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Free Vibration of Functionally Graded Beams with Piezoelectric Layers Subjected to Axial Load

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ABSTRACT

This paper studies free vibration of simply supported functionally graded beams with piezoelectric layers subjected to axial compressive loads. The Young's modulus of beam is assumed to be graded continuously across the beam thickness. Applying the Hamilton's principle, the governing equation is established. Resulting equation is solved using the Euler's Equation. The effects of the constituent volume fractions, the influences of applied voltage and axial compressive loads on the vibration frequency are presented. To investigate the accuracy of the present analysis, a compression study is carried out with a known data.

Keywords: Free vibration; Functionally graded beam; Piezoelectric layer

1 INTRODUCTION

THE mechanics of piezoelectric materials has been an important branch of solid mechanics in recent years. In particular, the research of piezoelectric beam, a plate and shell structure has attracted much attention from both engineers and scientists [1-3]. Shu'lga [4] utilized separation formulae for displacements and stresses to simplify the basic equations of piezoelasticity for spherical isotropy, and obtained two independent classes of vibrations. Chen and Ding [5] exactly analyzed a rotating piezoelectric hollow sphere by introducing displacement functions. Chen et al. [6] recently investigated the coupled-free vibration problem of a submerged piezoelectric spherical shell.

Studies on functionally graded material (FGM) have been extensive in the last decade [7]. The dynamic analysis of FGM elastic beam, plates and shells has been of particular research interest recently [8, 9]. Based on the three-dimensional elasticity equations for spherical isotropy, Chen et al. [10] exactly analyzed the coupled-free vibration of a fluid-filled FGM hollow sphere. By introducing displacement functions and using the Frobenius power-series method, Chen [11] recently considered the vibration problem of spherically isotropic piezoelastic spheres with a functionally graded property that the material constants vary with the radial co-ordinate in a power law. It should be noted that laminated models have been widely employed to analyze functionally graded materials or structures in the study of FGM [7, 12, 13]. However, with the increasing number of involved layers, conventional methods used by many authors usually lead to lower numerical exigency. The state-space method has shown to be very effective in the analysis of laminated structures because of the associated lower order solving matrix. Its recent applications in piezoelectric materials and structures can be found in references [14-17].

In the present work, the free vibration of a functionally graded beam with piezoelectric actuators subjected to axial compressive loads is studied. The elasticity modulus of functionally graded layer is assumed to vary as a power form of the thickness coordinate variable. Applying the Hamilton's principle, the governing equation of beam is derived and solved. The effects of the applied voltages, axial compressive loads and functionally graded index on the vibration frequency of beam are also discussed.

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2 FORMULATIONS

Consider a functionally graded beam with piezoelectric actuators and rectangular cross-section as shown in Fig. 1. The thickness, length, and width of the beam are denoted, respectively, by $h, L,$ and b. Also, h_T and h_B are the thickness of top and bottom of piezoelectric actuators, respectively. The $x-y$ plane coincides with the midplane of the beam and the $z-$axis located along the thickness direction. The Young's modulus E is assumed to vary as a power form of the thickness coordinate variable z $(-h/2 \le z \le h/2)$ as follow [18]

$$E(z) = (E_c - E_m)\left(\frac{2z+h}{2h}\right)^k + E_m \tag{1}$$

where k is the power law index and the subscripts m and c refer to the metal and ceramic constituents, respectively. The Poisson's ratio υ is assumed to be constant. The beam is assumed to be slender, thus, the Euler-Bernoulli beam theory is adopted. The piezoelectric layers are also assumed to be polarized along the thickness direction. The axial stress and electrical displacement can be written as

$$\sigma_{xx} = \frac{E(z)}{1-\nu^2}\varepsilon_{xx} - e_{31}E_z, \qquad D_z = e_{31}\varepsilon_{xx} + \eta_{33}E_z \tag{2}$$

where

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{3}$$

and

$$E_z = \frac{V}{h} \tag{4}$$

where $\sigma_{xx}, D_z, e_{31},$ and η_{33} are the normal stress, electrical displacement, piezoelectric elastic stiffness, and permittivity coefficient, respectively, and u and w are the displacement components in the $x-$ and $z-$ directions, respectively. The potential energy can be expressed as

$$U = \frac{1}{2}\int_v (\sigma_{xx}\varepsilon_{xx} - D_z E_z)\,\mathrm{d}v \tag{5}$$

Substituting Eqs. (2)-(4) into Eq. (5) and neglecting the higher-order terms, we obtain

$$\begin{aligned} U = {} & \frac{1}{2}\int_v \frac{E(z)}{1-\nu^2}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}v + \int_v \frac{1}{2}\left(\frac{E(z)}{1-\nu^2}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right) - e_{31}\frac{V}{h}\right)\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}v \\ & - \int_v e_{31}\frac{V}{h}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}v - \frac{1}{2}\int_v \eta_{33}\left(\frac{V}{h}\right)^2 \mathrm{d}v \end{aligned} \tag{6}$$

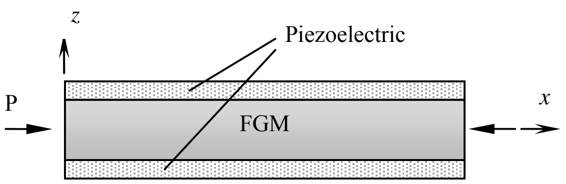

Fig. 1
Schematic view of the problem studied.

The width of beam is assumed to be constant, which is obtained by integrating along y over v. Then Eq. (6) becomes

$$U=\frac{b}{2}\int_0^L\left(A_{11}\left(\frac{\partial u}{\partial x}\right)^2-2B_{11}\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right)\mathrm{d}x+\frac{b}{2}\int_0^L P'\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x-\frac{b}{2}\int_0^L \eta_{33}\left(\frac{V_B^2}{h_B}+\frac{V_T^2}{h_T}\right)\mathrm{d}x$$
$$-b\int_0^L\left(\int_{-h_B-h/2}^{-h/2}e_{31}\frac{V_B}{h_B}\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}z+\int_{h/2}^{h_T+h/2}e_{31}\frac{V_T}{h_T}\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}z\right)\mathrm{d}x \tag{7}$$

where

$$(A_{11},B_{11},D_{11})=\frac{1}{1-\nu^2}\int_{-h_B-h/2}^{h_T+h/2}(1,z,z^2)E(z)\,\mathrm{d}z \tag{8}$$

and

$$P'=\int_{-h_B-h/2}^{h_T+h/2}\left(\frac{E(z)}{1-\nu^2}\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)-e_{31}E_z\right)\mathrm{d}z=A_{11}\frac{\partial u}{\partial x}-B_{11}\frac{\partial^2 w}{\partial x^2}-\int_{-h_B-h/2}^{h_T+h/2}e_{31}\frac{V}{h}\,\mathrm{d}z \tag{9}$$

where $A_{11}, B_{11}, D_{11}, V_T, V_B$ and P' are the extensional stiffness, coupling stiffness, bending stiffness, applied voltages on the top and bottom actuators and piezoelectric force, respectively. When the applied voltage is negative, the piezoelectric force is tensile. Note that, no residual stresses due to the piezoelectric actuator are considered in the present study and the extensional displacement is neglected. Thus, the potential energy can be written as

$$U=\frac{b}{2}\int_0^L D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}x+\frac{P'b}{2}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x-\frac{b}{2}\int_0^L\eta_{33}\left(\frac{V_B^2}{h_B}+\frac{V_T^2}{h_T}\right)\mathrm{d}x$$
$$+b\int_0^L\left(\int_{-h_B-h/2}^{-h/2}ze_{31}\frac{V_B}{h_B}\left(\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}z+\int_{h/2}^{h_T+h/2}ze_{31}\frac{V_T}{h_T}\left(\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}z\right)\mathrm{d}x \tag{10}$$

The beam is subjected to the axial compressive loads P, as shown in Fig. 2. The work done by the axial compressive load can be expressed as

$$W=\frac{1}{2}\int_0^L P\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x \tag{11}$$

The kinetic energy can be expressed as

$$T=\frac{1}{2}\int_0^L m\left(\frac{\partial w}{\partial t}\right)^2\mathrm{d}x \tag{12}$$

where m is the mass per unit length of the beam. We apply the Hamilton's principle to derive the dynamic equation of beam, that is

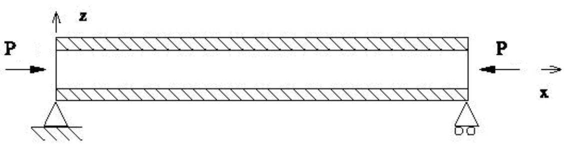

Fig. 2
Simply supported beam under periodic loads.

$$\delta\int_0^t (T - U + W)\,\mathrm{d}t = 0 \tag{13}$$

Substitution from Eqs. (10), (11), and (12) into Eq. (13) leads to the following dynamic equation

$$m\frac{\partial^2 w}{\partial t^2} + bD_{11}\frac{\partial^4 w}{\partial x^4} + (P - bP')\frac{\partial^2 w}{\partial x^2} = 0 \tag{14}$$

3 VIBRATION ANALYSIS

For the simply supported boundary condition, the solution of the dynamic equation is assumed to be in the following form:

$$w(x,t) = f(t)\sin\frac{n\pi x}{L}, \qquad n = 1,2,3,... \tag{15}$$

where $f(t)$ is the function of time. Substituting expression (15) into Eq. (14) leads to the following equation:

$$bD_{11}\left(\frac{k\pi}{L}\right)^4 + (p - bp')\left(\frac{k\pi}{L}\right)^2 - m\omega^2 = 0 \tag{16}$$

Thus, the nth free vibration frequency of functionally graded beam with piezoelectric actuators loaded by a constant axial force P, can be obtained as

$$\Omega_a = \overline{\omega_k}\sqrt{1 - \frac{p \;\; - bp'}{p^*{}_k}} \tag{17}$$

where

$$\overline{\omega_n} = \left(\frac{n\pi}{L}\right)^2\sqrt{\frac{bD_{11}}{m}} \tag{18}$$

and

$$P_n^* = \left(\frac{n\pi}{L}\right)^2 bD_{11} \tag{19}$$

Eq. (17) is referred to as the nth free vibration frequency of functionally graded beam with piezoelectric actuators loaded by a constant axial force. By setting the power law index equal to zero $(k = 0)$ and neglecting the piezoelectric effect, the nth free vibration frequency of homogeneous beams can be written as

$$\Omega_a = \overline{\omega_k}\sqrt{1 - \frac{p}{p^*{}_k}} \tag{20}$$

where

$$\overline{\omega_n} = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{bD}{m}}$$

$$P_n^* = \left(\frac{n\pi}{L}\right)^2 bD \tag{21}$$

and

$$D = \frac{1}{1-\nu^2}\int_{-h/2}^{h/2} z^2 E \,\mathrm{d}z \tag{22}$$

Eq. (20) has been reported by Bolotin [19].

4 NUMERICAL RESULTS

The free vibrations of functionally graded beams with piezoelectric actuators subjected to axial compressive loads are studied in this paper. It is assumed that both the top and bottom piezoelectric layers have the same thickness; $h_T = h_B$ and the same voltages are applied to both actuators. The material properties of the beam are listed in Table 1. The effect of power law index k on the free vibration frequency for the axial compressive load $10\mathrm{KN}$ is shown in Fig. 3.

Table 1
Material properties

Property	Piezoelectric layer	FGM layer	
		Aluminium	Alumina
Young's modulus E (GPa)	63	70	380
Poisson's ratio ν	0.3	0.3	0.3
Length L (m)	0.3	0.4	0.4
Thickness h (m)	0.00005	0.01	0.01
Density ρ (Kgm^{-3})		2707	3800
Piezoelectric constant e_{31}, e_{32} (Cm^{-2})	17.6	-	-

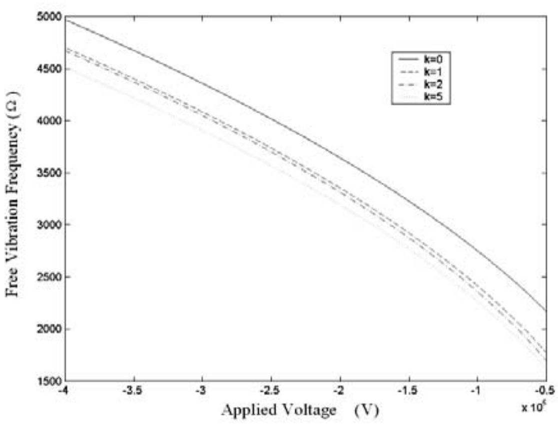

Fig. 3
Effect of power law index on the free vibration frequency.

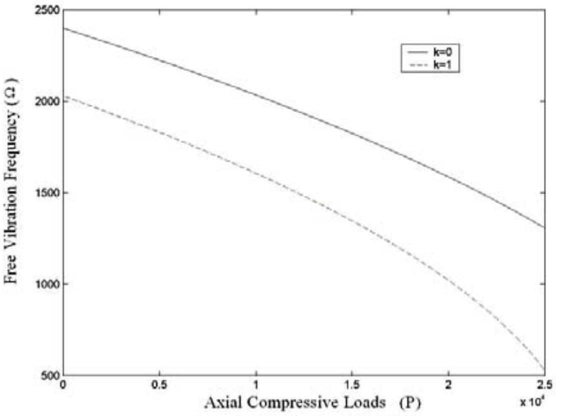

Fig. 4
Effect of axial compressive load on the free vibration frequency.

It is found that, as k increases, the free vibration frequency decreases. Fig. 4 illustrates the effect of the axial compressive load on the free vibration frequency for the applied voltage -40 KV. As the axial compressive load increases, the free vibration frequencies decrease. Also, Comparisons of the free vibration frequency for the functionally graded beam and isotropic beam are shown in Fig. 4. It is evident that the free vibration frequencies decrease when the beam is made of functionally graded materials.

5 CONCLUSION

The free vibration analysis of a functionally graded beam with piezoelectric actuators subjected to the axial compressive loads has been presented. It was shown that the piezoelectric actuators induce tensile piezoelectric force produced by applying negative voltages that significantly affect the free vibration of the functionally graded beam with piezoelectric actuators. The free vibration frequency increases when the applied voltage is negative. The functionally graded beam with a smaller axial compressive load is more efficient in reducing the chance of resonance. The comparison of the stability for the functionally graded beam and isotropic beam shows that the functionally graded beam is more stable.

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