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Transverse Vibration of Clamped and Simply Supported Circular Plates with an Eccentric Circular Perforation and Attached Concentrated Mass

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ABSTRACT
In this investigation Rayleigh-Ritz variational method has been applied to determine the least natural frequency coefficient for the title problem. Classical plate theory assumptions have been used to calculate strain energy and kinetic energy. Coordinate functions are combination of polynomials which satisfy boundary conditions at the outer boundary and trigonometric terms. In the second part of this study ABAQUS software is used to compute vibration natural frequency for some special combinations of geometrical and mechanical parameters. Then results of Rayleigh-Ritz method have been obtained for the mentioned special cases. It can be seen that the agreement between them is acceptable.

Keywords: Vibration; Circular plate; Eccentric circular perforation; Concentrated mass

1 INTRODUCTION

DRING the past four decades, vibration of plates has become an important subject in engineering applications. There are several papers about plate vibrations in open technical literature. Circular plates have many engineering applications. These are commonly found in spacecrafts, missiles, land base vehicles, off-shore platforms, and underwater acoustic transducers. In many situations there are mechanical or electro mechanical or electronic systems attached to circular plates. Hole eccentricity may be caused by human inaccuracy. In other cases it may have practical reasons. Consequently, it is considerable to compute fundamental dynamic parameters such as lower natural frequencies of these structures.

Various methods have been applied to determine natural frequency coefficients of vibrating plates. Jacout and Lindsay [1] presented the influence of Poisson's Ratio on the lower natural frequencies of vibrating circular plates. Laura and Grossi [2] expanded the previous study by changing thickness and edge type. Circular plates supporting masses distributed over a finite area has been studied by Gutierrez and Laura in 1977 [3]. Transverse vibration of simply supported circular plates having partial elastic constraints is considered in the work done by Navita and Leissa [4]. Laura et al. [5] analyzed the vibration and stability of circular plates elastically restrained against rotation. The effect of support flexibility on free and forced vibration of plates has been investigated by Laura et al. [6]. Irie et al. [7] considered the case of circular plates elastically restrained along some radial segments. Grossi and Laura [8] have used Rayleigh-Schmidt technique to compute lower frequency coefficients of polar orthotropic circular plates carrying concentrated masses. Free vibration of solid circular plates of linearly varying thickness and attached to a Winkler foundation has been solved by Laura and Gutierrez [9]. Bercin [10] have obtained the natural frequency coefficients for clamped orthotropic plates by applying Kantorovich method. Ranjan and Gosh have discussed transverse vibration of thin solid and annular circular plate with attached discrete masses using finite element analysis [11]. Vibration of polar orthotropic circular plates, making use of Rayleigh-Ritz method, has been studied by Kang et al. in 2004 [12]. Using the Hamilton's principle, Park [13] derived frequency equation for the in-

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plane vibration of a clamped circular plate. Avalos and Larrondo [14] studied circular plates with a concentric square hole in 1997. Bambill et al. [15] have applied Rayleigh-Ritz method to investigate about vibration of circular and annular plates with an attached concentrated mass. Vibration of circular plates with an eccentric perforation has been considered by Laura et al [16]. To the authors' knowledge, circular plates with eccentric circular perforation and attached concentrated mass have not been apparently studied in open literature.

2 FORMULATION

Fig. 1 shows the vibrating system of the studied problem. The plate transverse displacement can be written as:

$$W'(r', \theta, t) = W'(r', \theta)\, e^{i\omega t} \tag{1}$$

where the displacement amplitude is assumed to be:

$$W'(r', \theta) \cong W'_a(r', \theta) = \sum_{j=0}^{J} A_{j0}(\alpha_{j0} r'^4 + \beta_{j0} r'^2 + 1) r'^{2j} + \sum_{k=1}^{K} \cos(k\theta) \sum_{j=0}^{J} A_{jk}(\alpha_{jk} r'^4 + \beta_{jk} r'^2 + 1) r'^{j+k} \tag{2}$$

Here, A_{jk} are constant coefficients. The coefficient α and β are determined by applying the governing boundary conditions at the outer boundary. For clamped edge, the boundary conditions are:

$$W'(a, \theta) = 0 \quad , \quad \frac{\partial W'(r', \theta)}{\partial r'}(a, \theta) = 0 \tag{3}$$

Simply supported edge boundary conditions can be written as:

$$W'(a, \theta) = 0 \quad , \quad \frac{\partial^2 W'(r', \theta)}{\partial r'^2}(a, \theta) = 0 \tag{4}$$

where a is the radius of circular plate. To study the frequency response of the plate the Rayleigh-Ritz variational method is applied. This method needs to minimize energy functional.

$$J[W'] = U[W'] - T_1[W'] - T_2[W'] \tag{5}$$

where $U[W']$ is the strain energy, $T_1[W']$ is the plate kinetic energy, and $T_2[W']$ is the concentrated mass kinetic energy. We can compute strain energy with the next relation:

$$U = \frac{1}{2} \iiint_v (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\, dx\, dy\, dz \tag{6}$$

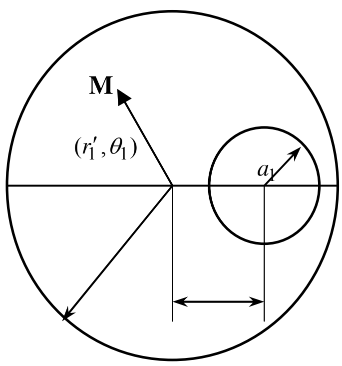

Fig. 1
Vibrating system.

Using the classical plate theory assumptions we have:

$$U=\frac{1}{2}\iiint_{v}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\tau_{xy}\gamma_{xy})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{7}$$

The strains are expressed as:

$$\varepsilon_x=\frac{1}{E}(\sigma_x-\upsilon\sigma_y),\quad \varepsilon_y=\frac{1}{E}(\sigma_y-\upsilon\sigma_x),\quad \gamma_{xy}=\frac{\tau_{xy}}{G} \tag{8}$$

Additionally the stresses are defined by:

$$\sigma_x=\frac{-Ez}{1-\upsilon^2}\left(\frac{\partial^2 W'}{\partial x^2}+\upsilon\frac{\partial^2 W'}{\partial y^2}\right),\ \sigma_y=\frac{-Ez}{1-\upsilon^2}\left(\frac{\partial^2 W'}{\partial y^2}+\upsilon\frac{\partial^2 W'}{\partial x^2}\right),\ \tau_{xy}=\frac{-Ez}{1+\upsilon}\left(\frac{\partial^2 W'}{\partial x\partial y}\right) \tag{9}$$

Substituting Eqs. (8) and (9) into Eq. (7) leads to the following equation:

$$U=\frac{1}{2}\iint_A D\left[\left(\frac{\partial^2 W'}{\partial y^2}+\frac{\partial^2 W'}{\partial x^2}\right)^2-2(1-\upsilon)\left(\frac{\partial^2 W'}{\partial x^2}\frac{\partial^2 W'}{\partial y^2}-\left(\frac{\partial^2 W'}{\partial x\partial y}\right)^2\right)\right]\mathrm{d}x\,\mathrm{d}y \tag{10}$$

Substituting the transformation relations from rectangular coordinates to polar coordinates into (10) we obtain:

$$U[W']=\frac{D}{2}\iint_A\{\left[\left(\frac{\partial^2 W'}{\partial r'^2}+\frac{1}{r'}\frac{\partial W'}{\partial r'}\right)+\left(\frac{1}{r'^2}\frac{\partial^2 W'}{\partial\theta^2}\right)\right]^2-2(1-\upsilon)\{\frac{\partial^2 W'}{\partial r'^2}\left(\frac{1}{r'}\frac{\partial W'}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2 W'}{\partial\theta^2}\right)$$
$$-\left(\frac{1}{r'}\frac{\partial^2 W'}{\partial\theta\partial r'}-\frac{1}{r'^2}\frac{\partial W'}{\partial\theta}\right)^2\}\}r'\mathrm{d}r'\mathrm{d}\theta \tag{11}$$

where A is the area of the plate under study, D is plate flexural rigidity, and υ is the Poisson's ratio. For the plate kinetic energy we have:

$$T_1[W']=\frac{1}{2}\rho h\omega^2\iint_A W'^2 r'\mathrm{d}r'\mathrm{d}\theta \tag{12}$$

where ρ is the density of plate, h is its thickness, and ω is the natural frequency. The concentrated mass kinetic energy can be expressed as:

$$T_2[W']=\frac{1}{2}M\omega^2[W'(r_1',\theta_1)]^2 \tag{13}$$

where M is the quantity of concentrated mass, and (r_1',θ_1) is the concentrated mass position. Substituting Eqs. (11)-(13) into (5) results in the energy functional relation:

$$J[W']=\frac{D}{2}\iint_A\{\left[\left(\frac{\partial^2 W'}{\partial r'^2}+\frac{1}{r'}\frac{\partial W'}{\partial r'}\right)+\left(\frac{1}{r'^2}\frac{\partial^2 W'}{\partial\theta^2}\right)\right]^2-2(1-\upsilon)[\frac{\partial^2 W'}{\partial r'^2}\left(\frac{1}{r'}\frac{\partial W'}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2 W'}{\partial\theta^2}\right)$$
$$-\left(\frac{1}{r'}\frac{\partial^2 W'}{\partial\theta\partial r'}-\frac{1}{r'^2}\frac{\partial W'}{\partial\theta}\right)^2]\}r'\mathrm{d}r'\mathrm{d}\theta-\frac{1}{2}\rho h\omega^2\iint_A W'^2 r'\mathrm{d}r'\mathrm{d}\theta-\frac{1}{2}M\omega^2[W'(r_1',\theta_1)]^2 \tag{14}$$

Minimizing the governing function using the Rayleigh-Ritz variational method with respect to A_{jk} produce linear system of equations.

$$\frac{\partial J[W']}{\partial A_{jk}} = 0 \tag{15}$$

Non-triviality condition yields an equation in natural frequency coefficients. For generality and convenience of the mathematical formulation, the following dimensionless parameters are introduced as:

$$W = \frac{W'}{a}, r = \frac{r'}{a}, r_1 = \frac{r_1'}{a}, \Omega_i = \sqrt{\frac{\rho h}{D}}\omega_i a^2 \tag{16}$$

In this study the least natural frequency coefficient is determined for different cases.

3 RESULTS AND DISCUSSION

By employing the Rayleigh-Ritz variational method, numerical solutions have been obtained. To calculate the natural frequency coefficients for the title problem there are several parameters which should be considered. a_1 :radius of perforation, e: eccentricity, M: quantity of concentrated mass, (r_1, θ_1) : position of concentrated mass. The Poisson's ratio coefficient υ is assumed to be 0.3 in all cases. Two boundary conditions are considered to analyze the vibrating system. Table 1 shows the variation of non-dimensional natural frequency coefficients, Ω_1 for a simply supported circular plate. Table 2 demonstrated the same results for a clamped circular plate. In these tables we took $a_1 / a = 0.1$, $e / a = 0.1$. Note that M_p refers to the plate mass $(M_p = \rho \pi h (a^2 - a_1^2))$.

In the second part of this study (tables 3-6), to investigate the accuracy of the present formulation, comparison studies are carried out with the finite element analysis. A computer program is developed and the results are obtained by ABAQUS software. The material of the circular plate under study in this part is assumed to be stainless steel with the following material properties: the Young's modulus is considered $E = 200\,\text{GPa}$ and the density is considered as $\rho = 7800\,\text{kg/m}^3$. Tables 3 and 4 show the frequency variations for circular plates under simply supported and clamped boundary conditions, respectively and $a = 1\,\text{m}$, $a_1 = 0.1\,\text{m}$, $e = 0.1\,\text{m}$ and $h = 2\,\text{cm}$. Tables 5 and 6 show the frequency variations for circular plates under simply supported and clamped boundary conditions, respectively and $a = 1\,\text{m}$, $a_1 = 0.1\,\text{m}$, $e = 0.2\,\text{m}$, $M = 100\text{kg}$ and $h = 2\,\text{cm}$.

Table 1
Non-dimensional natural frequency coefficients Ω_1 for a simply supported circular plate

$\frac{M}{M_p}$	Mass Position, r_1	θ_1 (°) 0	45	90	135	180
0.05	0.2	4.504	4.510	4.519	4.522	4.521
	0.4	4.608	4.615	4.620	4.619	4.615
	0.6	4.738	4.741	4.742	4.741	4.738
	0.8	4.840	4.840	4.841	4.841	4.840
0.1	0.2	4.169	4.210	4.227	4.230	4.225
	0.4	4.368	4.386	4.397	4.394	4.380
	0.6	4.603	4.613	4.615	4.613	4.603
	0.8	4.802	4.804	4.805	4.804	4.803
0.2	0.2	3.721	3.747	3.776	3.777	3.763
	0.4	3.965	4.008	4.030	4.021	3.985
	0.6	4.347	4.378	4.385	4.378	4.347
	0.8	4.726	4.732	4.732	4.733	4.727

Table 2
Non-dimensional natural frequency coefficients Ω_1 for a clamped circular plate

$\frac{M}{M_p}$	Mass Position, r_1	θ_1 (°)				
		0	45	90	135	180
0.05	0.2	9.050	9.094	9.139	9.142	9.121
	0.4	9.521	9.563	9.583	9.574	9.540
	0.6	9.989	9.998	9.996	9.994	9.984
	0.8	10.175	10.177	10.173	10.174	10.175
0.1	0.2	8.156	8.251	8.340	8.332	8.275
	0.4	8.890	9.008	9.054	9.031	8.924
	0.6	9.768	9.802	9.801	9.795	9.757
	0.8	10.158	10.156	10.153	10.150	10.157
0.2	0.2	6.902	7.060	7.203	7.174	7.062
	0.4	7.831	8.100	8.197	8.137	7.886
	0.6	9.282	9.418	9.420	9.407	9.264
	0.8	10.119	10.123	10.122	10.123	10.120

Table 3
Natural frequency coefficient ω_1 in hertz for simply supported circular plate $e = 0.1$ m

Mass (kg)	Mass position, r_1' (m)		θ_1 (°)		
			0	90	180
100	0.2	ABAQUS	110.735	113.016	113.656
		Analytical	113.374	115.122	114.723
		Error (%)	2.3	1.8	0.9
	0.4	ABAQUS	119.983	120.618	120.932
		Analytical	121.012	123.067	121.595
		Error (%)	0.8	1.9	0.5
	0.6	ABAQUS	131.695	131.984	132.129
		Analytical	132.914	133.803	132.883
		Error (%)	0.9	1.3	0.5
200	0.2	ABAQUS	91.156	94.053	94.832
		Analytical	94.233	96.779	95.889
		Error (%)	3.2	2.8	1.1
	0.4	ABAQUS	101.957	102.787	103.163
		Analytical	103.067	106.748	103.865
		Error (%)	1	3.7	0.7
	0.6	ABAQUS	117.420	117.870	118.061
		Analytical	119.141	122.177	119.141
		Error (%)	1.4	3.5	0.9

In order to show the deflection of the circular plate of title problem, 3D mode shapes are depicted in Figs. 2-4. From tables 1-2 it is found that in most cases the maximum and minimum natural frequencies have happened at $\theta_1 = 90^\circ$ and $\theta_1 = 0^\circ$, respectively. The results show that the natural frequencies are decreased with an increase of the ratio M / M_p and this decrease for clamped circular plates is more than the simply supported circular plates. It is clear that in all cases by increasing r_1 (concentrated mass position) natural frequency coefficient increases. As it can be seen in tables 3-6 most of minimum frequencies occurred at $\theta_1 = 0^\circ$ while the maximum frequencies always happened at $\theta_1 = 90^\circ$. The error percentage for clamped plates are higher than the simply supported ones.

Table 4
Natural frequency coefficient ω_1 in hertz for clamped circular plate $e = 0.1$ m

Mass (kg)	Mass position, r_1' (m)		θ_1 (°)		
			0	90	180
100		ABAQUS	200.151	210.298	212.497
	0.2	Analytical	209.877	219.263	214.907
		Error (%)	4.6	4	1
		ABAQUS	235.097	237.328	237.900
	0.4	Analytical	238.558	250.030	240.214
		Error (%)	1.4	5	0.9
		ABAQUS	279.168	280.110	279.985
	0.6	Analytical	283.282	288.220	283.282
		Error (%)	1.4	2.8	1.2
200	0.2	ABAQUS	155.665	166.667	169.005
		Analytical	166.073	177.668	171.472
		Error (%)	6.3	6.1	1.4
	0.4	ABAQUS	190.192	192.743	193.289
		Analytical	194.846	212.852	196.871
		Error (%)	1.8	9.5	1.8
		ABAQUS	242.279	243.944	243.385
	0.6	Analytical	252.668	265.981	252.02
		Error (%)	4.1	8.3	3.4

Table 5
Natural frequency coefficient ω_1 in hertz for simply supported circular plate $e = 0.2$ m , $M = 100$ kg

Mass position, r_1' (m)		θ_1 (°)		
		0	90	180
0.1	ABAQUS	109.321	111.564	112.193
	Analytical	113.639	114.659	113.901
	Error (%)	3.8	2.7	1.5
0.3	ABAQUS	113.411	116.698	117.251
	Analytical	115.489	120.557	118.555
	Error (%)	1.8	3.2	1.1
0.6	ABAQUS	131.451	132.110	132.387
	Analytical	134.271	135.776	133.589
	Error (%)	2.1	2.7	0.9

Table 6
Natural frequency coefficient ω_1 in hertz for clamped circular plate $e = 0.2$ m , $M = 100$ kg

Mass position, r_1' (m)		θ_1 (°)		
		0	90	180
0.1	ABAQUS	195.922	205.686	207.822
	Analytical	206.451	215.378	212.279
	Error (%)	5.1	4.5	2.1
0.3	ABAQUS	209.311	223.361	224.750
	Analytical	214.677	243.577	228.636
	Error (%)	2.5	8.3	1.7
0.6	ABAQUS	278.980	280.877	281.355
	Analytical	284.093	292.885	285.060
	Error (%)	1.8	4	1.3

By increasing the concentrated mass the error percentage increases, while the natural frequencies decrease. This decrease for far positions from the plate center is less than nearer positions to plate center.

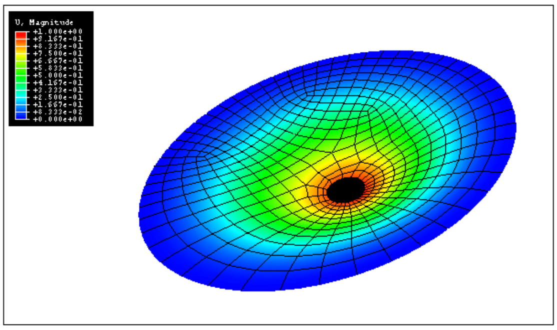

Fig. 2
First deformed mode shape and frequency parameter of a circular plate with clamped outer boundary
$(E = 200\,\mathrm{GPa},\ \rho = 7800\ \mathrm{kg/m^3},\ a = 1\,\mathrm{m},\ a_1 = 0.1\,\mathrm{m}$
$e = 0.1\,\mathrm{m},\ h = 0.02\,\mathrm{m}, (r_1', \theta_1) = (0.2\,\mathrm{m}, 0^\circ),\ M = 100\ \mathrm{kg})$.

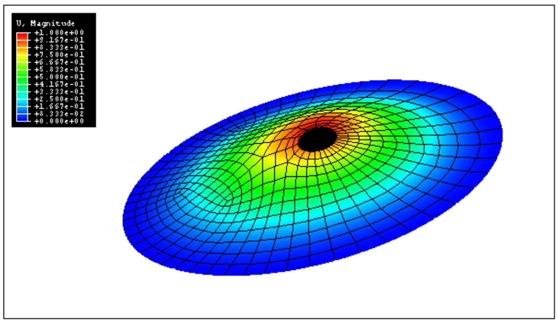

Fig. 3
First deformed mode shape and frequency parameter of a circular plate with clamped outer boundary
$(E = 200\,\mathrm{GPa},\ \rho = 7800\ \mathrm{kg/m^3},\ a = 1\,\mathrm{m},\ a_1 = 0.1\,\mathrm{m}$
$e = 0.1\,\mathrm{m},\ h = 0.02\,\mathrm{m}, (r_1', \theta_1) = (0.2\,\mathrm{m}, 90^\circ),\ M = 100\ \mathrm{kg})$.

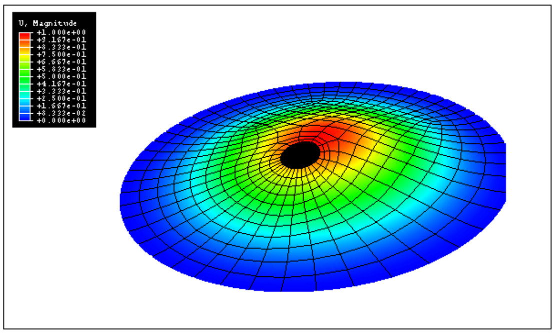

Fig. 4
First deformed mode shape and frequency parameter of a circular plate with clamped outer boundary
$(E = 200\,\mathrm{GPa},\ \rho = 7800\ \mathrm{kg/m^3},\ a = 1\,\mathrm{m},\ a_1 = 0.1\,\mathrm{m}$
$e = 0.1\,\mathrm{m},\ h = 0.02\,\mathrm{m}, (r_1', \theta_1) = (0.2\,\mathrm{m}, 180^\circ),\ M = 100\ \mathrm{kg})$.

Tables 3 and 4 depict that if $r_1' = 0.4\,\mathrm{m}$ or $0.6\,\mathrm{m}$, there are the maximum and the minimum error percentages at $\theta_1 = 90^\circ$ and $\theta_1 = 180^\circ$, respectively. These two tables also show that when $r_1' = 0.2$, as θ_1 increases, error percentage decreases. Two results can be concluded from tables 5 and 6. Firstly, when $r_1' = 0.3m$ or $0.6m$, the maximum errors happen at $\theta_1 = 90^\circ$ and the minimum ones occur at $\theta_1 = 180^\circ$. Secondly, when $r_1' = 0.1\,\mathrm{m}$, by increasing θ_1 error percentage decreases.

4 CONCLSIONS

Small amplitude transverse vibration of circular plates with an eccentric perforation and a concentrated mass placed at any arbitrary position has been investigated in this paper. The following results can be made from this work:

1. The natural frequency coefficients for clamped circular plate are higher than those for simply supported ones.
2. When the quantity of concentrated mass increases, the fundamental frequency coefficients decrease.

3. It seems that the mathematical model is more suitable for simply supported circular plates than the clamped circular plates. That is because of the fact that the difference between results obtained in analytical study and ABAQUS software for simply supported plates is less than clamped Circular plates.

As a general conclusion, it can be expressed that the mathematical model is quite acceptable for the title problem.

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