

Research Paper

# Fractional Cattaneo Heat Equation in A Multilayer Elliptic Ring Membrane and Its Thermal Stresses

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## ABSTRACT

A fractional Cattaneo model from the generalized Cattaneo model with two fractional derivatives of different orders is considered for studying the thermoelastic response for a multilayer elliptic ring membrane with source function. The solution is obtained by applying an integral transform technique analogous to Vodicka's approach considering series expansion functions in terms of an eigenfunction to the generalized fractional Cattaneo-type heat conduction equation within an elliptic coordinates system. The analytical expressions of displacement and stress components employing Airy's stress function approach are investigated. The results are obtained as a series solution in terms of Mathieu functions and hold convergence test. The effects of fractional parameters on the temperature fields and their thermal stresses are also discussed. The findings are depicted graphically for different kinds of surface temperature gradients, and it is distinguished that the higher the fractional-order parameter, the higher the thermal response.

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**Keywords:** Fractional Cattaneo-type equation; Fractional calculus; Non-Fourier heat conduction; Thermal stress; Integral transform.

## 1 INTRODUCTION

**M**ANY engineering applications necessitate a thorough understanding of the transient temperature distribution and heat flux inside elliptic structures comprising two or more rigid layers with internal source functions. Usually, these multilayer structural elements are subjected to a high-temperature environment, and due to the presence of interface boundary conditions, the layer coefficient follows a discontinuity of temperature at the corresponding interface. Hence, a precise evaluation of their thermal characteristic is of great interest for engineering design and manufacture due to their wide applications as structural members in modern industries such as mechanical, aerospace, nuclear engineering, and reactors.

Most heat conduction problems are described and analyzed using classical Fourier's law concept. In this context, many researchers have solved the transient heat conduction problem in composite layers with different analytical approaches, such as the orthogonal expansion technique [1-5], quasi-orthogonal expansion technique [6-8], Laplace

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transform method [9-11], Green's function approach [12-14], Galerkin procedure [15-16], finite integral transform technique [17-21], the line heat-source method [22,23], the method of separation of variables [24], and so on. This Fourier law is widely used and has been demonstrated to produce acceptable results in various contexts. This law's fundamental flaw is that it assumes that the heat flux is instantaneously produced (or vanished) when a temperature gradient is applied (or removed) [25]. This law needs to be amended because no procedure is instantaneous. To put it simply, the Fourier law implies that thermal disturbances propagate at an infinite speed, and hence it is unreasonable. Mathematically, this is caused by the parabolic nature of the heat equation. To prevent this, a relaxation of the flux is introduced, leading to the so-called Cattaneo's equation [26]. Such a hyperbolic equation characterizes heat conduction's combined diffusion and wave-like behavior and predicts a finite heat propagation speed [27].

Fractional calculus has recently become an increasingly growing field with applications in many areas, including physical sciences, engineering, biological sciences, and economics [28-33]. Compte and Metzler [34] examined four varieties of potential Cattaneo equation generalizations, three of which are validated by a different scheme: continuous-time random walks, nonlocal transport theory, and the delayed flux-force relationship. In this background, Povstenko [35] demonstrated that the time-nonlocal generalization of Fourier law with kernels of Mittag-Leffler would yield all fractional generalizations of the Cattaneo equations proposed by Compte and Metzler. Povstenko [36] also published highly cited literature reviews on recently published articles on fractional thermoelasticity. As the fractional-order Cattaneo model is compared to the classical Cattaneo-Vernotte and Fourier models, it is realized by Xu and Wang [37] that the heat flux expected by the fractional Cattaneo model always transports from high to low temperature, which is consistent with the second law of thermodynamics. Maillet [38] reviewed a series of previous experimental papers to validate whether the Cattaneo and Vernotte hyperbolic heat equation (non-Fourier models) could be applied to bioheat transfer simulation.

Recently, few researchers have obtained mathematical solutions to the fractional Cattaneo-Vernotte heat conduction problem with different boundary conditions on a finite or semi-infinite medium, which can be briefly summarized below to know the emerging trends in the current scenario. Qi et al. [39-40] used the time fractional Cattaneo model as the heat conduction model having a volumetric heat source and obtained the analytical solution for the temperature distribution using the Laplace transformation method. Choi et al. [41] brought a new analytical solution for the hyperbolic partial differential equations of the non-Fourier heat conduction in terms of a simple exponential function explicitly compared with the existing data for justification. Povstenko [35,36,42-44] has investigated the time-fractional heat conduction equation with Caputo derivative under mathematical and physical Robin-type boundary conditions. Jiang and Qi [45] studied the thermal wave model of bio-heat transfer with the modified Riemann Liouville fractional derivative and demonstrated that the fractional models could provide a unified approach to examining the heat transfer of biological tissues. Xu et al. [46] investigated the heat conduction process in metal materials irradiated by non-Gauss laser pulses using the transient temperature field based on the fractional heat conduction equation for laser heating.

Most studies on the fractional Cattaneo-Vernotte model have focused on heat conduction problems in semi-infinite spaces with the Neumann boundary condition [35,46-48] and finite thickness problems with the Dirac boundary condition [49]. Nevertheless, based on the Cattaneo-Vernotte fractional model, the multilayered elliptic membrane's heat conduction with convective-type mathematical boundary condition has yet to be studied. The heat conduction mechanism that differs from the fractional-order parameters is analyzed. The time-fractional thermoelastic analysis of the Cattaneo-type for an elliptic membrane under prescribed boundary conditions has not been investigated to the best of the author's knowledge.

The following is an outline of the remaining section of the paper: Section 2 presents the mathematical modeling of the generalized heat conduction equation in the fractional Cattaneo-type framework. Section 3 describes the mathematical formulation of the problem for a multilayer elliptic ring membrane and its associated thermal stresses. In Section 4, the solution of time-fractional Cattaneo analysis is obtained as a series solution in terms of Mathieu functions. Section 5 depicts the graphical outcomes. Finally, conclusive comments are summarized in Section 6.

## 2 GENERALIZED FRACTIONAL CATTANEO-TYPE EQUATION

The classical Cattaneo model [26] as

$$q + \tau \frac{\partial q}{\partial t} = -k \nabla T \quad (1)$$

By combining Eq. (1) with the continuity equation

$$\rho C_v \frac{\partial T}{\partial t} = -k \nabla \cdot q + g \tag{2}$$

leads to the hyperbolic heat conduction equation

$$\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} = \kappa \nabla T \tag{3}$$

in which  $g$  is the rate of the heat generation per unit volume,  $q$  is the heat flux vector,  $\tau$  is the relaxation time,  $k$  is the heat conductivity of a solid,  $\kappa$  is the thermal diffusivity coefficient,  $\nabla$  is the gradient operator,  $T$  is the temperature, and  $t$  is the time, respectively.

The fractional generalization [40] of Eq. (1) is written as

$$\frac{\partial^{\alpha-1} q}{\partial t^{\alpha-1}} + \tau \frac{\partial^{\beta-1} q}{\partial t^{\beta-1}} = -k \nabla T \tag{4}$$

Taking the Laplace transform and its inversion, by neglecting the initial value, one obtains the time-nonlocal constitutive relationship of Eq. (4) as

$$q(t) = -\frac{k}{\tau} \int_0^t (t-\gamma)^{\alpha-2} E_{\alpha-\beta, \alpha-1} \left[ -\frac{(t-\gamma)^{\alpha-\beta}}{\tau} \right] \nabla T(\gamma) d\gamma \tag{5}$$

where  $E_{\alpha, \beta}$  is the generalized Mittag-Leffler functions in two parameters  $\alpha$  and  $\beta$  [28].

By combining Eq. (4) with Eq. (2) leads to the fractional generalized Cattaneo equation as

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \tau \frac{\partial^\beta T}{\partial t^\beta} = \kappa \nabla^2 T + \frac{\kappa}{k} g \tag{6}$$

in which we take  $0 < \beta \leq \alpha \leq 2$ , and the fractional Caputo derivative [28] of order  $\alpha$  with lower limit zero in Eq. (6) is taken as

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \left\{ \int_0^t \frac{\partial^m}{\partial \gamma^m} \left[ \frac{f(r, \gamma)}{(t-\gamma)^{\alpha+1-m}} \right] d\gamma \right\}, m-1 < \alpha < m, m \in N \tag{7}$$

whereas the Riemann-Liouville fractional derivative of order  $\beta$  is defined as

$$\frac{\partial^\beta f}{\partial t^\beta} = \frac{\partial^m}{\partial t^m} \left\{ \frac{1}{\Gamma(m-\beta)} \int_0^t \left[ \frac{f(r, \gamma)}{(t-\gamma)^{\alpha+1-m}} \right] d\gamma \right\}, m-1 < \beta < m \tag{8}$$

where  $\Gamma$  is the Euler gamma function and  $m$  is an integer satisfying  $m-1 < \beta < m$ .

For the limiting case: (i) Taking  $\tau = 0$ ,  $\alpha = 0$ , Eq. (6) reduces to classical Fourier heat conduction [50], (ii) Taking  $\alpha = 2, \beta = 1$ , Eq. (6) reduces to the telegraph equation [35], (iii) Taking  $\alpha = \beta = 2$ , Eq. (6) reduces the classical wave equation [35], and (iv) Otherwise stated, all four fractional generalizations of the Cattano constitutive relationship given by Povstenko [35] can be viewed as exceptional cases of Eq. (6).

### 3 FORMULATION OF THE PROBLEM

Consider a  $k$ -layer composite elliptic membrane occupying space  $D = \{(\xi, \eta, z) \in R^3 : \xi_i < \xi < \xi_{i+1}, 0 < \eta < 2\pi, 1 \leq i \leq k\}$  in elliptical coordinates, related to the rectangular coordinate system by the relation  $x = c \cosh \xi \cos \eta$ ,  $y = c \sinh \xi \sin \eta$ ,  $c = (a^2 - b^2)^{1/2}$ , where  $c$  is the semi-focal length of the ellipse,  $a$  and  $b$  are semi-major and semi-minor axis, as shown in Figure 1.

Let us consider the time-fractional heat conduction differential equation as

$$\frac{\partial^\alpha T_i}{\partial t^\alpha} + \tau \frac{\partial^\beta T_i}{\partial t^\beta} = \kappa_i \left[ h^2 \left( \frac{\partial^2 T_i}{\partial \xi^2} + \frac{\partial^2 T_i}{\partial \eta^2} \right) + \frac{g_i(\xi, \eta) f(t)}{\lambda_i} \right] \quad (9)$$

with the initial and boundary conditions as

$$T_i(\xi, \eta, z, t=0) = 0, \quad \frac{\partial}{\partial t} T_i(\xi, \eta, z, t=0) = 0, \quad (10)$$

$$\begin{aligned} \alpha_1 \frac{\partial T_1}{\partial \xi}(\xi = \xi_1, \eta, t) - h_0 T_1(\xi = \xi_1, \eta, t) &= f(t), \quad h_0 \geq 0, \\ \alpha_k \frac{\partial T_k}{\partial \xi}(\xi = \xi_{k+1}, \eta, t) + h_k T_k(\xi = \xi_{k+1}, \eta, t) &= f(t), \\ \alpha_i \frac{\partial T_i}{\partial \xi}(\xi = \xi_{i+1}, \eta, t) + h_i [T_i(\xi = \xi_{i+1}, \eta, t) & \\ - T_{i+1}(\xi = \xi_{i+1}, \eta, t)] &= \alpha_{i+1} \frac{\partial T_{i+1}}{\partial \xi}(\xi = \xi_{i+1}, \eta, t) & (11) \\ + h_i [T_i(\xi = \xi_{i+1}, \eta, t) - T_{i+1}(\xi = \xi_{i+1}, \eta, t)] &= 0 \end{aligned}$$

in which the function  $T_i = T_i(\xi, \eta, t)$  represents the temperature,  $\kappa_i$  be the thermal diffusivity,  $\lambda_i$  denotes the thermal conductivity,  $\rho_i$  for density,  $C_i$  as specific heat for the  $i^{\text{th}}$  layer,  $f(t) = \delta(t)$  and  $\delta(\cdot)$  is the Dirac distribution, respectively.

The differential equation governing the Airy stress function  $\chi(\xi, \eta, t)$  is given as [5]

$$h^2 \nabla^2 h^2 \nabla^2 \chi^{(i)} = -h^2 \nabla^2 T_i \quad (12)$$

where

$$\nabla^2 = h^2 (\partial^2 / \partial \xi^2 + \partial^2 / \partial \eta^2). \quad (13)$$

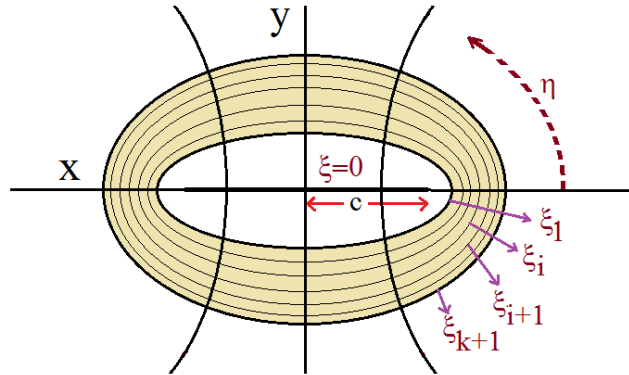
The stress distribution for each layer as [5]

$$\begin{aligned} \sigma_{\xi\xi}^{(i)} &= h^4 \left\{ \frac{\partial^2}{h^2 \partial \eta^2} + \frac{c^2}{2} \left( \sinh 2\xi \frac{\partial}{\partial \xi} - \sin 2\eta \frac{\partial}{\partial \eta} \right) \right\} \chi^{(i)}, \\ \sigma_{\eta\eta}^{(i)} &= h^4 \left\{ \frac{\partial^2}{h^2 \partial \xi^2} + \frac{c^2}{2} \left( -\sinh 2\xi \frac{\partial}{\partial \xi} + \sin 2\eta \frac{\partial}{\partial \eta} \right) \right\} \chi^{(i)}, \\ \sigma_{\xi\eta}^{(i)} &= h^4 \left\{ -\frac{\partial^2}{h^2 \partial \xi \partial \eta} + \frac{c^2}{2} \left( \sin 2\eta \frac{\partial}{\partial \xi} + \sinh 2\xi \frac{\partial}{\partial \eta} \right) \right\} \chi^{(i)} \end{aligned} \quad (14)$$

where  $c$  is the semi-focal length of the ellipse.

The traction-free surfaces

$$\sigma_{\xi\xi}^{(1)} = \sigma_{\xi\eta}^{(1)} = \sigma_{\xi\xi}^{(k)} = \sigma_{\xi\eta}^{(k)} = \sigma_{\xi\xi}^{(i)} = \sigma_{\xi\eta}^{(i)} = 0 \quad (15)$$



**Fig.1**  
Schematic sketch of the k-layer confocal elliptic membrane.

**4 THE SOLUTION OF THE PROBLEM**

Initially, we recall an integral transform proposed by Dhakate et al. [5], which states that if  $f_i(\xi, \eta)$  is continuous and has piecewise continuous first and second derivatives in the region  $0 \leq \xi \leq \xi_0, 0 \leq \eta \leq 2\pi$ ; satisfy interfacial and other boundary conditions, then the Sturm-Liouville transform for the composite region of order  $n$  and  $m$  over the variable  $\xi$  and  $\eta$  is defined as

$$\bar{f}_i(q_{2n,m}) = \beta_i \int_0^{2\pi} \int_{\xi_i}^{\xi_{i+1}} f_i(\xi, \eta) (\cosh 2\xi - \cos 2\eta) \psi_{i,2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) d\xi d\eta \tag{16}$$

and its inversion theorem is given by

$$f_i(\xi, \eta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{i=1}^k \frac{\bar{f}_i(\xi, \eta) \psi_{i,2n}(\xi, q) ce_{2n}(\eta, q)}{\pi \int_{\xi_i}^{\xi_{i+1}} \psi_{i,2n}^2(\xi, q) (\cosh 2\xi - \Theta_{2n,m}) d\xi} \tag{17}$$

in which  $ce_{2n}(\eta, q)$  is the ordinary Mathieu function of the first kind of order  $n$  [51, pp. 21],  $Ce_{2n}(\xi, q)$  is a modified Mathieu function of the first kind of order  $n$  [51, pp.27],  $\psi_{i,2n}(\xi, q_{2n,m})$  is the kernel of the transform,  $\bar{f}_i(q_{2n,m})$  is Sturm-Liouville transform for the composite region,  $\psi_{i,2n}(\xi)$  is the eigenfunction of the  $i^{th}$  layer,  $h_0$  is the surface coefficient at  $\xi = \xi_1, q_{2n,m}$  is the eigenvalue of  $[A_{i,2n} Ce_{2n}(\xi, q_{2n,m}) + B_{i,2n} Fey(\xi, q_{2n,m})] ce_{2n}(\eta) = 0$ ,  $(\alpha_i, \beta_i)$  is the characteristics of the  $i^{th}$  layer,  $h_i$  is the surface coefficient at  $\xi = \xi_{i+1}, h_k$  is the surface coefficient at  $\xi = \xi_{k+1}$  and  $L = \partial^2 / \partial \xi^2 + \partial^2 / \partial \eta^2$ . The arbitrary constants  $A_{i,2n}$  and  $B_{i,2n}$  can be obtained using a system of  $2k$  simultaneous equations. Here we consider the recurrence relations for the Bessel functions  $Y_{2r}$  (are identical to those for  $J_{2r}$ ) that can be defined as

$$Fey_{2n}(\xi, q) = \frac{ce_{2n}(0, q)}{A_0^{(2n)}} = \sum_{r=0}^{\infty} A_{2r}^{(2n)} Y_{2r}(2k' \sinh \xi) \begin{cases} |\sinh \xi| > 1 \\ R(\xi) > 0 \end{cases} \tag{18}$$

with  $y$  in  $Fey$  [51, pp.159] signifies the  $Y$ -Bessel function and  $q = k'^2 = \lambda c^2 / 4$ .

Applying the Sturm-Liouville transform defined in Eq. (16), one obtains

$$\frac{\partial^\alpha \bar{T}_i}{\partial t^\alpha} + \tau \frac{\partial^\beta \bar{T}_i}{\partial t^\beta} + \alpha_{2n,m}^2 \bar{T}_i(q_{2n,m}, t) = [\mu_i \bar{g}_i(q_{2n,m}) + \bar{F}(q_{2n,m})] f(t) \tag{19}$$

in which

$$\mu_i = \kappa_i / \lambda_i \beta_i, \alpha_{2n,m}^2 = 4\kappa_i q_{2n,m} / (\alpha_i c^2), \tag{20}$$

$$\bar{F}(q_{2n,m}) = \frac{4\pi\kappa_i}{c^2 \alpha_i} A_0^{(2n)} [\psi_{k,n,m}(\xi_{k+1}) - \psi_{1,n,m}(\xi_1)], \tag{21}$$

$$\bar{g}_i(q_{2n,m}) = \beta_i \int_0^{\xi_i} \int_0^{\xi_{i+1}} g_i(\xi, \eta) (\cosh 2\xi - \cos 2\eta) \psi_{i,2n}(\xi, q_{2n,m}) c e_{i,2n}(\eta, q_{2n,m}) d\xi d\eta \tag{22}$$

and

$$\bar{H}(q_{2n,m}) = \mu_i \bar{g}_i(q_{2n,m}) + \bar{F}(q_{2n,m}) \tag{23}$$

Now, applying the Laplace transform to the Eq. (19), one obtains

$$\bar{\bar{T}}_i(q_{2n,m}, s) = \frac{\bar{H}(q_{2n,m})}{s^\beta + \tau s^\alpha + \alpha_{2n,m}^2} F(s) \tag{24}$$

in which the double overbar is the transformed function,  $s$  is the Laplace parameter,

$$\bar{\bar{T}}_i(q_{2n,m}, s) = \int_0^\infty \bar{T}_i(q_{2n,m}, t) \exp(-st) dt, F(s) = \int_0^\infty f(t) \exp(-st) dt. \tag{25}$$

Extending the right side of (24) in a power series of  $s$  [28, pp 155], one gets

$$\bar{\bar{T}}_i(q_{2n,m}, s) = \bar{H}(q_{2n,m}) \frac{1}{\alpha_{2n,m}^2} \frac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau} \frac{1}{1 + \frac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau}} \tag{26}$$

Assuming  $\beta > \alpha$ , one obtains

$$\frac{1}{1 + \frac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau}} = \sum_{j=0}^\infty (-1)^j \left( \frac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau} \right)^j \tag{27}$$

Substituting Eq. (27) into Eq. (26), one obtains

$$\bar{\bar{T}}_i(q_{2n,m}, s) = \bar{H}(q_{2n,m}) \sum_{j=0}^\infty (-1)^j \frac{(\alpha_{2n,m}^2)^j s^{-(1+j\alpha)}}{(s^{\beta-\alpha} + \tau)^{j+1}} \tag{28}$$

The series in (27) is convergent [52] if we choose  $s_0 > 0$ , then

$$\left| \frac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau} \right| = \left| \frac{\alpha_{2n,m}^2}{s^\beta + s^\alpha \tau} \right| < 1 \tag{29}$$

hold if  $\text{Re } s > s_0$  and it determines  $s_0 > 0$ .

Thus, the term-by-term inversion, based on the general expansion theorem for the Laplace transform [28, pp 21], one obtains

$$\mathcal{L}^{-1} \left\{ \sum_{j=0}^\infty \frac{(-1)^j (\alpha_{2n,m}^2)^j}{j!} \frac{j! s^{-(1+j\alpha)}}{(s^{\beta-\alpha} + \tau)^{j+1}} \right\} = \sum_{j=0}^\infty (-1)^j (\alpha_{2n,m}^2)^j t^{\beta(j+1)-1} E_{\beta-\alpha, \beta+j\alpha}^{(j)}(-\tau t^{\beta-\alpha}) / j! \tag{30}$$

where  $E_{\alpha, \beta}^{(j)}$  is the generalized Mittag-Leffler functions [28, pp 21] defined as

$$E_{\alpha, \beta}^{(j)}(z) = \frac{d^n}{dz^n} E_{\alpha, \beta}(z) = \sum_{j=0}^\infty \frac{(j+n)! z^j}{j! \Gamma(\alpha j + \alpha n + \beta)} \tag{31}$$

Finally, applying the inversion theorems, one obtains

$$T_i = \sum_{n=0}^\infty \sum_{m=1}^\infty \frac{\bar{H}(q_{2n,m})}{C_{2n,m}} \psi_{i,2n}(\xi, q_{2n,m}) c e_{2n}(\eta, q_{2n,m}) \left[ \int_0^t \bar{T}(u) f(t-u) du \right] \tag{32}$$

where

$$\begin{aligned}
 C_{2n,m} &= \pi \int_{\xi_i}^{\xi_{i+1}} \psi_{i,2n}^2(\xi, q_{2n,m}) (\cosh 2\xi - \Theta_{2n,m}) d\xi, \\
 \Theta_{2n,m} &= \frac{1}{\pi} \int_0^{2\pi} \cos 2\eta c e_{2n}^2(\eta, q_{2n,m}) d\eta, \\
 \bar{T}(u) &= \sum_{j=0}^{\infty} (-1)^j (\alpha_{2n,m}^2)^j u^{\beta(j+1)-1} E_{\beta-\alpha, \beta+j\alpha}^{(j)}(-\tau u^{\beta-\alpha}) / j!
 \end{aligned}
 \tag{33}$$

Now assume Airy's stress function, which satisfies condition (12) as

$$\chi^{(i)} = h^2 \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \psi_{i,2n}(\xi, q_{2n,m}) c e_{2n}(\eta, q_{2n,m}) \phi_1 / C_{2n,m}
 \tag{34}$$

in which  $A_{2n,m}$  is the arbitrary constant that can be determined finally by using condition (9).

$$A_{2n,m} = \frac{c e_{2h} a(x) - \cot 2x b(h)}{c e_{2h} g(x) - b(h)}
 \tag{35}$$

in which

$$\begin{aligned}
 \alpha(\xi) &= \cot 2\xi \sinh 2\xi \psi(\xi) / \psi(\xi)_{,\xi} - 2 \sinh 2\xi, \\
 \beta(\eta) &= \sinh 2\eta c e(\eta, q_{2n,m})_{,\eta} - 2 c e(\eta, q_{2n,m})_{,\eta\eta} / c^2 h^2, \\
 \gamma(\xi) &= 2 \cot 2\xi \sinh 2\xi - \sinh 2\xi \psi(\xi) / \psi(\xi)_{,\xi},
 \end{aligned}
 \tag{36}$$

$$\phi_1 = \sum_{i=1}^k \left\{ \left[ \int_0^t \bar{T}(u) f(t-u) du \right] \mu_i \bar{g}_i(q_{2n,m}) (A_{2n,m} \sin 2\xi - \cos 2\xi) \right\}$$

Substituting  $A_{2n,m}$  in Eq. (27), we get

$$\chi^{(i)} = h^2 \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\psi_{i,2n}(\xi, q_{2n,m}) c e_{2n}(\eta, q_{2n,m}) \phi_2}{C_{2n,m} [\beta(\eta) - c e_{2n}(\eta, q_{2n,m}) \gamma(\xi)]}
 \tag{37}$$

where

$$\phi_2 = \sum_{i=1}^k \left\{ \left[ \int_0^t \bar{T}(u) f(t-u) du \right] \mu_i \bar{g}_i(q_{2n,m}) [-\sin 2\xi \alpha(\xi) + \cos 2\xi \gamma(\xi)] \right\}
 \tag{38}$$

Substituting Eq. (30) into Eq. (14), one obtains the stress components as

$$\sigma_{\xi\xi}^{(i)} = h^6 \left\{ \frac{\partial^2}{h^2 \partial \eta^2} + \frac{c^2}{2} \left( \sinh 2\xi \frac{\partial}{\partial \xi} - \sin 2\eta \frac{\partial}{\partial \eta} \right) \right\} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\psi_{i,2n}(\xi, q_{2n,m}) c e_{2n}(\eta, q_{2n,m}) \phi_2}{C_{2n,m} [\beta(\eta) - c e_{2n}(\eta, q_{2n,m}) \gamma(\xi)]}
 \tag{39}$$

$$\sigma_{\eta\eta}^{(i)} = h^6 \left\{ \frac{\partial^2}{h^2 \partial \xi^2} + \frac{c^2}{2} \left( -\sinh 2\xi \frac{\partial}{\partial \xi} + \sin 2\eta \frac{\partial}{\partial \eta} \right) \right\} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\psi_{i,2n}(\xi, q_{2n,m}) c e_{2n}(\eta, q_{2n,m}) \phi_2}{C_{2n,m} [\beta(\eta) - c e_{2n}(\eta, q_{2n,m}) \gamma(\xi)]}
 \tag{40}$$

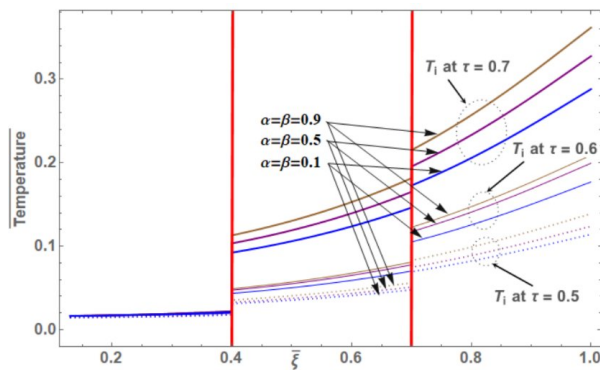
$$\sigma_{\xi\eta}^{(i)} = h^6 \left\{ -\frac{\partial^2}{h^2 \partial \xi \partial \eta} + \frac{c^2}{2} \left( \sin 2\eta \frac{\partial}{\partial \xi} + \sinh 2\xi \frac{\partial}{\partial \eta} \right) \right\} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\psi_{i,2n}(\xi, q_{2n,m}) c e_{2n}(\eta, q_{2n,m}) \phi_2}{C_{2n,m} [\beta(\eta) - c e_{2n}(\eta, q_{2n,m}) \gamma(\xi)]}
 \tag{41}$$

### 5 NUMERICAL RESULTS, DISCUSSION, AND REMARKS

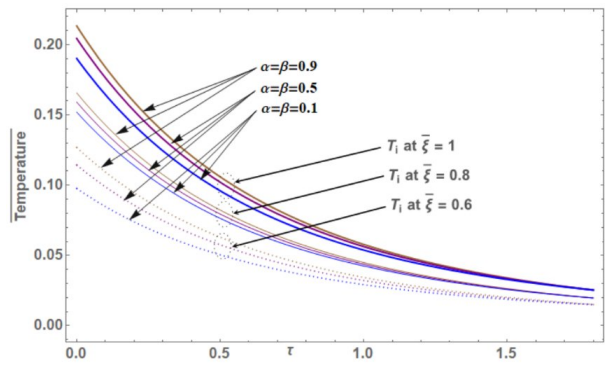
We introduce the following dimensionless values

$$\begin{aligned}
 \bar{\xi}_i &= \xi_i / b, \bar{a} = a / b, \bar{b} = b / b, e = c / b, \tau = \kappa t / b^2, \\
 T &= T / T_0, \bar{\sigma}_{ij} = \sigma_{ij} / E \alpha_i \theta_0, (i, j = \xi, \eta)
 \end{aligned}
 \tag{42}$$

For the sake of simplicity of numerical calculations, we consider a three-layered composite elliptic annulus membrane. The numerical computations have been carried out for Aluminium and Tin metal with initial temperature and for ( $t > 0$ ), the temperature raised to a finite value. The physical parameters are considered as  $\xi_1 = a = 0.1$  cm,  $\xi_2 = 0.5$  cm,  $\xi_3 = b = 1$  cm, and a case of  $\alpha = \beta$ . The mechanical material properties are considered as specific heat at constant pressure in the  $i^{\text{th}}$  section,  $C_{v1} = 0.181$  cal/g $^{\circ}$ C and  $C_{v2} = 0.054$  cal/g $^{\circ}$ C; Modulus of Elasticity,  $E_1 = 6.9 \times 10^6$  N/cm $^2$  and  $E_2 = 4.7 \times 10^6$  N/cm $^2$ ; Thermal expansion coefficient,  $\alpha_1 = 24.8 \times 10^{-6}$  cm/cm $^{\circ}$ C,  $\alpha_2 = 23.0 \times 10^{-6}$  cm/cm $^{\circ}$ C; Thermal conductivity  $\lambda_1 = 0.52$  cal sec $^{-1}$ /cm $^{\circ}$ C and  $\lambda_2 = 0.15$  cal sec $^{-1}$ /cm $^{\circ}$ C;  $f_1(t) = f_2(t) = 20$  cm $^{\circ}$ C. To examine the influence of heating on the membrane, we performed the numerical calculation for all variables, and numerical calculations are depicted in the following figures with the help of MATHEMATICA software. Figs. 2-3 illustrate the numerical results of temperature distribution and stresses of the elliptical membrane due to internal heat generation within the solid. It can be seen that Fig. 2(a) and Fig. 2(c) both exhibit relatively similar patterns. Fig. 2(a) and Fig. 2(c) show the temperature distribution along  $\bar{\xi}$  - direction for  $\alpha = \beta$  and a different time and  $\eta$ , respectively. In both the figures, temperature increases gradually towards the outer end of each layer due to the combined effect of sectional heat supply and internal heat energy, as shown in Fig. 2(b). Initially, the temperature approaches a maximum, whereas it attains a minimum as time increases. Fig. 2(d) shows that the temperature approaches a minimum at both extreme ends, i.e., at  $\eta = 0$  and  $\eta = \pi$  due to more compressive force. In contrast, due to a tensile force, the temperature is high at the central, i.e., at  $\eta = \pi/2$ , which gives an overall bell-shaped curve for all three layers of different materials. The given result in Eq. (32) was in good agreement with the result [53] with a three-layer plate of finite dimension. Fig. 3(a) illustrates the axial stress decreases with time. Initially, the axial stress attains maximum expansion due to the accumulation of thermal energy dissipated by sectional heat supply and internal heat energy, further decreasing time. Fig. 3(b) indicated that the axial stress along the  $\xi$ - direction is initially on the negative side due to more compressive stress occurring at the inner edge, which goes on increasing along  $\xi$ - direction and attains a maximum at the outer edge. Fig. 3(c) depicts tangential stress along the radial direction achieving minimum at the inner and maximum at the outer edge. In Fig. 3(d), shear stress is of negative magnitude. Initially, it reaches a minimum, decreasing at the middle core and suddenly increasing towards the outer region. Fig. 3(e) observed the minimum tangential stress with a negative magnitude which increases gradually with time.

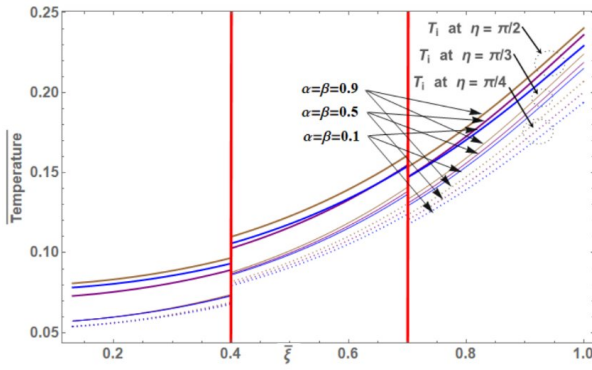


**Fig.2a**  
Temperature distribution along  $\bar{\xi}$ -direction for  $\alpha = \beta$  and different  $\tau$ .

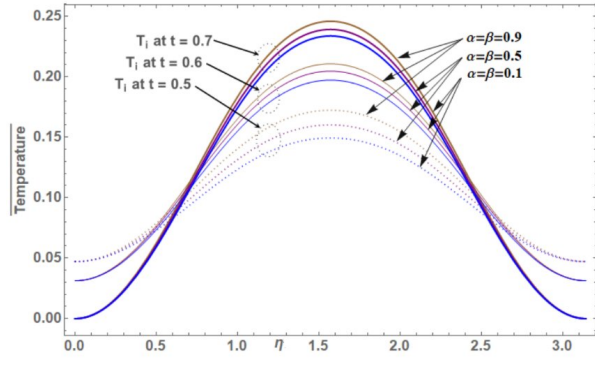


**Fig.2b**  
Temperature distribution along  $\tau$  for  $\alpha = \beta$  and different  $\bar{\xi}$ .

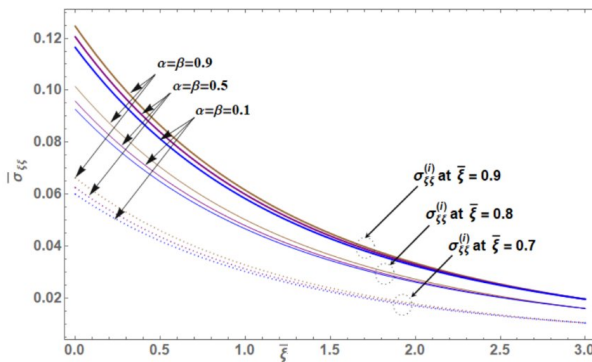




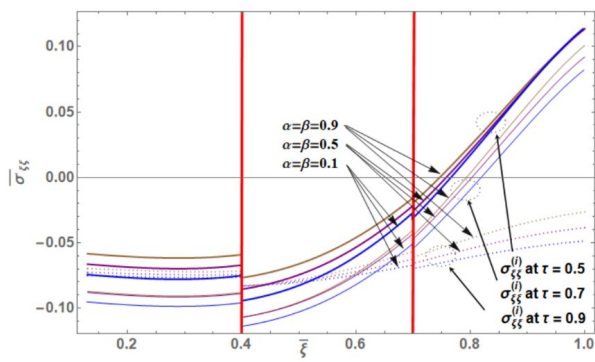
**Fig.2c**  
Temperature distribution along  $\bar{\xi}$ -direction for  $\alpha = \beta$  and different  $\eta$ .



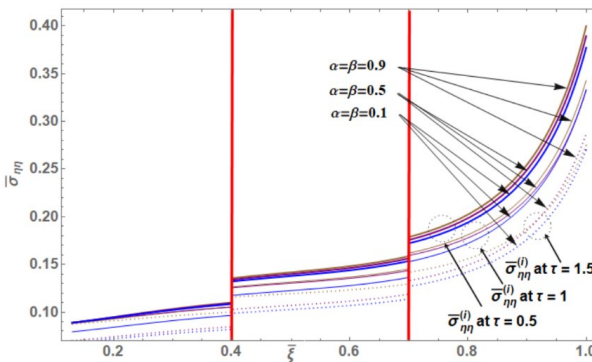
**Fig.2d**  
Temperature distribution along  $\eta$ -direction for  $\alpha = \beta$  and different  $\tau$ .



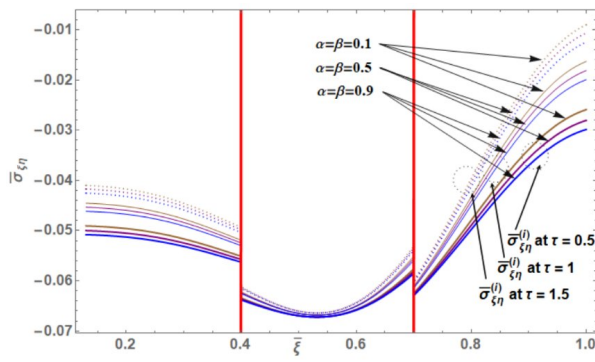
**Fig.3a**  
Radial stress along  $\tau$  for  $\alpha = \beta$  and different  $\bar{\xi}$ .



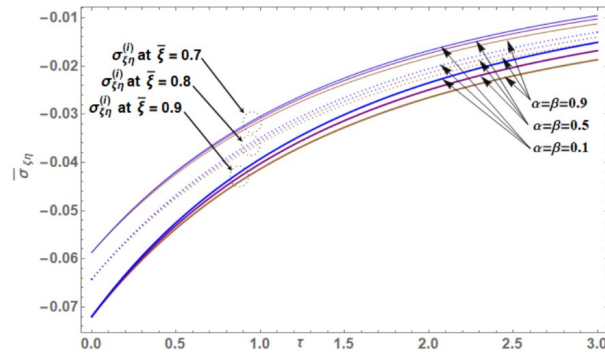
**Fig.3b**  
Radial stress along  $\bar{\xi}$ -direction for  $\alpha = \beta$  and different  $\tau$ .



**Fig.3c**  
Tangential stress along  $\bar{\xi}$ -direction for  $\alpha = \beta$  and different  $\tau$ .



**Fig.3d**  
Shear stress along  $\bar{\xi}$ -direction for  $\alpha = \beta$  and different  $\tau$ .



**Fig.3e**

Shear stress along  $\tau$  for different  $\alpha = \beta$  and different  $\bar{\xi}$ .

The key that was derived by Dhakate et al. [5] for an isotropic, homogeneous, elastic multilayer elliptic annulus is consistent with the determined thermoelastic solutions. This piece of research brings a fractional-order constitutive model and the standard continuity equation together. Recent research [54,55] shows that it is possible for a non-Fourier constitutive model and a non-trivial continuity equation based on the Boltzmann transport theory to coexist. The findings demonstrate that the constitutive model and the continuity equation are not independent of one another, which this work does not consider.

## 6 CONCLUSIONS

We introduced the fractional Cattaneo constitutive model and the corresponding fractional Cattaneo equation with two fractional derivatives of different orders to investigate the thermoelastic response of a multilayer elliptic ring membrane with a source function. The solution is investigated using the integral transform method establishing the Sturm-Liouville integral transform, which is analogous to Vodicka's approach considering series expansion function in terms of an eigenfunction to solve the heat conduction partial differential equation in elliptical coordinates. The temperature and thermal stresses for the three-layer elliptic region have been computed numerically and exhibited graphically. The following results were obtained during our research:

- The advantage of this method is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions.
- The maximum tensile stress shifting from the central core to the outer region may be due to heat, stress, concentration, or internal heat sources under-considered temperature field.
- Finally, the maximum tensile stress occurring in the circular core on the major axis compared to the elliptical central part indicates the distribution of weak heating. It might be due to insufficient heat penetration through the elliptical inner surface.

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