natural frequencies of DLGNR embedded in an elastic matrix are significantly influenced by nonlocal effects, the aspect ratio of DLGNRs and the Winkler foundation modulus. Also, he and his co-workers [18] investigated nonlocal elasticity theory for the buckling of DLGNRs based on a continuum model. Their results indicated that the nonlocal effect has an inverse relationship with the buckling stress, and the nonlocal effect decreases with increasing aspect ratio of DLGNRs.

With respect to developmental works on mechanical behavior analysis of GNRs, it should be noted that none of the papers mentioned above, have considered a coupled DGNRS. But there are studies about coupled systems such as double-carbon nanotube and double-graphene sheet. For example, axial instability of double-nanobeam-systems (DNBS) was studied by Murmu and Adhikari [19] who showed the small scale effects are higher with increasing values of nonlocal parameter for the case of in-phase buckling modes than the out-of-phase buckling modes. Besides, they found that the increase of the stiffness of the coupling elastic medium in DNBS reduces the small-scale effects during the out-of-phase buckling modes. In another work, he and his co-worker [20] analyzed nonlocal elasticity based vibration of initially pre-stressed coupled DNBS. Their study indicated that increasing the stiffness of the springs in coupled nanosystems reduces the nonlocal effects. Furthermore, their results revealed that the small-scale effects in coupled nano systems are more prominent with the increasing nonlocal parameter in the in-phase vibration than in the out-of-phase motion condition. These works are one-dimensional analysis. Analysis on coupled systems of two dimensional graphene sheets was taken up by several researchers lately. Murmu and Adhikari [21] examined nonlocal vibration of bonded double-nanoplate-systems (DNPS). They concluded that the small-scale effects in DNPS are higher with the increasing values of nonlocal parameter for the case of synchronous modes of vibration than in the asynchronous modes. In addition, analytical results clarified that the increase of the stiffness of the coupling springs in DNPS reduces the small-scale effects during the asynchronous modes of vibration. The papers [19-21] have considered the Winkler model for simulation of elastic medium between two nanostructures. In this simplified model, a proportional interaction between pressure and deflection of nanostructure is assumed, which is carried out in the form of discrete and independent vertical springs. Whereas, Pasternak suggested considering not only the normal stresses but also the transverse shear deformation and continuity among the spring elements, and subsequent applications for developing the model for vibration/buckling analysis, which proved to be more accurate than the Winkler model. Recently, Ghorbanpour Arani et al. [22] examined buckling analysis and smart control of single layered GS using elastically coupled polyvinylidene fluoride (PVDF) nanoplate based on the nonlocal Mindlin plate theory. In this study, single layered GS is coupled with PVDF nanoplate by the Pasternak model. This study aims to couple two GNRs by an elastic medium which simulated by the Pasternak model.

However, to date, no report has been found in the literature on vibration and stability analysis of a Pasternak coupled DGNRS. Motivated by these considerations, in order to improve optimum design of nanostructures, we aim to investigate the vibration and stability of a Pasternak coupled DGNRS based on different nonlocal theories including Eringen's nonlocal, strain gradient and modified couple stress. Herein, three cases of out-of-phase vibration, in-phase vibration and one GNR being stationary are considered. The influences of small scale parameter, elastic medium, mode number and axial load on vibration and critical buckling load of coupled DGNRS have been taken into account.

2 MODELING OF PROBLEM

Consider double-GNR-system as shown in Fig. 1. GNRs are modeled with nanobeam so that denoted as nanoribbon-1 and nanoribbon- 2. The two nanoribbons with identical length L and width b are bonded by a Pasternak medium which is characterized by Winkler modulus, k_w and shear modulus, k_g. The elastic modulus, mass density, cross section area, and second moment of inertia of the i th nanobeam are denoted by E_i, ρ_i, A_i, and I_i ($i=1,2$), respectively. The both nanoribbons in coupled system are considered to be identical (i.e. $E_1 = E_2 = E$, $\rho_1 = \rho_2 = \rho$, $A_1 = A_2 = A$,and $I_1 = I_2 = I$). Also, denote the deflections of two nanoribbons by $w_1(x,t)$ and $w_2(x,t)$, respectively. Consider the nanoribbons be subjected to axial forces (i.e. $N_{x1} = N_{x2} = N_x$).

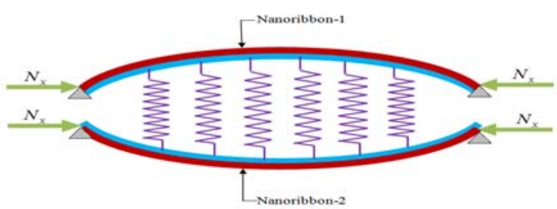

Fig. 1
Schematic of a Pasternak coupled double-nanoribbons-system subjected to axial load.

3 FORMULATION
3.1 Eringen's nonlocal theory

In the Eringen's nonlocal elasticity model, the stress state at a reference point in the body is regarded to be dependent not only on the strain state at this point but also on the strain states at all of the points throughout the body. The basic equations for homogeneous, isotropic and nonlocal elastic solid with zero body forces are given by [10]

$$\begin{aligned} &\sigma_{ij,j} = 0, \\ &\sigma_{ij}(x) = \int \alpha(|x-x'|,\tau) C_{ijkl}\varepsilon_{kl}(x')dV(x'), \qquad \forall x \in V \\ &\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}), \end{aligned} \tag{1}$$

where C_{ijkl} is the elastic module tensor of classical (local) isotropic elasticity; σ_{ij} and ε_{ij} are stress and strain tensors, respectively, and u_i is displacement vector. $\alpha(|x-x'|,\tau)$ is the nonlocal modulus. $|x-x'|$ is the Euclidean distance, and $\tau = e_0 a / l$ is defined that l is the external characteristic length, e_0 denotes a constant appropriate to each material, and a is an internal characteristic length of the material (e. g., length of C-C bond, lattice spacing, granular distance). Consequently, $e_0 a$ is a constant parameter which is obtained with molecular dynamics, experimental results, experimental studies and molecular structure mechanics. The constitutive equation of the nonlocal elasticity can be written as [10]

$$(1-(e_0a)^2\nabla^2)\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{2}$$

where the parameter e_0a denotes the small scale effect on the response of structures in nanosize, and ∇^2 is the Laplace operator in the above equation.

On the basis of Refs. [23-26] and considering the following dimensionless parameter

$$\begin{aligned} &W_i = \frac{w_i}{L} \ \ (i=1,2), \quad X = \frac{x}{L}, \quad \mu = \left(\frac{e_0a}{L}\right)^2, \ K_W = \frac{k_w L^4}{EI}, \\ &K_G = \frac{k_g L^2}{EI}, \ P = \frac{N_x L^2}{EI}, \quad \tau = \frac{t}{L}\sqrt{\frac{E}{\rho}}, \ \alpha = \frac{AL^2}{I}, \end{aligned} \tag{3}$$

The motion equations of coupled DNBS for Raleigh beam model can be introduced as:

For nanoribbon-1,

$$\begin{aligned} &(1+\mu K_G)\frac{\partial^4 W_1}{\partial X^4} + \alpha(\frac{\partial^2 W_1}{\partial \tau^2} - \mu\frac{\partial^4 W_1}{\partial X^2 \partial \tau^2}) - (\frac{\partial^4 W_1}{\partial X^2 \partial \tau^2} - \mu\frac{\partial^6 W_1}{\partial X^4 \partial \tau^2}) + K_W(W_1 - W_2) \\ &-(K_G + \mu K_W)(\frac{\partial^2 W_1}{\partial X^2} - \frac{\partial^2 W_2}{\partial X^2}) + P(\frac{\partial^2 W_1}{\partial X^2} - \mu\frac{\partial^4 W_1}{\partial X^4}) = 0, \end{aligned} \tag{4}$$

For nanoribbon-2,

$$\begin{aligned} &(1+\mu K_G)\frac{\partial^4 W_2}{\partial X^4} + \alpha(\frac{\partial^2 W_2}{\partial \tau^2} - \mu\frac{\partial^4 W_2}{\partial X^2 \partial \tau^2}) - (\frac{\partial^4 W_2}{\partial X^2 \partial \tau^2} - \mu\frac{\partial^6 W_2}{\partial X^4 \partial \tau^2}) + K_W(W_2 - W_1) \\ &-(K_G + \mu K_W)(\frac{\partial^2 W_2}{\partial X^2} - \frac{\partial^2 W_1}{\partial X^2}) + P(\frac{\partial^2 W_2}{\partial X^2} - \mu\frac{\partial^4 W_2}{\partial X^4}) = 0. \end{aligned} \tag{5}$$

For simplifying analysis, the relative movement of two nanoribbons is introduced as:

$$W(X,\tau) = W_1(X,\tau) - W_2(X,\tau). \tag{6}$$

Such that

$$W_1(X,\tau) = W(X,\tau) + W_2(X,\tau), \tag{7}$$

By subtracting Eq. (4) from Eq. (5) and considering Eq. (6) gives:

$$\begin{aligned} &(1+2\mu K_G)\frac{\partial^4 W}{\partial X^4} + \alpha(\frac{\partial^2 W}{\partial \tau^2} - \mu\frac{\partial^4 W}{\partial X^2 \partial \tau^2}) - (\frac{\partial^4 W}{\partial X^2 \partial \tau^2} - \mu\frac{\partial^6 W}{\partial X^4 \partial \tau^2}) + 2K_W W \\ &-2(K_G + \mu K_W)\frac{\partial^2 W}{\partial X^2} + P(\frac{\partial^2 W}{\partial X^2} - \mu\frac{\partial^4 W}{\partial X^4}) = 0 \end{aligned} \tag{8}$$

Considering Eq. (7), Eq. (5) becomes:

$$\begin{aligned} &(1+\mu K_G)\frac{\partial^4 W_2}{\partial X^4} + \alpha(\frac{\partial^2 W_2}{\partial \tau^2} - \mu\frac{\partial^4 W_2}{\partial X^2 \partial \tau^2}) - (\frac{\partial^4 W_2}{\partial X^2 \partial \tau^2} - \mu\frac{\partial^6 W_2}{\partial X^4 \partial \tau^2}) - K_W(W) \\ &+(K_G + \mu K_W)(\frac{\partial^2 W}{\partial X^2}) + P(\frac{\partial^2 W_2}{\partial X^2} - \mu\frac{\partial^4 W_2}{\partial X^4}) = 0 \end{aligned} \tag{9}$$

3.2 Strain gradient theory

Based on the strain gradient theory, the strain energy density is considered as function of the symmetric strain tensor, the dilatation gradient vector, the deviatoric stretch gradient tensor, and the symmetric rotation gradient tensor. In order to characterize these tensors and vectors, there are three independent material length scale parameters in addition to four classical material constants for isotropic linear elastic materials. So the strain energy U that occupying region Ω is represented by [14]

$$U=\frac{1}{2}\int_{\Omega}(\sigma_{ij}\varepsilon_{jk}+p_i\gamma_i+\tau_{ijk}^{(1)}\eta_{ijk}^{(1)}+m_{ij}\chi_{ij})dV, \tag{10}$$

where γ_i represent the dilatation gradient vector, ε_{ij} , $\eta_{ijk}^{(1)}$ and χ_{ij} denote the strain, the deviatoric stretch gradient and the symmetric rotation gradient tensors, respectively which are defined by:

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right), \tag{11}$$

$$\gamma_i=\frac{\partial\varepsilon_{mm}}{\partial x_i}, \tag{12}$$

$$\eta_{ijk}^{(1)}=\frac{1}{3}\left(\frac{\partial\varepsilon_{jk}}{\partial x_i}+\frac{\partial\varepsilon_{ki}}{\partial x_j}+\frac{\partial\varepsilon_{ij}}{\partial x_k}\right)-\frac{1}{15}\left[\delta_{ij}(\frac{\partial\varepsilon_{mm}}{\partial x_k}+2\frac{\partial\varepsilon_{mk}}{\partial x_m})+\delta_{jk}(\frac{\partial\varepsilon_{mm}}{\partial x_i}+2\frac{\partial\varepsilon_{mi}}{\partial x_m})+\delta_{ki}(\frac{\partial\varepsilon_{mm}}{\partial x_j}+\frac{\partial\varepsilon_{mj}}{\partial x_m})\right], \tag{13}$$

$$\chi_{ij}=\frac{1}{2}\left(e_{ipq}\frac{\partial\varepsilon_{qj}}{\partial x_p}+e_{jpq}\frac{\partial\varepsilon_{qi}}{\partial x_p}\right), \tag{14}$$

where u_i,δ_{ij}, and e_{jpq} are the displacement vector, the knocker delta, and the alternate tensor, respectively. The classical stress tensor, σ_{ij} , the higher-order stresses, p_i , $\tau_{ijk}^{(1)}$ and m_{ij} are given by:

$$\sigma_{ij}=k\delta_{ij}\varepsilon_{mm}+2G(\varepsilon_{ij}-\frac{1}{3}\varepsilon_{mm}\delta_{ij})-e_{mij}E_m, \tag{15}$$

$$p_i=2l_0^2G\gamma_i, \tag{16}$$

$$\tau_{ijk}^{(1)}=2l_1^2G\eta_{ijk}^{(1)}, \tag{17}$$

$$m_{ij}=2l_2^2G\chi_{ij}, \tag{18}$$

where k and G are the bulk modulus and the shear modulus, respectively. (l_0,l_1,l_2) are independent material length scale parameters.

On the basis of Ref. [23, 27 and 28] and considering the following and above dimensionless parameters

$$L_0=\frac{l_0}{L},\ L_1=\frac{l_1}{L},\ L_2=\frac{l_2}{L},\lambda=\frac{G}{E}, \tag{19}$$

The motion equations of DNBS for Raleigh beam model can be introduced as:

For nanoribbon-1,

$$
\begin{aligned}
&(1+2L_0^2\lambda\alpha+\frac{8}{15}L_1^2\lambda\alpha+L_2^2\lambda\alpha)\frac{\partial^4 W_1}{\partial X^4}-(2L_0^2\lambda+\frac{4}{5}L_1^2\lambda)\frac{\partial^6 W_1}{\partial X^6}+\alpha(\frac{\partial^2 W_1}{\partial \tau^2})-(\frac{\partial^4 W_1}{\partial X^2\partial \tau^2})\\
&+K_W(W_1-W_2)-K_G(\frac{\partial^2 W_1}{\partial X^2}-\frac{\partial^2 W_2}{\partial X^2})+P\frac{\partial^2 W_1}{\partial X^2}=0
\end{aligned}
\tag{20}
$$

For nanoribbon-2,

$$
\begin{aligned}
&(1+2L_0^2\lambda\alpha+\frac{8}{15}L_1^2\lambda\alpha+L_2^2\lambda\alpha)\frac{\partial^4 W_2}{\partial X^4}-(2L_0^2\lambda+\frac{4}{5}L_1^2\lambda)\frac{\partial^6 W_2}{\partial X^6}+\alpha(\frac{\partial^2 W_2}{\partial \tau^2})-(\frac{\partial^4 W_2}{\partial X^2\partial \tau^2})\\
&+K_W(W_2-W_1)-K_G(\frac{\partial^2 W_2}{\partial X^2}-\frac{\partial^2 W_1}{\partial X^2})+P\frac{\partial^2 W_2}{\partial X^2}=0
\end{aligned}
\tag{21}
$$

Subtracting Eq. (20) from Eq. (21) and considering Eq. (6) gives:

$$
\begin{aligned}
&(1+2L_0^2\lambda\alpha+\frac{8}{15}L_1^2\lambda\alpha+L_2^2\lambda\alpha)\frac{\partial^4 W}{\partial X^4}-(2L_0^2\lambda+\frac{4}{5}L_1^2\lambda)\frac{\partial^6 W}{\partial X^6}+\alpha(\frac{\partial^2 W}{\partial \tau^2})-(\frac{\partial^4 W}{\partial X^2\partial \tau^2})\\
&+2K_W W-2K_G\frac{\partial^2 W}{\partial X^2}+P\frac{\partial^2 W}{\partial X^2}=0
\end{aligned}
\tag{22}
$$

Considering Eq. (7), Eq. (21) becomes:

$$
\begin{aligned}
&(1+2L_0^2\lambda\alpha+\frac{8}{15}L_1^2\lambda\alpha+L_2^2\lambda\alpha)\frac{\partial^4 W_2}{\partial X^4}-(2L_0^2\lambda+\frac{4}{5}L_1^2\lambda)\frac{\partial^6 W_2}{\partial X^6}+\alpha(\frac{\partial^2 W_2}{\partial \tau^2})-(\frac{\partial^4 W_2}{\partial X^2\partial \tau^2})\\
&-K_W(W)+K_G(\frac{\partial^2 W}{\partial X^2})+P\frac{\partial^2 W_2}{\partial X^2}=0
\end{aligned}
\tag{23}
$$

3.3 Modified couple stress theory

The modified couple stress theory was first proposed by Yang et al. [15] in which the strain energy density is the function of both the strain tensor and curvature tensor. When the additional scale parameters l_0 and l_1 in the strain gradient elasticity theory equal to zero, then the relation of the modified couple stress theory is obtained. In fact, only one length scale parameter l_2 is needed for this theory.

By setting $l_0=l_1=0$ into Eqs. (22) and (23) , the equations related to modified couple stress are obtained as:

$$
(1+L_2^2\lambda\alpha)\frac{\partial^4 W}{\partial X^4}+\alpha(\frac{\partial^2 W}{\partial \tau^2})-(\frac{\partial^4 W}{\partial X^2\partial \tau^2})+2K_W W-2K_G\frac{\partial^2 W}{\partial X^2}+P\frac{\partial^2 W}{\partial X^2}=0 \tag{24}
$$

$$
(1+L_2^2\lambda\alpha)\frac{\partial^4 W_2}{\partial X^4}+\alpha(\frac{\partial^2 W_2}{\partial \tau^2})-(\frac{\partial^4 W_2}{\partial X^2\partial \tau^2})-K_W(W)+K_G(\frac{\partial^2 W}{\partial X^2})+P\frac{\partial^2 W_2}{\partial X^2}=0 \tag{25}
$$

Noted that by setting $\mu=l_0=l_1=l_2=0$ in aforementioned nonlocal theories, the classical theory is obtained.
By setting $\mu=0$ into Eqs. (8) and (9) , the equations related to classical theory are obtained as:

$$
\frac{\partial^4 W}{\partial X^4}+\alpha(\frac{\partial^2 W}{\partial \tau^2})-(\frac{\partial^4 W}{\partial X^2\partial \tau^2})+2K_W W-2K_G\frac{\partial^2 W}{\partial X^2}+P\frac{\partial^2 W}{\partial X^2}=0 \tag{26}
$$

$$\frac{\partial^4 W_2}{\partial X^4}+\alpha(\frac{\partial^2 W_2}{\partial \tau^2})-(\frac{\partial^4 W_2}{\partial X^2 \partial \tau^2})-K_W(W)+K_G(\frac{\partial^2 W}{\partial X^2})+P\frac{\partial^2 W_2}{\partial X^2}=0 \tag{27}$$

4 VIBRATIONAL STATES OF THE COUPLED SYSTEM

4.1 Out-of-phase vibration

In this case, both of the nanoribbons vibrate asynchronously (i.e. $W_1(X,\tau)-W_2(X,\tau)\neq 0$) as shown in Fig.2(a). Based on the simply supported boundary condition, the following solution can be expressed as [20]

$$W=\sum_{m=1}^{\infty} W_m \sin(m\pi X)\, e^{i\Omega\tau}, \tag{28}$$

where m is half wave number.

Substitiuting Eq. (28) into Eqs. (8), (22), (24) and (26), the out-of-phase frequency for different theories can be written as:

4.1.1 Eringen's nonlocal theory

$$\Omega_{out-of-phase}=\sqrt{\frac{(1+2\mu K_G)m^4\pi^4+2(K_G+\mu K_W)m^2\pi^2-P(m^2\pi^2+\mu m^4\pi^4)+2K_W}{\alpha+m^2\pi^2+\mu m^2\pi^2\alpha+\mu m^4\pi^4}} \tag{29}$$

4.1.2 Strain gradient theory

$$\Omega_{out-of-phase}=\sqrt{\frac{(1+2L_0^2\lambda\alpha+\frac{8}{5}L_1^2\lambda\alpha+L_2^2\lambda\alpha)m^4\pi^4+(2L_0^2\lambda+\frac{4}{5}L_1^2\lambda)m^6\pi^6-Pm^2\pi^2+2K_W+2m^2\pi^2K_G}{\alpha+m^2\pi^2}} \tag{30}$$

4.1.3 Modified couple stress theory

$$\Omega_{out-of-phase}=\sqrt{\frac{(1+L_2^2\lambda\alpha)m^4\pi^4-Pm^2\pi^2+2K_W+2m^2\pi^2K_G}{\alpha+m^2\pi^2}} \tag{31}$$

4.1.4 Classical theory

$$\Omega_{out-of-phase}=\sqrt{\frac{m^4\pi^4-Pm^2\pi^2+2K_W+2m^2\pi^2K_G}{\alpha+m^2\pi^2}} \tag{32}$$

4.2 In-phase vibration

In this case, both of the nanoribbons vibrate synchronously (i.e. $W_1(X,\tau)-W_2(X,\tau)=0$) as shown in Fig. 2(b). Hence, the coupled system can treat as a single nanoribbon without elastic medium. Substitiuting Eq. (28) to Eqs. (9), (23), (25) and (27), the out-of-phase frequency for different theories can be written as:

4.2.1 Eringen's nonlocal theory

$$\Omega_{in-phase}=\sqrt{\frac{m^4\pi^2-P(m^2\pi^2+\mu m^4\pi^4)}{\alpha+m^2\pi^2+\mu m^2\pi^2\alpha+\mu m^4\pi^4}} \tag{33}$$

4.2.2 Strain gradient theory

$$\Omega_{in-phase}=\sqrt{\frac{(1+2L_0^2\lambda\alpha+\frac{8}{5}L_1^2\lambda\alpha+L_2^2\lambda\alpha)m^4\pi^4+(2L_0^2\lambda+\frac{4}{5}L_1^2\lambda)m^6\pi^6-Pm^2\pi^2}{\alpha+m^2\pi^2}} \tag{34}$$

4.2.3 Modified couple stress theory

$$\Omega_{in-phase}=\sqrt{\frac{(1+L_2^2\lambda\alpha)m^4\pi^4-Pm^2\pi^2}{\alpha+m^2\pi^2}} \tag{35}$$

4.2.4 Classical theory

$$\Omega_{in-phase}=\sqrt{\frac{m^4\pi^4-Pm^2\pi^2}{\alpha+m^2\pi^2}} \tag{36}$$

4.3 One nanoribbon being stationary

In this case, one nanoribbon is fixed (i.e. $W_2(X,\tau)=0$) as shown in Fig. 2(c) . Hence, the coupled system can treat as a single nanoribbon with elastic medium. For this case, the vibration Eqs. (8), (22), (24), and (26) for Eringen's, strain gradient, modified couple stress and classical theories, respectively, reduce to:

$$\begin{aligned}&(1+\mu K_G)\frac{\partial^4 W}{\partial X^4}+\alpha(\frac{\partial^2 W}{\partial\tau^2}-\mu\frac{\partial^4 W}{\partial X^2\partial\tau^2})-(\frac{\partial^4 W}{\partial X^2\partial\tau^2}-\mu\frac{\partial^6 W}{\partial X^4\partial\tau^2})+K_W W\\&-(K_G+\mu K_W)\frac{\partial^2 W}{\partial X^2}+P(\frac{\partial^2 W}{\partial X^2}-\mu\frac{\partial^4 W}{\partial X^4})=0\end{aligned} \tag{37}$$

$$\begin{aligned}&(1+2L_0^2\lambda\alpha+\frac{8}{15}L_1^2\lambda\alpha+L_2^2\lambda\alpha)\frac{\partial^4 W}{\partial X^4}-(2L_0^2\lambda+\frac{4}{5}L_1^2\lambda)\frac{\partial^6 W}{\partial X^6}+\alpha(\frac{\partial^2 W}{\partial\tau^2})-(\frac{\partial^4 W}{\partial X^2\partial\tau^2})\\&+K_W W-K_G\frac{\partial^2 W}{\partial X^2}+P\frac{\partial^2 W}{\partial X^2}=0\end{aligned} \tag{38}$$

$$(1+L_2^2\lambda\alpha)\frac{\partial^4 W}{\partial X^4}+\alpha(\frac{\partial^2 W}{\partial\tau^2})-(\frac{\partial^4 W}{\partial X^2\partial\tau^2})+K_W W-K_G\frac{\partial^2 W}{\partial X^2}+P\frac{\partial^2 W}{\partial X^2}=0 \tag{39}$$

$$\frac{\partial^4 W}{\partial X^4}+\alpha(\frac{\partial^2 W}{\partial\tau^2})-(\frac{\partial^4 W}{\partial X^2\partial\tau^2})+K_W W-K_G\frac{\partial^2 W}{\partial X^2}+P\frac{\partial^2 W}{\partial X^2}=0 \tag{40}$$

Substitiuting Eq. (28) to Eqs. (37), (38), (39) and (40), the system frequency for different theories can be written as:

4.3.1 Eringen's nonlocal theory

$$\Omega_{one\,GNR\,fixed}=\sqrt{\frac{(1+\mu K_G)m^4\pi^2+(K_G+\mu K_W)m^2\pi^2-P(m^2\pi^2+\mu m^4\pi^4)+K_W}{\alpha+m^2\pi^2+\mu m^2\pi^2\alpha+\mu m^4\pi^4}} \tag{41}$$

4.3.2 Strain gradient theory

$$\Omega_{oneGNR\,fixed}=\sqrt{\frac{(1+2L_0^2\lambda\alpha+\frac{8}{5}L_1^2\lambda\alpha+L_2^2\lambda\alpha)m^4\pi^4+(2L_0^2\lambda+\frac{4}{5}L_1^2\lambda)m^6\pi^6-Pm^2\pi^2+K_W+m^2\pi^2K_G}{\alpha+m^2\pi^2}} \tag{42}$$

4.3.3 Modified couple stress theory

$$\Omega_{one\,GNR\,fixed}=\sqrt{\frac{(1+L_2^2\lambda\alpha)m^4\pi^4-Pm^2\pi^2+K_W+m^2\pi^2K_G}{\alpha+m^2\pi^2}} \tag{43}$$

4.3.4 Classical theory

$$\Omega_{one\,GNR\,fixed}=\sqrt{\frac{m^4\pi^4-Pm^2\pi^2+K_W+m^2\pi^2K_G}{\alpha+m^2\pi^2}} \tag{44}$$

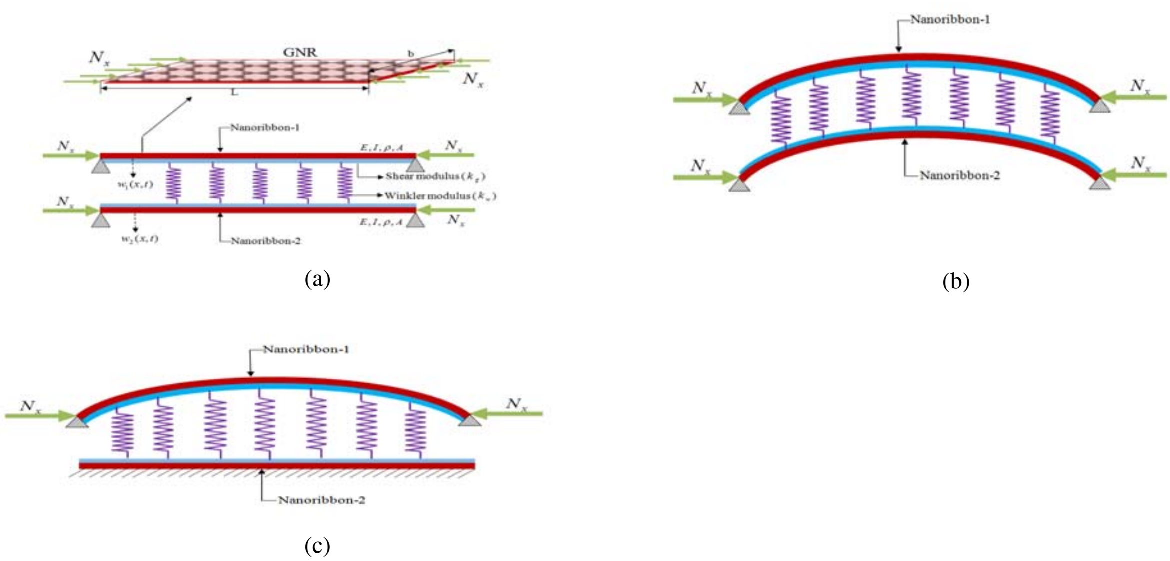

Fig.2
(a) out-of-phase vibration of coupled DGNRS (b) in-phase vibration of coupled DGNRS (c) vibration of coupled DGNRS when one nanoribbon is fixed.

5 RESULTS AND DISCUSSIONS

The material and geometrical properties of the GNRs are assumed same as the graphene sheets. Hence, the thickness, Young's modulus and mass density of the GNR are $0.34\ \text{nm}$, $1.02\ \text{TPa}$ and $2250\ \text{Kg/m}^3$, respectively [17,18]. In this section, vibration behavior and critical buckling load of the coupled nanoribbons are studied for three cases of out-of-phase vibration, one nanoribbon fixed and in-phase vibration so that the effects of nonlocal parameter, axial load, elastic medium and mode number are considered. To this end, frequency ratio (FR) and frequency reduction percent (FRP) are defined as follows:

$$\mathrm{FR}=\frac{\Omega_{\mathrm{Nonlocal}}}{\Omega_{\mathrm{local}}}$$
$$\mathrm{FRP}=\frac{\left(\Omega_{\mathrm{local}}-\Omega_{\mathrm{Nonlocal}}\right)}{\Omega_{\mathrm{local}}}\times 100$$

where $\Omega_{\mathrm{Nonlocal}}$ and Ω_{local} can be obtained from the Eringen's nonlocal and local theories, respectively.

In the absence of similar publications in the literature covering the same scope of the problem, one can not directly validate the results found here. However, due to use nanobeam for modeling of nanoribbons in the present work, the results could be partially validated based on a simplified analysis suggested by Murmu and Adhikari [19,20] on the buckling and vibration of double coupled-SWCNT-systems modeled by Euler-Bernoulli beam. For this purpose, a SWCNT with $E=0.971\text{TPa}$, $h=0.34\ \text{nm}$ and $\rho=2300\ \text{Kg/m}^3$ is considered. Furthermore, the coefficient of spring foundation between two SWCNT is $K_W=30$. A comparison between the present results and Refs. [19, 20] for buckling and vibration analysis of double coupled-SWCNT-systems using Eringen's nonlocal theory are presented in Fig. 3 and 4. In this Figures, buckling load reduction percent and FRP versus dimensionless nonlocal parameter are plotted for three cases of out-of-phase vibration, in-phase vibration and one SWCNT fixed. As can be seen, the present results based on Eringen's nonlocal theory match with those reported by Murmu and Adhikari [19, 20], indicating validation of our work.

Table 1 shows the comparison of strain gradient, modified couple stress, Eringen's and classical theories in studying of coupled system frequency for different mode numbers and three cases of out-of-phase vibration, one nanoribbon fixed and in-phase vibration. It is clear that the frequency of the DGNRS increases with increasing mode number for all cases and theories. Irrespective of theory type, the effect of vibrational states (i.e. three cases) on the frequency follows the order.

Out-of-phase vibration > Vibration with one nanoribbon fixed > In-phase vibration

For three cases mentioned above, it is also concluded that the frequency predicted by Eringen's nonlocal theory and strain gradient theories are minimum and maximum, respectively. However, the frequency obtained by different theories follows the order.

Strain gradient theory > Modified couple stress theory > Classical theory > Eringen theory

It is important to note that the results obtained here, are the same as those expressed in [14, 28], indicating validation of our work.

For three aforementioned cases in Table 1. , the frequency of the DGNRS for classical, Eringen's, modified couple stress and strain gradient theories is illustrated in Table 2. Here, for showing the effects of elastic medium between two nanoribbons, three states of without elastic medium, Winkler medium and Pasternak medium are considered. The same as Table 2. , the values of frequencies for strain gradient theory are higher than those for classical, Eringen's and couple stress theories due to the three additional material parameters. With regard to elastic medium effect, it is clear that this effect on the frequency of the coupled system follows the order.

Pasternak medium > Winkler medium >Without elasic medium

It is due to the fact that the Pasternak medium consider not only the normal stresses (i.e. Winkler medium) but also the transverse shear deformation and continuity among the spring elements. It is also observed that the frequency of the coupled system in the case of in-phase vibration is unchangeable with internal elastic medium. This is because in this case, the elastic medium terms in governing equations are eliminated and the coupled system behaves as a single nanoribbon without elastic medium.

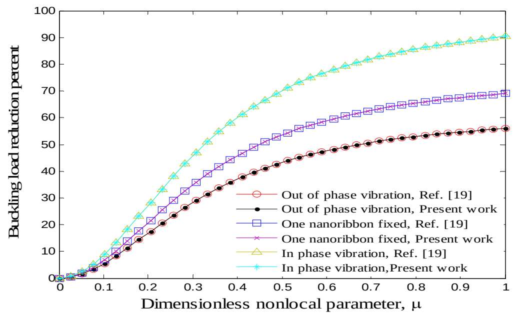

Fig. 3
Comparison of buckling load reduction percent versus dimensionless nonlocal parameter between present work and Ref. [19].

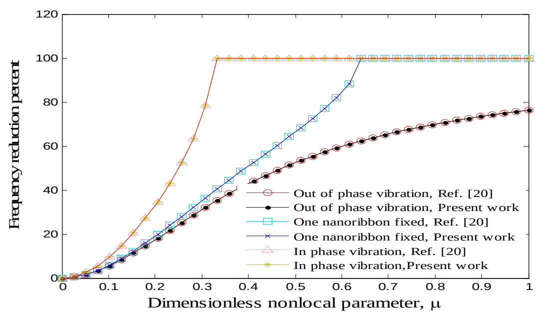

Fig. 4
Comparison of vibration reduction percent versus dimensionless nonlocal parameter between present work and Ref. [20].

Table 1
Effect of mode number on frequency of the coupled system for different theories

Vibrational states	Mode number	Classical theory	Eringen's nonlocal theory	Modified couple stress theory	Strain gradient theory
Out-of-phase vibration	Mode 1	4.0676	3.9573	4.1042	4.1269
	Mode 2	6.4467	5.5141	6.8078	7.0240
	Mode 3	9.4875	6.9473	10.6883	11.3747
	Mode 4	12.5990	7.8857	15.3397	16.8275
One nanoribbon fixed	Mode 1	3.6335	3.5096	3.6744	3.6998
	Mode 2	6.3639	5.4170	6.7294	6.9481
	Mode 3	9.4538	6.9032	10.6585	11.3466
	Mode 4	12.5795	7.8545	15.3237	16.8129
In-phase vibration	Mode 1	3.1400	2.9958	3.1873	3.2165
	Mode 2	6.2800	3.3182	6.6501	6.8713
	Mode 3	9.4200	6.8568	10.6285	11.3184
	Mode 4	12.5600	7.8233	15.3077	16.7984

Table 2
Effect internal elastic medium on frequency of the coupled system for different theories

Vibrational states	Mode number	Classical theory	Eringen's nonlocal theory	Modified couple stress theory	Strain gradient theory
Out-of-phase vibration	Without elastic edium	3.1400	2.9958	3.1873	3.2165
	With Winkler medium	3.9931	3.8807	4.0304	4.0535
	With Pasternak medium	4.0676	3.9573	4.1042	4.1269
One nanoribbon fixed	Without elastic medium	3.1400	2.9958	3.1873	3.2165
	With Winkler medium	3.5920	3.4666	3.6334	3.6590
	With Pasternak medium	3.6335	3.5096	3.6744	3.6998
In-phase vibration	Without elastic medium	3.1400	2.9958	3.1873	3.2165
	With Winkler medium	3.1400	2.9959	3.1873	3.2165
	With Pasternak medium	3.1400	2.9958	3.1873	3.2165

Fig. 5 demonstrates the relationship between the dimensionless frequency (Ω) and the dimensionless axial load (P) for three cases of out-of-phase vibration, one nanoribbon fixed and in-phase vibration which are obtained by different theories, including classical, Eringen's, modified couple stress and strain gradient. As can be seen, Ω decreases with increasing P . When the dimensionless frequency becomes zero, critical buckling load is reached, which the system loses its stability. Therefore, with increasing axial load, system stability decreases and becomes susceptible to buckling. Comparing classical theory with Eringen's nonlocal theory, it can be concluded that the dimensionless frequency and critical buckling load of the classical theory are higher than Eringen's nonlocal theory. This is because in Eringen's nonlocal theory, the interaction force between nanoribbon atoms decreases, and that leads to a softer structure. The dimensionless frequency and critical buckling load obtained by the strain gradient theory are higher than the Eringen's and modified couple stress theories. This is due to the fact that the strain gradient theory expresses the three additional dilatation gradient tensor, the deviatoric stretch gradient tensor and the rotation gradient tensor. Furthermore, for all classical and nonlocal theories, observation of Fig. 5 implies that the Ω and P for the case of out-of-phase vibration are higher than two other cases including in-phase vibration and one nanoribbon fixed. The reason is that in the cases of one nanoribbon fixed and in-phase vibration, the coupled system behaves as the single GNR with and without elastic medium effects, respectively.

The effect of enclosing elastic medium between two GNRs is shown in Figs.6(a-d) on the dimensionless frequency with respect to the dimensionless axial load for Eringen's nonlocal theory. The three vibrational states including out-of-phase vibration, one nanoribbon fixed and in-phase vibration are considered in this figure. It can be found that the vibrational states are equal in the case of without elastic medium and this effect becomes more prominent with increasing elastic medium coefficient. It is also worth mentioning that the elastic medium has not any effect on the vibration of coupled system in the case of in-phase vibration, since in this case, the DGNRS treats as a single GNR without internal elastic medium. Furthermore, the stability region of the coupled system in the cases of out-of-phase vibration and one nanoribbon fixed increases with increasing dimensionless axial load. Although not presented here, the same conclusions are also valid for other theories.

Figs.7(a-d) and Figs. 8(a-d) illustrate the FR and FRP versus dimensionless Eringen's nonlocal parameter for different dimensionless axial loads including $P=0$, $P=2$, $P=4$ and $P=8$. In these figures, similar to pervious figures, three cases for vibrational states are considered (i.e. out-of-phase vibration, in-phase vibration and vibration with one nanoribbon fixed). It can be observed that the FR and FRP for the case of in-phase vibration are, respectively smaller and higher than cases of out-of-phase vibration and one nanoribbon fixed. This is due to the absence of coupling effect of the elastic medium between the two GNRs in the case of in-phase vibration. Meanwhile, the small scale effect in the case of in-phase vibration is higher than that in the other cases. As mentioned in discussion of Fig. 5, the coupled system becomes unstable when the FR and FRP are equal to zero and 100%, respectively. In the case of $P=0$, the coupled system is stable in three aforementioned cases but with increasing dimensionless axial load, the system stability decreases. In other words, when $P=2$, the DGNRS becomes unstable in the case of in-phase vibration at $\mu=0.66$. With increasing P from 2 to 4, the coupled system buckles in two cases of in-phase vibration and one nanoribbon fixed at $\mu=0.41$ and $\mu=0.87$, respectively. When $P=8$, the DGNRS becomes unstable in three cases mentioned above. However, it can be concluded that the coupled system in the case of out-of-phase vibration buckles later with respect to two other cases. It should be noted that, the same conclusions are also valid for other theories, although not presented here.

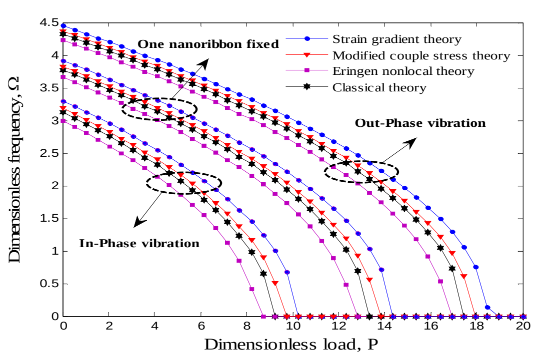

Fig. 5
Dimensionless frequency versus dimensionless axial load for different theories.

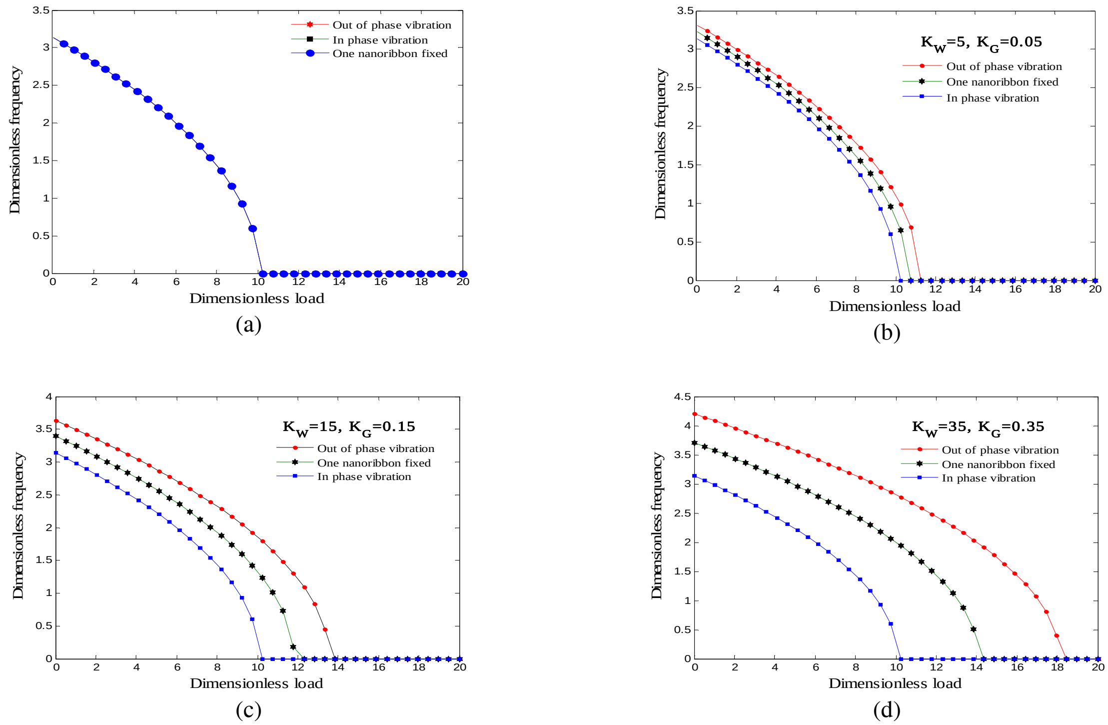

Fig. 6
Dimensionless frequency versus dimensionless axial load for Eringen's nonlocal theory (a) $K_W = K_G = 0$ (b) $K_W = 5, K_G = 0.05$ (c) $K_W = 15, K_G = 0.15$ (d) $K_W = 35, K_G = 0.35$.

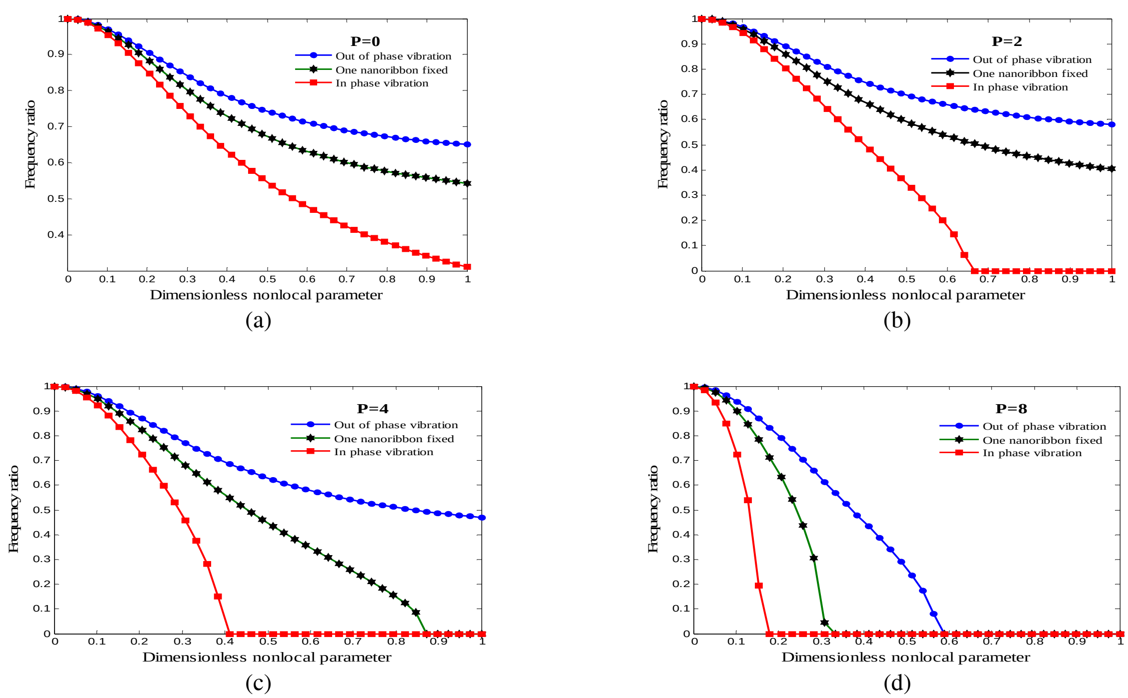

Fig. 7
Frequency ratio versus dimensionless nonlocal parameter for Eringen's nonlocal theory (a) $P = 0$ (b) $P = 2$ (c) $P = 4$ (d) $P = 8$.

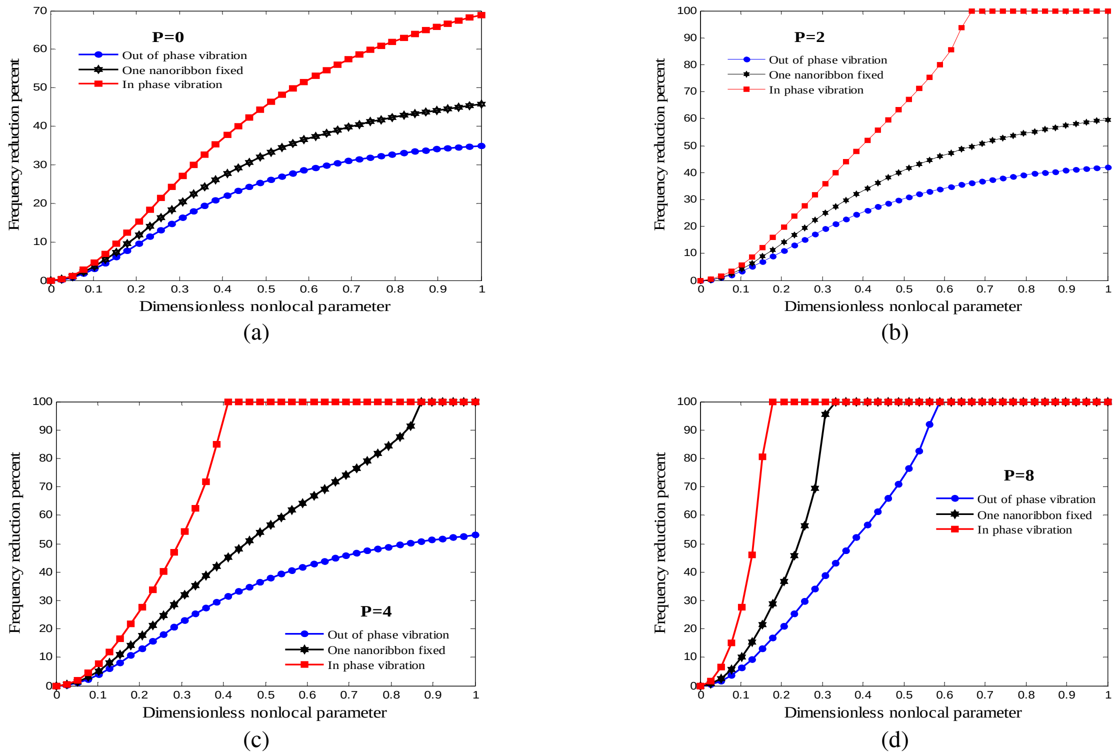

Fig. 8
Frequency reduction percent versus dimensionless nonlocal parameter for Eringen's nonlocal theory (a) $P=0$ (b) $P=2$ (c) $P=4$ (d) $P=8$.

6 CONCLUSION

Using Rayleigh beam model, vibration and stability of the DGNRS were investigated in this manuscript based on the classical, Eringen, modified couple stress and strain gradient theories. An analytical approach was used for obtaining the frequency and critical buckling load of the coupled system for three cases of out-of-phase vibration, one nanoribbon fixed and in-phase vibration. The proposed strain gradient and Eringen's theories contains three and one material length scale parameters to capture the size effect, respectively. Hence, the frequency and critical buckling load predicted by strain gradient theory was higher than those predicted by Eringen's theory. It was observed that the elastic medium has not any effect on the vibration of coupled system in the case of in-phase vibration. Meanwhile, the small scale effect in the case of in-phase vibration was higher than that in the other cases. It was also concluded that the coupled system in the case of out-of-phase vibration buckles later with respect to two other cases. The results of this study are validated as far as possible by Refs. [19,20]. This investigation on the vibration and critical buckling load of coupled-GNRs may be used as a useful reference for the study and design of nano- devices in which GNRs act as basic elements.

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