Buckling Analyses of Rectangular Plates Composed of Functionally Graded Materials by the New Version of DQ Method Subjected to Non-Uniform Distributed In-Plane Loading

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ABSTRACT

In this paper, the new version of differential quadrature method (DQM), for calculation of the buckling coefficient of rectangular plates is considered. At first the differential equations governing plates have been calculated. Later based on the new version of differential quadrature method, the existing derivatives in equation are converted to the amounts of function in the grid points inside the region. Having done that, the equation will be converted to an eigen value problem and the buckling coefficient is obtained. Solving this problem requires two kinds of loading: (1) unaxial half-cosine distributed compressive load and (2) uni-axial linearly varied compressive load. Having considered the answering in this case and the analysis of the effect of number of grid points on the solution of the problem, the accuracy of answering is considered, and also the effect of power law index over the buckling coefficient is investigated. Finally, if the case is an isotropic type, the results will be compared with the existing literature.

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1 INTRODUCTION

D URING 1980s, most of the research on buckling has been concentrated upon isotropic materials. The book by Brush and Almroth [1] in 1975, clearly illustrates the buckling equations of beams, cylinder shells and general shells from isotropic materials under mechanical loading. The book also discusses the buckling of plates and cylinders that have longitudinal and transversal reinforcement and also the effect of initial fault up on buckling. The deployment of composed materials especially in avian and military industries and the importance of recognition of mechanical behavior attracted the attention of researchers to analysis of buckling of composed plates subjected to thermal and mechanical loading and different boundary conditions have been preformed recently. In the researches performed, different methods have been suggested for obtaining critical load or critical thermal of buckling of plate and also has higher order shear deformation theory of plates for analysis of buckling has been deployed. A review has been done over recent works on the buckling of layer composed materials by Leissa [2] and Tauchert [3]. A number of published articles in the field of buckling of layer composed materials under the sort of mechanical and thermals loading on the base of classical theory of plates [4] have been presented in the reference section.

In recent years, DQ method has been used for consideration of the buckling of rectangular plates. This method was introduced by Bellman and Casti [6] as a form of simple and effective numerical method for solving linear and non-linear partial differential equations from boundary value problems for the first time in 1971. The DQ method is based on approximation and estimating of partial derivatives from one function ratio to a variable in every separation point. Wang [7] independently proposed a method for applying multiple boundary conditions by

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assigning two degree of freedom to each end point for a fourth-order differential equation in 1996. The method was called differential quadrature element method (DQEM). Liu and Wu [8] proposed a generalized differential quadrature rule (GDQR) that was based on the analysis of higher order approximation polynomials. This method can obtain all buckling loads. All boundary conditions can be easily applied in the DQEM or GDQR and accurate solution can be obtained. In this article, the buckling coefficient of rectangular plate made of functionally graded materials has been obtained under in-plane loading by the new version of DQ numerical method. Bert and Devarakonda [9] presented an analytical solution for in-plane stresses for the case of a half sine load distribution on two opposite sides. The buckling load is calculated by the use of Galerkin method. Much more accurate buckling load has been obtained for a rectangular plate simply supported along all edges. There are many methods such as Rayleigh-Ritz method, finite-element method, finite-difference method and fourier series method. However, the method we employ here is the DQ method. It was found that solutions were very sensitive to grid spacing [10] when DQ method is used for solving buckling problems of anisotropic rectangular plates even under uniform edge loading. With the proposed non-uniform grid spacing [11] and a new way to apply the boundary conditions [12-13] the difficulty has been removed and some isotropic plate buckling problems have been successfully solved by the DQ method [14].

A new method was proposed recently for applying the boundary conditions which is called the new version of DQ [15]. This method is successfully used to obtain the buckling loads for anisotropic plates with uniform in-plane loading [16]. In this method, instead of solving the second-order partial differential equation in terms of displacement, the fourth-order partial differential equation (the compatibility equation) in terms of Airy stress functions is solved by the DQ method and accurate stress distributions were obtained for the case of non-uniform distributed in-plane loadings. This method has only been successfully used to obtain buckling loads for the cases of uniform or linearly distributed loadings. In this paper, the new version of differential quadrature method (DQM), for calculation of the buckling coefficient of rectangular plates is considered and the results are compared with the data of finite-element and finite-difference [17].

2 MATERIAL PROPERTIES

Consider a rectangular plate made of a mixture of metal and ceramic as shown in Fig. 1. The material in top surface and in bottom surface is metal and ceramic, respectively. The modulus of elasticity E, the coefficient of thermal expansion α, and the Poisson's ratio υ are assumed as [18]

$$E(z) = E_c V_c + E_m (1 - V_c),$$
$$\alpha(z) = \alpha_c V_c + \alpha_m (1 - V_c),$$
$$\upsilon(z) = \upsilon_\circ$$

(1)

Subscripts m and c denote the properties of metal and ceramic, respectively. The relation between volume fraction of ceramic v_c and metal v_m by the power law is represented as follows:

$$V_c = \left(\frac{2z + h}{2h} \right)^p, \qquad V_m = 1 - V_c$$

(2)

where z ($-h/2 \leq z \leq h/2$) is the thickness coordinate variable and, where h is the thickness of plate and p is the power law index that takes values greater than or equal to zero. The material properties of the FGM plate with attention to the equations proposed by Reddy is as follow.
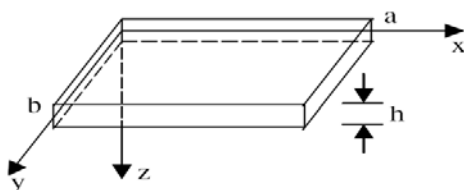
Fig. 1
Configuration and coordinate system of a rectangular plate.

$$E(z) = E_m + E_{cm}\left(\frac{2z+h}{2h}\right)^p$$

$$\alpha(z) = \alpha_m + \alpha_{cm}\left(\frac{2z+h}{2h}\right)^p,$$

$$\upsilon(z) = \upsilon_\circ$$

(3)

That

$$E_{cm} = E_c - E_m, \qquad \alpha_{cm} = \alpha_c - \alpha_m$$

$$(E_1, E_2, E_3) = \int_{-h/2}^{h/2} (1, z, z^2) E(z)\, \mathrm{d}z$$

(4)

3 STABLE EQUATIONS

Consider a rectangular plate of FGM material with length a and width b and thickness h, that are subjected to mechanical loading. According to the classical theory of plates, the components of normal strain, $\bar{\varepsilon}_y, \bar{\varepsilon}_x$ and shear strain $\bar{\varepsilon}_{xy}$ in every point of the thickness of plate with distance z from mid-plane, are presented as follow [1].

$$\bar{\varepsilon}_x = \varepsilon_x + zk_x$$

$$\bar{\varepsilon}_y = \varepsilon_y + zk_y$$

$$\bar{\varepsilon}_{xy} = \varepsilon_{xy} + 2zk_{xy}$$

(5)

That $\varepsilon_x, \varepsilon_y$ and ε_{xy} are normal strains and shear strain respectively in mid plane. The k_x, k_y and k_{xy} are the indications of curvature and torsion respectively. The Kinematic equations of mid-plane that is known as Sander՚s nonlinear equations is defined as:

$$\varepsilon_x = u_{,x} + \frac{1}{2}\beta_x^2, \varepsilon_y = v_{,y} + \frac{1}{2}\beta_y^2$$

$$\varepsilon_{xy} = (u_{,y} + v_{,x}) + \beta_x\beta_y$$

$$k_x = \beta_{x,x}, \qquad k_y = \beta_{y,y}$$

$$k_{xy} = \frac{1}{2}(\beta_{x,y} + \beta_{y,x})$$

$$\beta_x = -w_{,x}, \qquad \beta_y = -w_{,y}$$

(6)

As that w, v, u denote the displacement of plane in z, y, x directions, respectively. The subscript (,) is the sign of partial derivatives. The stress relations without the effective of thermal for above plate are defined as:

$$\bar{\sigma}_x = \frac{E}{1-v^2}[\bar{\varepsilon}_x + v\bar{\varepsilon}_y]$$

$$\bar{\sigma}_y = \frac{E}{1-v^2}[\bar{\varepsilon}_y + v\bar{\varepsilon}x]$$

$$\bar{\sigma}_{xy} = \frac{E}{2(1+v)}\bar{\varepsilon}_{xy}$$

(7)

So E and υ are assumed as the modulus of elasticity and Poisson's ratio for FGM material, respectively. That was represented by Eq. (3), where $\overline{\sigma}_x$, $\overline{\sigma}_y$ and $\overline{\sigma}_{xy}$ are the normal and shear stresses in every point of plate thickness with distance z from mid-plane. The forces and moments per unit length of the plate are expressed in terms of the stress components (through the thickness):

$$N_{ij} = \int_{-h/2}^{h/2} \overline{\sigma}_{ij}\, dz, \qquad M_{ij} = \int_{-h/2}^{h/2} z\overline{\sigma}_{ij}\, dz, \qquad i,j = x,y,xy \tag{8}$$

The constitutive equation of FGM plate can be obtained by substituting Eqs. (3), (5) and (7) into Eq. (8).

$$N_x = \frac{E_1}{1-\upsilon_\circ^2}(\varepsilon_x + \upsilon_\circ\varepsilon_y) + \frac{E_2}{1-\upsilon_\circ^2}(k_x + \upsilon_\circ k_y)$$

$$N_y = \frac{E_1}{1-\upsilon_\circ^2}(\varepsilon_y + \upsilon_\circ\varepsilon_x) + \frac{E_2}{1-\upsilon_\circ^2}(k_y + \upsilon_\circ k_x)$$

$$N_{xy} = \frac{E_1}{2(1+\upsilon_\circ)}\varepsilon_{xy} + \frac{E_2}{1+\upsilon_\circ}k_{xy}$$

$$M_x = \frac{E_2}{1-\upsilon_\circ^2}(\varepsilon_x + \upsilon_\circ\varepsilon_y) + \frac{E_3}{1-\upsilon_\circ^2}(k_x + \upsilon_\circ k_y) \tag{9}$$

$$M_y = \frac{E_2}{1-\upsilon_\circ^2}(\varepsilon_y + \upsilon_\circ\varepsilon_x) + \frac{E_3}{1-\upsilon_\circ^2}(k_y + \upsilon_\circ k_x)$$

$$M_{xy} = \frac{E_2}{2(1+\upsilon_\circ)}\varepsilon_{xy} + \frac{E_3}{1+\upsilon_\circ}k_{xy}$$

That N_x, N_y, N_{xy}, N_{yx} are the normal and shear forces in plane and M_x, M_y, M_{xy}, M_{yx} are torsion and bending momentums, also from Eqs. (3) and (4) will have

$$E_1 = E_m h + \frac{E_{cm}h}{p+1}$$

$$E_2 = E_{cm}h^2\left(\frac{1}{p+1} - \frac{1}{2p+2}\right) \tag{10}$$

$$E_3 = \frac{E_m h^3}{12} + E_{cm}h^3\left(\frac{1}{p+3} - \frac{1}{p+2} + \frac{1}{4(p+1)}\right)$$

Finally the nonlinear equations of equilibrium according to Von Karman's theory are given by:

$$N_{x,x} + N_{xy,y} = 0$$

$$N_{y,y} + N_{xy,x} = 0 \tag{11}$$

$$M_{x,xx} + M_{y,yy} + 2M_{xy,xy} - N_x\beta_{x,x} - N_y\beta_{y,y} - 2N_{xy}\beta_{x,y} + P_n = 0$$

When Eq. (9) are substituted into Eq .(11), the equations of equilibrium in terms of displacements are as follow:

$$\frac{E_1}{1-\upsilon_\circ^2}(u_{,xx} + \upsilon_\circ v_{,yx}) - \frac{E_2}{1-\upsilon_\circ^2}(w_{,xxx} + \upsilon_\circ w_{,yyx}) + \frac{E_1}{2(1+\upsilon_\circ)}(u_{,yy} + v_{,xy}) - \frac{E_2}{1+\upsilon_\circ}w_{,xyy} = 0 \tag{12a}$$

$$\frac{E_1}{1-\upsilon_\circ^2}(v_{,yy} + \upsilon_\circ u_{,xy}) - \frac{E_2}{1-\upsilon_\circ^2}(w_{,yyy} + \upsilon_\circ w_{,xxy}) + \frac{E_1}{2(1+\upsilon_\circ)}(u_{,yx} + v_{,xx}) - \frac{E_2}{1+\upsilon_\circ}w_{,xyx} = 0 \tag{12b}$$

$$\frac{E_2}{1-\upsilon_\circ^2}(u_{,xxx}+\upsilon_\circ v_{,yxx})-\frac{E_3}{1-\upsilon_\circ^2}(w_{,xxxx}+\upsilon_\circ w_{,yyxx})+\frac{E_2}{1-\upsilon_\circ^2}(v_{,yyy}+\upsilon_\circ u_{,xyy})$$

$$-\frac{E_3}{1-\upsilon_\circ^2}(w_{,yyyy}+\upsilon_\circ w_{,xxyy})+\frac{E_2}{(1+\upsilon_\circ)}(u_{,yxy}+v_{,xxy})-\frac{2E_3}{1+\upsilon_\circ}w_{,xyxy} \tag{13}$$

$$+N_{x\circ}w_{,xx}+N_{y\circ}w_{,yy}+2N_{xy\circ}w_{,xy}=0$$

With deriving from Eq. (12a) ratio the variable of *x* and Eq. (12b) ratio y and after pulsing them will have:

$$\frac{E_1}{1-\upsilon_\circ^2}(u_{,xxx}+\upsilon_\circ v_{,yxx})-\frac{E_2}{1-\upsilon_\circ^2}(w_{,xxxx}+\upsilon_\circ w_{,yyxx})+\frac{E_1}{1-\upsilon_\circ^2}(v_{,yyy}+\upsilon_\circ u_{,xyy})$$

$$-\frac{E_2}{1-\upsilon_\circ^2}(w_{,yyyy}+\upsilon_\circ w_{,xxyy})+\frac{E_1}{(1+\upsilon_\circ)}(u_{,yxy}+v_{,xyx})-\frac{2E_2}{1+\upsilon_\circ}w_{,xyxy}=0 \tag{14}$$

Next multiply E_2 in Eq. (14) and then substituting it in Eqs. (13) after simplification followed equation obtained:

$$\frac{E_2^2}{E_1(1-\upsilon_\circ^2)}(w_{,xxxx}+\upsilon_\circ w_{,yyxx})+\frac{E_2^2}{E_1(1-\upsilon_\circ^2)}(w_{,yyyy}+\upsilon_\circ w_{,xxyy})+\frac{2E_2^2}{E_1(1+\upsilon_\circ)}w_{,xyxy}$$

$$-\frac{E_1E_3}{E_1(1-\upsilon_\circ^2)}(w_{,xxxx}+v_\circ w_{,yyxx})-\frac{E_1E_3}{E_1(1-\upsilon_\circ^2)}(w_{,yyyy}+\upsilon_\circ w_{,xxyy})-\frac{2E_1E_3}{E_1(1+\upsilon_\circ)}w_{,xyxy} \tag{15}$$

$$+N_{x\circ}w_{,xx}+N_{y\circ}w_{,yy}+2N_{xy\circ}w_{,xy}=0$$

With the using of definition of delfour the above equation will be as follow:

$$\frac{E_2^2-E_1E_3}{E_1(1-\upsilon_\circ^2)}\nabla^4 w+N_{x\circ}w_{,xx}+N_{y\circ}w_{,yy}+2N_{xy\circ}w_{,xy}=0 \tag{16}$$

According to definition

$$\nabla^4 w=w_{,xxxx}+w_{,yyyy}+2w_{,xxyy}$$

which is related to the classical theory of plates, the displacement field (u,v,w) are assumed in coordinate $(u_\circ,v_\circ,w_\circ)$. The arbitrarily small increment of displacement performed as (u_1,v_1,w_1). The displacement components in neighboring of equilibrium state is

$$u=u_\circ+u_1$$
$$v=v_\circ+v_1 \tag{17}$$
$$w=w_\circ+w_1$$

With substituting Eqs. (17) into Eqs.(16), the stability equation of FGM rectangular plate according to the classical theory obtained as follow

$$\frac{E_2^2-E_1E_3}{E_1(1-v_\circ^2)}\nabla^4 w_1+N_{x\circ}w_{1,xx}+N_{y\circ}w_{1,yy}+2N_{xy\circ}w_{1,xy}=0 \tag{18}$$

Where $N_{x\circ}, N_{y\circ}, N_{xy\circ}$ refer to the pre-buckling force resultants.

4   THE NEW VERSION OF DQ METHOD

Q_i^k is the kth-order derivative of the solution function at grid point i in ordinary DQ method [19],

$$Q_i^k = \sum_{j=1}^{n} C_{ij}^{(k)} Q_j, \qquad i = 1,2,...,n \tag{19}$$

Q_j is the solution values at grid point *j* and *N* is the total number of grid points in the entire domain including the end points and the solution values at grid point j. $C_{ij}^{(k)}$ are called the weighting coefficients of k*th*-order derivative. $C_{ij}^{(1)}, C_{ij}^{(2)}, C_{ij}^{(3)}, C_{ij}^{(4)}$ (*i*=1,2,…,*n*; *j*=1,2,…,*n*) the weighting coefficients of the first-,second-,third- and fourth order derivatives for the ordinary DQ method. In the new version of DQ $Q_1^{(1)}, Q_n^{(1)}$ the first-order derivatives of *Q* at two end points are introduced as additional independent variables.

$$\{u\}^T = \{Q_1, Q_2, ..., Q_{n-1}, Q_n, Q_1^{(1)}, Q_n^{(1)}\} \tag{20}$$

where $A_{ij}, B_{ij}, C_{ij}, D_{ij}$ $(i=1,2,...,n; j=1,2,...,n+2)$ are the weighting coefficients of first-, second-, third- and fourth- order derivatives in the new version of DQ method [19], therefore

$$A_{ij} = C_{ij}^{(1)}(i,j=1,2,...,n)$$
$$A_{ij} = 0(i=1,2,...,n; j=n+1,n+2) \tag{21a}$$
$$B_{ij} = C_{ij}^{(2)}(i,j=1,2,...,n)$$
$$B_{ij} = 0(i=1,2,...,n; j=n+1,n+2) \tag{21b}$$

The weighting coefficient of third- and fourth- orders are calculated differently:

$$Q_i^{(2)} = \sum_{j=1}^{n} \sum_{k=2}^{n-1} C_{ik}^{(1)} C_{kj}^{(1)} Q_j + C_{i1}^{(1)} Q_1^{(1)} + C_{in}^{(1)} Q_n^{(1)} = \sum_{j=1}^{n+2} B_{ij}^* u_j (i=1,n)$$

$$Q_i^{(2)} = \sum_{j=1}^{n+2} C_{ij}^{(2)} u_j = \sum_{j=1}^{n+2} B_{ij}^* u_j, \qquad (i=2,3,...,n-1) \tag{22}$$

$$Q_i^{(3)} = \sum_{k=1}^{n} C_{ik}^{(1)} \sum_{j=1}^{n+2} B_{kj}^* u_j = \sum_{j=1}^{n+2} C_{ij} u_j$$

$$Q_i^{(4)} = \sum_{k=1}^{n} C_{ik}^{(2)} \sum_{j=1}^{n+2} B_{kj}^* u_j = \sum_{j=1}^{n+2} D_{ij} u_j \tag{23}$$

For obtaining the weighting coefficient C_{ij}, D_{ij}, the value of B_{ij}^* is defined as:

$$B_{ij}^* = C_{ij}^{(2)}$$
$$B_{i,n+1}^* = 0; B_{i,n+2}^* = 0$$
$$(i = 2,3,...,n-1)$$
$$(j = 1,2,...,n)$$

(24a)

$$B_{ij}^* = \sum_{k=2}^{n-1} C_{ik}^{(1)} C_{kj}^{(1)}$$
$$B_{i,n+1}^* = A_{i,1}; B_{i,n+2}^* = A_{i,n}$$
$$(i = 1,n)$$
$$(j = 1,2,...,n)$$

(24b)

Consequently :

$$C_{ij} = \sum_{k=1}^{n} C_{ik}^{(1)} B_{kj}^* \, (i = 1,2,...,n; j = 1,2,...,n+2)$$

$$D_{ij} = \sum_{k=1}^{n} C_{ik}^{(2)} B_{kj}^* \, (i = 1,2,...,n; j = 1,2,...,n+2)$$

(25)

With substituting the new version of DQ weighting coefficient in equations, we can change those to new DQ form.

5   BUCKLING OF FGM RECTANGULAR PLATE TO THE NEW VERSION OF DQ

Consider first the problem of in-plane elasticity, anisotropic FGM thin rectangular plate with length *a* and width *b* subjected to a uniaxial non-uniform distributed in-plane edge load. When all boundary conditions are stress boundary conditions, methods based on stress function are most convenient to be used. The Airy stress function is used for solving the problems of plate reactionary theory. These functions without body forces are defined as

$$\sigma_x = \frac{\partial^2 Q}{\partial y^2}, \sigma_y = \frac{\partial^2 Q}{\partial x^2}, \tau_{xy} = -\frac{\partial^2 Q}{\partial x \partial y}$$

(26)
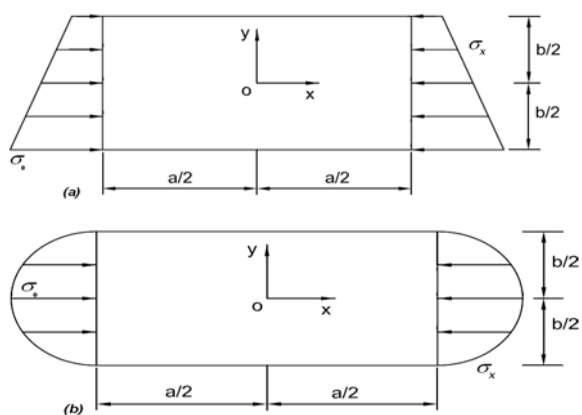
Fig. 2
Rectangular plate under unaxial edge compressions.

Airy stress function satisfied the governing differential equations and should satisfy the following compatibility differential equation.

$$\frac{\partial^4 Q}{\partial x^4} + 2\frac{\partial^4 Q}{\partial x^2 \partial y^2} + \frac{\partial^4 Q}{\partial y^4} = 0 \tag{27}$$

When Q is obtained, Eq. (27) calculated the in-plane stresses $\sigma_x, \sigma_y, \tau_{xy}$ in Eq. (26).

6   BOUNDARY CONDITIONS

simply supported (SS)

$$\begin{aligned}
x = -\frac{a}{2}, \frac{a}{2} &\longrightarrow w = M_x = 0 \\
y = -\frac{b}{2}, \frac{b}{2} &\longrightarrow w = M_y = 0
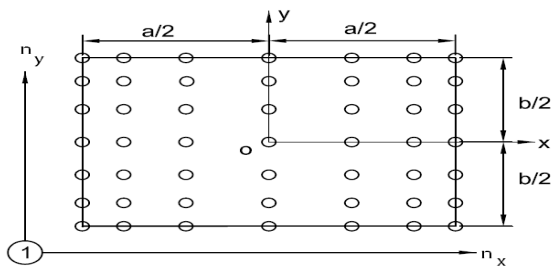\end{aligned} \tag{28}$$

clamped (C)

$$\begin{aligned}
x = -\frac{a}{2}, \frac{a}{2} &\longrightarrow w = w_x = 0 \\
y = -\frac{b}{2}, \frac{b}{2} &\longrightarrow w = w_y = 0
\end{aligned} \tag{29}$$

free (F)

$$\begin{aligned}
x = -\frac{a}{2}, \frac{a}{2} &\longrightarrow M_x = 0 \\
y = -\frac{b}{2}, \frac{b}{2} &\longrightarrow M_y = 0
\end{aligned} \tag{30}$$

7   EQUATIONS IN TERMS OF NEW VERSION OF DQ

Let n_x, n_y be the total number of grid points in the x and y directions shown in Fig. 3. As it can be seen, both Eq. (18) and Eq. (27) are fourth-order partial differential equations. The new version of DQ method is to be used to solve both equations. Each corner grid point has three degree of freedom, namely Q, Q_x, Q_y (or w, w_x, w_y), where

Fig. 3
Sketch of a rectangular plate with grid points.

$$Q_y = \frac{\partial Q}{\partial y}, Q_x = \frac{\partial Q}{\partial x}$$

$$w_y = \frac{\partial w}{\partial y}, w_x = \frac{\partial w}{\partial x},$$

(31)

The remaining boundary points have two degrees of freedom each, either Q, Q_x (or w, w_x) for points at edges parallel to the y – axis or Q, Q_y (or w, w_y) for points at edges parallel to the x-axis. Each inner grid point has only one degree of freedom Q (or w).

In terms of the new version of DQ method, the governing differential equation of Eq. (27) at inner grid point is expressed as:

$$\sum_{k=1}^{n_x} D_{ik}^x Q_{kl} + 2\sum_{j=1}^{n_x}\sum_{k=1}^{n_y} B_{ij}^x B_{lk}^y Q_{jk} + \sum_{k=1}^{n_y} D_{lk}^y Q_{ik} + D_{i(n_x+1)}^x (Q_x)_{1l} + D_{i(n_x+2)}^x (Q_x)_{n_x l} +$$

$$D_{l(n_y+1)}^y (Q_y)_{i1} + D_{l(n_y+2)}^y (Q_y)_{in_y} = 0, \qquad (i=2,3,...,n_x-1, \ l=2,3,...,n_y-1)$$

(32)

And the governing differential equation of Eq(18) at inner grid points is expressed as:

$$\sum_{k=1}^{n_x} D_{ik}^x w_{kl} + 2\sum_{j=1}^{n_x}\sum_{k=1}^{n_y} B_{ij}^x B_{lk}^y w_{jk} + \sum_{k=1}^{n_y} D_{lk}^y w_{ik} + D_{i(n_x+1)}^x (w_x)_{1l} + D_{i(n_x+2)}^x (w_x)_{n_x l}$$

$$+ D_{l(n_y+1)}^y (w_y)_{i1} + D_{l(n_y+2)}^y (w_y)_{in_y} = \frac{\sigma_\circ h}{D} \times \{ \sum_{k=1}^{n_x} B_{ik}^x w_{kl}(\alpha_x)_{il} + 2\sum_{j=1}^{n_x}\sum_{k=1}^{n_y} A_{ij}^x A_{lk}^y w_{jk}(\alpha_{xy})_{il}$$

$$+ \sum_{k=1}^{n_y} B_{lk}^y w_{ik}(\alpha_y)_{il} \}, \qquad (i=2,3,...,n_x-1, \ l=2,3,...,n_y-1)$$

(33)

where A_{ij}^x, A_{ij}^y are the weighting coefficients of the first-order derivatives, B_{ij}^x, B_{ij}^y are the weighting coefficients of the second-order derivatives and D_{ij}^x, D_{ij}^y are the weighting coefficients of the fourth-order derivatives with respect to the variables of x and y, Q_{ij} and w_{ij} are values of stress function and deflection at grid point ij, and

$$\alpha_x = \frac{\sigma_x}{\sigma_\circ}, \ \alpha_{xy} = \frac{\tau_{xy}}{\sigma_\circ}, \ \alpha_y = \frac{\sigma_y}{\sigma_\circ}$$

(34)

As it is pointed out the Eq. (32) cannot directly be applied to obtain stress function. First solving Eq. (32) when obtaining the values of Q_{ij}, the in-plane stresses can be computed by

$$(\sigma_x)_{il} = \sum_{k=1}^{n_y} B_{lk}^y Q_{ik}$$

$$(\tau_{xy})_{il} = -\sum_{j=1}^{n_x}\sum_{k=1}^{n_y} A_{ij}^x A_{lk}^y Q_{jk}$$

(35)

$$(\sigma_y)_{il} = \sum_{k=1}^{n_x} B_{lk}^x Q_{kl}, \qquad (i=1,2,...,n_x, \ l=1,2,...,n_y)$$

At each boundary point except for the corner points, i.e., at il (i=1 or n_x, $l = 2,3,...,n_y - 1$) on edges parallel to y-axis (x=-a/2 or a/2), the momentums can be expressed as

$$(M_x)_{il} = -\frac{E_3}{1-v_\circ^2}(\sum_{k=1}^{n_x} B_{ik}^x w_{kl} + v\sum_{k=1}^{n_y} B_{lk}^y w_{ik}) \tag{36}$$

Similarly at each boundary point except for the corner points, i.e., at il (l=1 or n_y, $i = 2,3,...,n_x - 1$) on edges parallel to x-axis (y=-b/2 or b/2), the momentums can be expressed as

$$(M_y)_{il} = -\frac{E_3}{1-v_\circ^2}(v\sum_{k=1}^{n_x} B_{ik}^x w_{kl} + \sum_{k=1}^{n_y} B_{lk}^y w_{ik}) \tag{37}$$

The buckling coefficient, (k) is defined as

$$k = \frac{\sigma_\circ hb^2}{\pi^2 D} \tag{38}$$

where $\sigma_\circ h / D$ is the eigen value of Eq. (33), this value is calculated by the computer programs of Maple and Matlab that are used in tables. To achieve convergence, set $n_x = n_y = n$ and used the following non-uniform grid spacing for results presented in the next section

$$x_i = -a\cos[\frac{(i-1)\pi}{n-1}]/2$$
$$y_i = -b\cos[\frac{(i-1)\pi}{n-1}]/2, \qquad i = 1,2,...,n \tag{39}$$

8 PROPERTIES OF SAMPLE FGM PLATE

A FGM (Aluminum-Alumina) rectangular plate is taken into consideration. The properties of plate [20] are defined as:

Alumina (Ceramic Constituent) $E_c = 380$ Gpa
Aluminum (Metal Constituent) $E_m = 70$ Gpa

9 RESULTS AND DISCUSSION

For obtaining buckling coefficient, first we obtained the Airy stress function after calculation of Eq. (32) and then with obtaining the stresses from Eq. (35) and substitute them in to Eq. (33), as the egen value $\sigma_\circ h / D$ has calculated, for obtaining buckling coefficient. A computer program was written by Maple and Matlab software, the above equations were solved and finally the buckling coefficient has been calculated. The buckling coefficient was calculated for different grid points under two kinds of loading when all edges are simply supported and clamped.
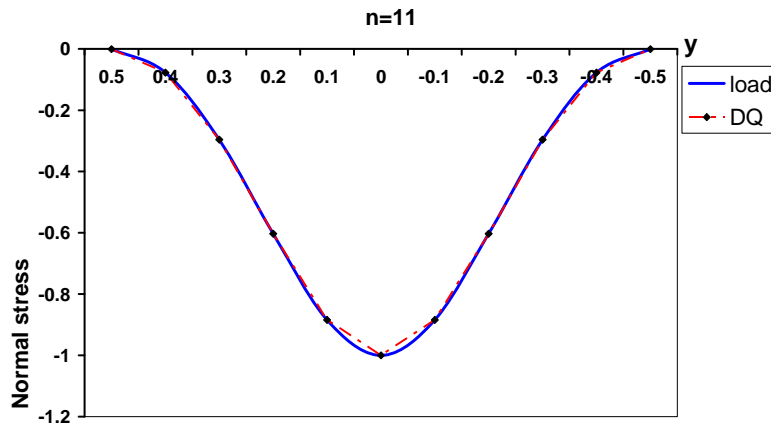
Fig. 4
Normal stress distributions at
$x = \pm a / 2$ for *n*=11 .
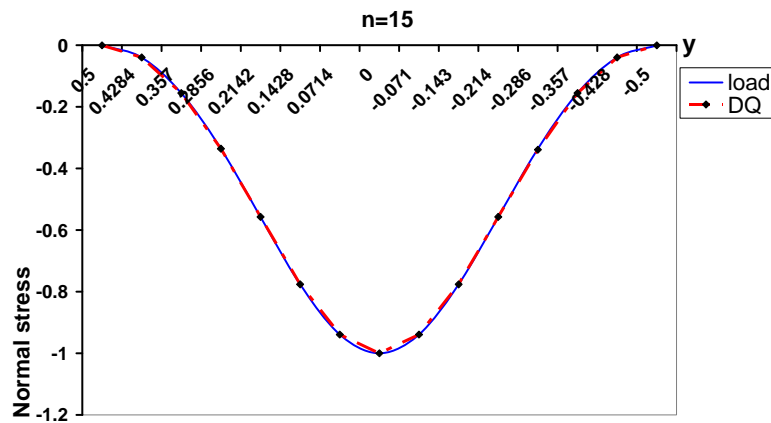
Fig. 5
Normal stress distributions at
$x = \pm a / 2$ for *n*=15.
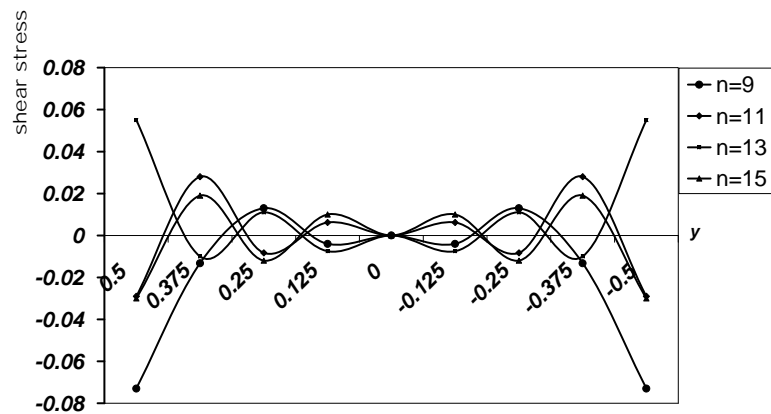
Fig. 6
Shear stress distributions at
$y = \pm b / 2$.

When power law index is equaled to zero, the FGM material is changed to uniform material and obtaining results were compared with principle reference. After getting to essential accuracy, the buckling coefficient was calculated for different value of power law index for sample FGM material and finally results were presented to table and figures. In the tables and figures, (*n*) is total number of grid points, (p) is power law index, (SS) and (C) are simply supported and clamped boundary conditions respectively and (K) is the buckling coefficient.

Two non-uniform distributed loading cases are studied, at first the rectangular plate is under uni-axial linearly varying compressive load

Table 1
Buckling coefficient for sample FGM plate, under uni-axial linearly varying compressive load, $\sigma_x = -2\sigma_0 / 3(1 - y/b)$ and different aspect ratios, all edges are (ss) for different value of power law index (n=11)

a/b	Present	Wanng[17]	FEM	Bert[9]	Benoy[22]	Van Der Neut[21]
0.5	7.421	7.452	7.409	7.841	7.08	4.68
1.0	5.408	5.419	5.383	5.146	4.59	4.68
3.0	5.847	5.849	5.818	5.748	4.53	4.68

Table 2
Convergence for buckling coefficient of rectangular plate with three different aspect ratios, under uniaxial half-cosine compressive load, $X = -\sigma_0 \cos(\pi y/b)$, all edges are simply supported (n=15).

p	n=9	n=11	n=13	n=15
2	36.15	36.17	36.15	36.14
4	41.55	41.55	41.57	41.57
6	43.71	43.70	43.71	43.74
8	45.39	45.37	45.40	45.40
10	46.93	46.91	46.91	46.93
12	48.37	48.37	48.34	48.36
14	49.70	49.69	49.72	49.71

Table 3
Buckling coefficient for sample FGM plate, under uniaxial half-cosine compressive load, $X = -\sigma_0 \cos(\pi y/b)$ and a/b=1, all edges are (C) for different value of power law index

| a/b	0.4	0.6	0.75	0.8	1.0
p=1	24.56	15.25	12.94	12.54	11.96
p=3	34.58	21.47	18.22	17.66	16.84
p=5	37.21	23.11	19.61	19.00	18.12
p=7	38.70	24.03	20.39	19.76	18.86
p=9	40.20	24.96	21.18	20.53	19.57
p=11	41.49	25.76	21.86	21.19	20.20
p=13	42.70	26.51	22.50	21.80	20.79
p=15	43.81	27.20	23.09	22.37	21.33

Table 4
Buckling coefficient for sample FGM plate, under uniaxial half-cosine compressive load, $X = -\sigma_0 \cos(\pi y/b)$ and a/b=1, all edges are (SS) for different value of power law index
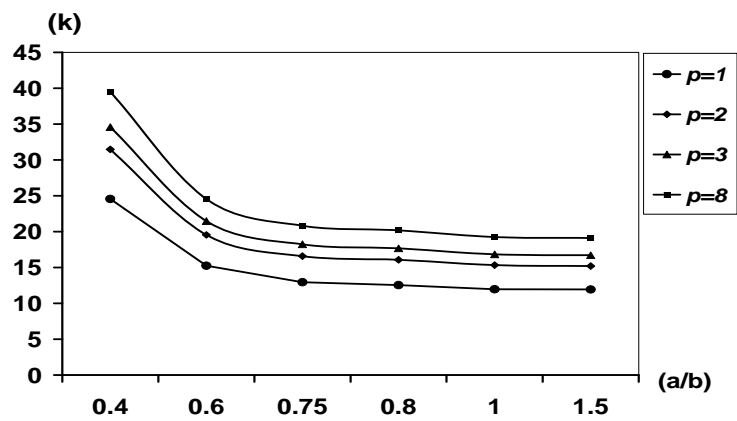
p	n=9	n=11	n=13	n=15
1	10.87	10.86	10.89	10.85
2	13.93	13.93	13.91	13.94
3	15.31	15.32	15.30	15.31
4	16.01	16.01	16.01	16.03
5	16.48	16.47	16.47	16.42
6	16.85	16.85	16.81	16.84
7	17.13	17.12	17.12	17.14
8	17.50	17.50	17.52	17.50
9	17.80	17.81	17.83	17.83
10	18.09	18.07	18.05	18.07
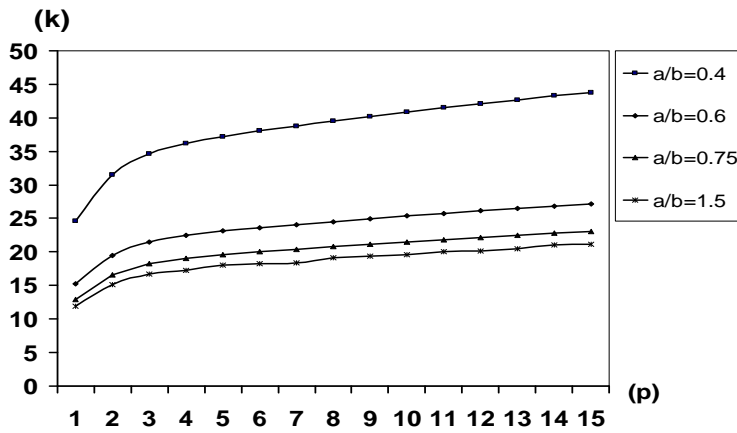
Fig. 7
Buckling coefficient for
$\sigma_x = -2\sigma_0 / 3(1 - y / b)$, all edges are
(SS), (n=11).
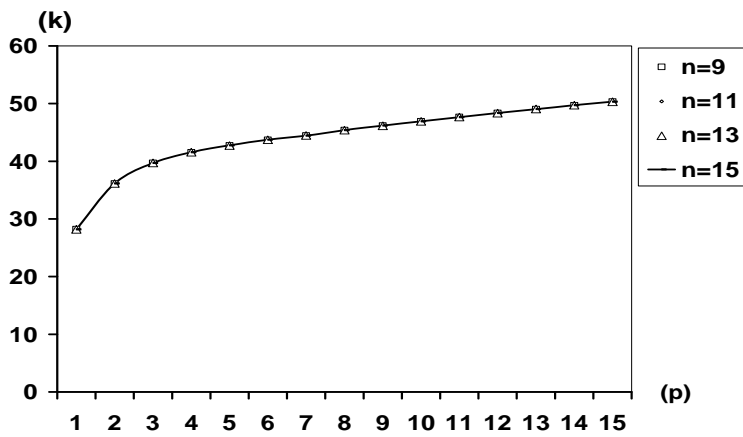
Fig. 8
Buckling coefficient for
$X = -\sigma_0 \cos(\pi y / b)$ and *a/b*=1, all
edges are (C).

Fig. 9
Buckling coefficient for
$\sigma_x = -2\sigma_0 / 3(1 - y / b)$, all edges are
(SS) (n=11).

$$X = -2\sigma_0 / 3(1 - y / b) \qquad (40a)$$

This load is presented in Fig. (2a), so we will have

$$\sigma_x = -2\sigma_0 / 3(1 - y / b)$$
$$\sigma_y = \tau_{xy} = 0$$

(40b)

Two boundary conditions are considered for buckling analysis: 1) All edges are simply supported. 2) All edges are clamped. It can be seen that DQ results are very close to the FE data that one may conclude that the numbers listed in the manual are not accurate enough for this loading case. Consider next a rectangular plate under non-uniformly distributed compressive load (Fig. 1b).

$$X = -\sigma_0 \cos(\pi y / b)$$

(41)

It was found that solutions to in-plane elasticity problem by DQ method are sensitive to grid spacing. The normal stresses in x and y directions are equal to zero, but shear stress is $\tau_{xy} \neq 0$. As presented in Fig. (6) $\tau_{xy} \neq 0$ along edges of $y = \pm b / 2$. The shear stress along these two edges are opposite in sign. $\tau_{xy} = 0$ can be approximately achieved with the increase of number of grid points. In Figs. 4 and 5, the calculated σ_x along $x = \pm a / 2$ by DQ method for n=11, n=15 as can be seen that symbols are close to the solid line (the applied load, $X = -\sigma_0 \cos(\pi y / b)$) , and the answers are very close together. It was found that the new version of DQ is comfortable for these problems. Table (1) shows the DQ results with n=11 for three different aspect ratios. It is found that differences are observed between the data obtained by DQ method and results cited from literature [21], but are close to the FE data, this shows the manual method is not accurate. In table (2) the buckling coefficient (k) obtained for a sample FGM plate. It can be seen with increasing aspect ratio, buckling coefficient decrease and with increasing the power law index it increases. And also in tables 3 and 4, the buckling coefficient obtained, it is found that the buckling coefficient is similar for different value of grid points numbers and increase with increasing the power law index. Figs (7-9) show the resultant of tables (2-4) alike curvature.

10   CONCLUSION

In this paper, the new version of differential quadrature method (DQM), for calculation of the buckling coefficient of rectangular plates is considered. At first the governing differential equation for plate has been calculated and then according to the new version of differential quadrature method the existence derivatives in equation are converted to the amounts of function in the grid points inside of the region is solved. Calculation performed and observed show that if the number of grid point was low then the results will be not convergence, therefore the manual are not accurate enough. It is also found that DQ solutions are sensitive to grid spacing, thus non-uniform grid spacing should be used to ensure the solution accuracy. The results of the new version of DQ are close to FE data. This indicates this method can be employed for obtaining buckling loads of plates under non–uniform distributed loading for all boundary conditions.

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