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Memory Response in Thermoelastic Plate with Three-Phase-Lag Model

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ABSTRACT

In this article, using memory-dependent derivative (MDD) on three-phase-lag model of thermoelasticity, a new generalized model of thermoelasticity theory with time delay and kernel function is constructed. The governing coupled equations of the new generalized thermoelasticity with time delay and kernel function are applied to two-dimensional problem of an isotropic plate. The two-dimensional equations of generalized thermoelasticity with MDD are solved using state space approach. Numerical inversion method is employed for the inversion of Laplace and Fourier transforms. The displacements, temperature and stress components for different thermoelastic models are presented graphically and the effect of different kernel and time delay on the considered parameters are observed.

Keywords: Memory-dependent derivative; Three-phase-lag model; State-space approach; Laplace-Fourier transform.

1 INTRODUCTION

TO overcome the paradox of infinite speed of thermal wave inherent in the classical coupled thermoelasticity theory [1] (CCTE), the subject of generalized thermoelasticity theory is developed. Lord and Shulman [2] formulated the generalized thermoelasticity theory introducing one relaxation time and thus transforming the heat conduction equation into a hyperbolic type. Green and Lindsay [3] introduced one more theory, called GL theory, which involves two relaxation times. Later Green and Naghdi [4, 5, 6] developed three models for generalized thermoelasticity of homogeneous isotropic materials, which are known as models I, II, III. The next generalization to the thermoelasticity is known as the dual-phase-lag model developed by Tzou [7]. Tzou [7] considered micro-structural effects into the delayed response in time in macroscopic formulation by taking into account that increase of the lattice temperature is delayed due to phonon-electron interactions on the macroscopic level. Tzou [7] introduced two-phase lags to both the heat flux vector and the temperature gradient and considered as constitutive equation to describe the lagging behavior in the heat conduction in solids. Roy Choudhuri [8] has established a generalized mathematical model of a coupled thermoelasticity theory that includes three-phase-lags in the heat flux vector, the temperature gradient and in the thermal displacement gradient. The more general model established reduces to the previous models as special cases. Wang and Li [9] proposed a memory-dependent derivative of which

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time delay and kernel function can be chosen freely according to the necessity of application and found it is better than the fractional calculus for reflecting the memory effect. Karamany and Ezzat [10] developed fractional thermoelasticity and Ezzat [11] considered thermoelastic MHD non-Newtonian fluid with fractional derivative heat transfer. Ezzat and Karamany [12] studied fractional order heat conduction law in magneto thermoelasticity involving two temperatures. Yu et al. [13] proposed a novel generalized thermoelasticity model based on memory dependent derivatives. Ezzat and Bary [14] discussed memory dependent derivatives theory of thermo-viscoelasticity involving two temperatures. Ezzat et al. [15] studied generalized thermo-viscoelasticity with MDD. Sherief and Al-Latief [16] examined the effect of variable thermal conductivity on a half space under fractional order theory of thermoelasticity. Ezzat and Bary [17] considered two temperature theory of magneto thermo-viscoelasticity with fractional derivative and integral orders heat transfer. A method for solving coupled thermoelastic problems by state space approach has been developed by Anwar and Sherief [18]. Ezzat et al. [19] considered state space approach to two-dimensional generalized thermoviscoelasticity with one relaxation time. Ezzat et al. [20] considered state space approach to two-dimensional electro-magneto thermoelastic problem with two relaxation times. Sherif and El-Sayed [21] studied two-dimensional generalized micropolar thermoelasticity using state space approach. A rather detailed account of diverse recent theoretical advances and applications of fractional calculus in the various fields can be found in the book of Diethelm [22], Sabatier et al. [23], Hilfer [24] and Atanackovic et al. [25]. Biswas [26] proposed memory-dependent derivatives in orthotropic medium with three-phase-lag model under the effect of magnetic field.

The present work is an attempt to derive a model of generalized thermoelasticity with three-phase-lag (TPL) heat conduction by using the methodology of memory-dependent-derivative. This model has been applied to a two-dimensional problem. State space approach is used to solve the problem and the inversion of Laplace transform is carried out using a numerical approach. The displacements, temperature and stress components are obtained and presented graphically.

2 DERIVATION OF THREE- PHASE-LAG MODEL OF THERMOELASTICITY WITH MEMORY-DEPENDENT DERIVATIVES

The conventional theory is based on the properties of the classical theory of heat conductivity, specifically on the classical Fourier's law which relates the heat flux $\vec{q}$ to the temperature gradient:

$$\vec{q} = -K\vec{\nabla}T \tag{1}$$

The energy equation in terms of the heat flux $\vec{q}$ is given by Biot [1]

$$\frac{\partial}{\partial t}\left(\rho C_e T + \gamma T_0 e\right) = -\vec{\nabla}\vec{q} \tag{2}$$

Under three-phase-lag model of thermoelasticity proposed by Roy Choudhuri [8], Fourier's law is replaced with $\vec{q}\left(x, t+\tau_q\right) = -K\vec{\nabla}T\left(x, t+\tau_T\right) - K^*\vec{\nabla}\nu\left(x, t+\tau_\nu\right)$, where $\frac{\partial \nu}{\partial t} = T$, K is the thermal conductivity, K^* is the material constant characteristic of the theory, τ_q, τ_T and τ_ν are phase lags of heat flux, temperature gradient and thermal displacement gradient respectively.

From a mathematical view point, the modified Fourier law in the theory of memory-dependent derivative three-phase-lag heat conduction is given by:

$$\left(1+\frac{\tau_q}{1!}D_{\omega_1}+\frac{\tau_q^2}{2!}D_{\omega_1}^2\right)\vec{q} = -K\left(1+\frac{\tau_T}{1!}D_{\omega_2}\right)\vec{\nabla}T - K^*\left(1+\frac{\tau_\nu}{1!}D_{\omega_3}\right)\vec{\nabla}\nu \tag{3}$$

Taking the divergence on both sides of Eq. (3), we get

$$\left(1+\frac{\tau_q}{1!}D_{\omega_1}+\frac{\tau_q^2}{2!}D_{\omega_1}^2\right)\vec{\nabla}\vec{q}=-K\left(1+\frac{\tau_T}{1!}D_{\omega_2}\right)\nabla^2 T-K^*\left(1+\frac{\tau_\nu}{1}D_{\omega_3}\right)\nabla^2\nu \tag{4}$$

Differentiating on both sides of Eq. (4) with respect to time and using Eq. (2), the heat conduction equation takes the form:

$$\left(1+\frac{\tau_q}{1!}D_{\omega_1}+\frac{\tau_q^2}{2!}D_{\omega_1}^2\right)\frac{\partial^2}{\partial t^2}\left(\rho C_e T+\gamma T_0 e\right)=K\frac{\partial}{\partial t}\left(1+\frac{\tau_T}{1!}D_{\omega_2}\right)\nabla^2 T+K^*\left(1+\frac{\tau_\nu}{1!}\right)\nabla^2 T \tag{5}$$

Eq. (5) is the generalized heat conduction equation of three-phase-lag model with memory-dependent derivative. The kernel function form $k\left(t-\xi\right)$ can be chosen freely as:

$$k\left(t-\xi\right)=1-\frac{2n}{\omega}\left(t-\xi\right)+\frac{m^2\left(t-\xi\right)^2}{\omega^2}=\begin{cases}1;m=n=0\\ 1-\left(\frac{t-\xi}{\omega}\right);m=0,n=\frac{1}{2}\\ 1-\left(t-\xi\right);m=0,n=\frac{\omega}{2}\\ \left(1-\frac{t-\xi}{\omega}\right)^2;m=n=1\end{cases}$$

3 LIMITING CASES

i. The dynamic coupled theory of heat conduction law follows as the limit case, since when $k=1$, we get $\frac{\partial f\left(x,y,t\right)}{\partial t}=\lim_{\omega\to 0}\frac{f\left(x,y,t+\omega\right)-f\left(x,y,t\right)}{\omega}$. The heat conduction Eq. (5), in this case $\tau_q=\tau_T=\tau_\nu=0$, transforms to the work of Biot [1].

ii. The heat Eq. (5) in the limiting case $\tau_q=\tau,\ \tau_T=\tau_\nu=0,\ D_\omega f=\frac{\partial f}{\partial t}$, and neglecting $\frac{1}{2}\tau_q^2$, transforms to Lord-Shulman model (LS) in generalized thermoelasticity with one relaxation time τ.

iii. The heat Eq. (5) in the limiting case $D_\omega f=\frac{\partial f}{\partial t}$ transforms to dual-phase-lag model (DPL).

iv. The heat Eq. (5) in the limiting case, $\tau_q=\omega>0,\ \tau_T=\tau_\nu=0$ and neglecting $\frac{1}{2}\tau_q^2$, transforms to the work of Ezzat et al. [15].

v. The case $q_i+\tau D_\omega q_i=-KT_{,i}$ derived by Yu et al. [13], where the relaxation time is not the same as the time delay.

4 FORMULATION OF THE PROBLEM

We consider a two-dimensional problem and assume that all causes producing the wave propagation is independent of the variable z, and that waves are propagated only in the $xy-$plane.

The equation of motion is

$$\left(\lambda+\mu\right)u_{j,ij}+\mu u_{i,jj}-\gamma T_{,i}=\rho\frac{\partial^2 u_i}{\partial t^2} \tag{6}$$

The constitutive equation is

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma\left(T - T_0\right)\delta_{ij} \tag{7}$$

and the strain-displacement relation is

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{8}$$

where λ, μ are Lame constants, ρ is mass density, C_e is specific heat at constant strain, σ_{ij} are components of stress tensor, e_{ij} are components of strain tensor, u_i are components of displacement vector, T is the absolute temperature, $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t is coefficient of linear thermal expansion, T_0 is reference temperature chosen so that $\left|T - T_0\right| << 1$.

All quantities appearing in Eqs. (6)-(8) are independent of the variable z and the displacement vector has components $\left[u(x,y,t), v(x,y,t), 0\right]$.

The heat conduction equation of three-phase-lag model with memory –dependent derivatives can be obtained as:

$$\begin{aligned}&K\left[\frac{\partial}{\partial t}\left(\nabla^2 T\left(x,y,t\right)\right) + \frac{\tau_T}{\omega_2}\int_{t-\omega_2}^{t} k\left(t-\xi\right)\frac{\partial^2}{\partial \xi^2}\left(\nabla^2 T\left(x,y,\xi\right)\right)d\xi\right] + K^*\left[\nabla^2 T\left(x,y,t\right) + \frac{\tau_v}{\omega_3}\int_{t-\omega_3}^{t} k\left(t-\xi\right)\frac{\partial}{\partial \xi}\left(\nabla^2 T\left(x,y,\xi\right)\right)d\xi\right]\\&= F\left(x,y,t\right) + \frac{\tau_q}{\omega_1}\int_{t-\omega_1}^{t} k\left(t-\xi\right)\frac{\partial F\left(x,y,\xi\right)}{\partial \xi}d\xi + \frac{\tau_q^2}{2\omega_1}\int_{t-\omega_1}^{t} k\left(t-\xi\right)\frac{\partial^2 F\left(x,y,\xi\right)}{\partial \xi^2}d\xi\end{aligned} \tag{9}$$

where $F\left(x,y,t\right) = \frac{\partial^2}{\partial t^2}\left(\rho C_e T + \gamma T_0 e\right)$

Thus, the heat conduction equation of three-phase-lag model with memory-dependent derivative is obtained as:

$$\begin{aligned}&K\left[\frac{\partial}{\partial t}\left(T_{,xx} + T_{,yy}\right) + \frac{\tau_T}{\omega_2}\int_{t-\omega_2}^{t} k\left(t-\xi\right)\frac{\partial^4 T}{\partial x^2 \partial \xi^2}d\xi + \frac{\tau_T}{\omega_2}\int_{t-\omega_2}^{t} k\left(t-\xi\right)\frac{\partial^4 T}{\partial y^2 \partial \xi^2}d\xi\right]\\&+K^*\left[\left(T_{,xx} + T_{,yy}\right) + \frac{\tau_v}{\omega_3}\int_{t-\omega_3}^{t} k\left(t-\xi\right)\frac{\partial^3 T}{\partial x^2 \partial \xi}d\xi + \frac{\tau_v}{\omega_3}\int_{t-\omega_3}^{t} k\left(t-\xi\right)\frac{\partial^3 T}{\partial y^2 \partial \xi}d\xi\right]\\&= \frac{\partial^2}{\partial t^2}\left(\rho C_e T + \gamma T_0 e\right) + \frac{\tau_q}{\omega_1}\int_{t-\omega_1}^{t} k\left(t-\xi\right)\left(\rho C_e \frac{\partial^3 T}{\partial \xi^3} + \gamma T_0 \frac{\partial^3 e}{\partial \xi^3}\right)d\xi + \frac{\tau_q^2}{2\omega_1}\int_{t-\omega_1}^{t} k\left(t-\xi\right)\left(\rho C_e \frac{\partial^4 T}{\partial \xi^4} + \gamma T_0 \frac{\partial^4 e}{\partial \xi^4}\right)d\xi\end{aligned} \tag{10}$$

where $e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and ω_1, ω_2 and ω_3 are delay times due to three-phase-lag model. The initial conditions are taken to be homogeneous. We shall use the following non-dimensional variables:

$$\left(x', y'\right) = \beta_0 \eta_0\left(x,y\right), \left(u', v'\right) = \beta_0 \eta_0\left(u,v\right), \left(t', \tau_q', \tau_T', \tau_v', \omega_i'\right) = \beta_0^2 \eta_0\left(t, \tau_q, \tau_T, \tau_i, \omega_i\right), \theta = \frac{\gamma(T - T_0)}{\rho \beta_0^2}, \sigma_{ij}' = \frac{\sigma_{ij}}{\mu},$$

$$\beta_0^2 = \frac{\lambda + 2\mu}{\rho},\ c_2^2 = \frac{\mu}{\rho},\ \beta^2 = \frac{\beta_0^2}{c_2^2},\ \eta_0 = \frac{\rho C_e}{K}.$$

where the dashed quantities denote non-dimensional variables.

In terms of these non-dimensional variables, the equations of motion has the form (dropping primes)

$$\beta^2 u_{,xx} + u_{,yy} + \left(\beta^2 - 1\right) v_{,xy} - \beta^2 \theta_{,x} = \beta^2 \frac{\partial^2 u}{\partial t^2} \tag{11}$$

$$\left(\beta^2 - 1\right) u_{,xy} + \beta^2 v_{,yy} + v_{,xx} - \beta^2 \theta_{,y} = \beta^2 \frac{\partial^2 v}{\partial t^2} \tag{12}$$

and the components of the stress are:

$$\sigma_{xx} = \beta^2 u_{,x} + \left(\beta^2 - 2\right) v_{,y} - \beta^2 \theta \tag{13a}$$

$$\sigma_{xy} = u_{,y} + v_{,x} \tag{13b}$$

$$\sigma_{yy} = (\beta^2 - 2) u_{,x} + \beta^2 v_{,y} - \beta^2 \theta \tag{13c}$$

The Eq. (10) in non-dimensional form is obtained as:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\theta_{,xx} + \theta_{,yy}\right) + \frac{\tau_T}{\omega_2} \int_{t-\omega_2}^{t} k\,(t-\xi) \frac{\partial^4 \theta}{\partial \xi^2 \partial x^2} d\xi + \frac{\tau_T}{\omega_2} \int_{t-\omega_2}^{t} k\,(t-\xi) \frac{\partial^4 \theta}{\partial \xi^2 \partial y^2} d\xi \\ &+ \bar{K}\left[\left(\theta_{,xx} + \theta_{,yy}\right) + \frac{\tau_v}{\omega_3} \int_{t-\omega_3}^{t} k\,(t-\xi) \frac{\partial^3 \theta}{\partial \xi \partial x^2} d\xi + \frac{\tau_v}{\omega_3} \int_{t-\omega_3}^{t} k\,(t-\xi) \frac{\partial^3 \theta}{\partial \xi \partial y^2} d\xi\right] \\ &= \frac{\partial^2}{\partial t^2}\left(\theta + \varepsilon e\right) + \frac{\tau_q}{\omega_1} \int_{t-\omega_1}^{t} k\,(t-\xi)\left(\frac{\partial^3 \theta}{\partial \xi^3} + \varepsilon \frac{\partial^3 e}{\partial \xi^3}\right) d\xi + \frac{\tau_q^2}{2\omega_1} \int_{t-\omega_1}^{t} k\,(t-\xi)\left(\frac{\partial^4 \theta}{\partial \xi^4} + \varepsilon \frac{\partial^4 e}{\partial \xi^4}\right) d\xi \end{aligned} \tag{14}$$

where $\bar{K} = \frac{K^*}{K \beta_0^2 \eta_0}, \varepsilon = \frac{\gamma^2 T_0}{\rho C_e (\lambda + 2\mu)}$.

Taking the Laplace transform defined by the relation

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt \tag{15}$$

to both sides of Eqs. (11)-(12) and using the homogeneous initial conditions, we obtain

$$\beta^2 \bar{u}_{,xx} + \bar{u}_{,yy} + \left(\beta^2 - 1\right) \bar{v}_{,xy} - \beta^2 \bar{\theta}_{,x} = \beta^2 s^2 \bar{u} \tag{16}$$

$$\left(\beta^2 - 1\right) \bar{u}_{,xy} + \beta^2 \bar{v}_{,yy} + \bar{v}_{,xx} - \beta^2 \bar{\theta}_{,y} = \beta^2 s^2 \bar{v} \tag{17}$$

and the components of the stress are:

$$\bar{\sigma}_{xx} = \beta^2 \bar{u}_{,x} + (\beta^2 - 2) \bar{v}_{,y} - \beta^2 \bar{\theta} \tag{18a}$$

$$\bar{\sigma}_{xy} = \bar{u}_{,y} + \bar{v}_{,x} \tag{18b}$$

$$\bar{\sigma}_{yy} = (\beta^2 - 2) \bar{u}_{,x} + \beta^2 \bar{v}_{,y} - \beta^2 \bar{\theta} \tag{18c}$$

and

$$\bar{\theta}_{,xx} + \bar{\theta}_{,yy} = P(\bar{\theta} + \varepsilon\bar{e}) \tag{19}$$

where $P = s\,\chi(s)$, with $\chi(s) = \dfrac{s\left(1+\dfrac{\tau_q}{\omega_1}G_1 + \dfrac{s\,\tau_q^2}{2\omega_1}G_1\right)}{s\left(1+\dfrac{\tau_T}{\omega_2}G_2\right)+\bar{K}\left(1+\dfrac{\tau_v}{\omega_3}G_3\right)}$, $L\left[D_{\omega_i}F(x,y,t)\right] = \dfrac{\bar{F}(x,y,s)}{\omega_i}G_i(s)$ and

$$G_i(s) = G_{\omega_i}(s) = (1-e^{-s\omega_i})(1-\frac{2n}{\omega_i s}+\frac{2m^2}{\omega_i^2 s^2}) - (m^2-2n+\frac{2m^2}{\omega_i s})e^{-s\omega_i},$$

In which m and n are constants, such that

$$G_i(s) = \begin{cases} \left(1-e^{-s\omega_i}\right), m = n = 0 \\ \left[1-\dfrac{1}{\omega_i s}\left(1-e^{-s\omega_i}\right)\right], m = 0, n = \dfrac{1}{2} \\ \left[\left(1-e^{-s\omega_i}\right)-\dfrac{1}{s}\left(1-e^{-s\omega_i}\right)+\omega_i e^{-s\omega_i}\right], m = 0, n = \dfrac{\omega_i}{2} \\ \left[\left(1-\dfrac{2}{s\omega_i}\right)+\dfrac{2}{s^2\omega_i^2}\left(1-e^{-s\omega_i}\right)\right], m = n = 1 \end{cases}$$

We now use the Fourier transform with respect to the space variable x , defined by

$$f^*(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iqx} f(x)dx$$

The inversion formula for this transform is

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iqx} f^*(q)dq$$

The operational property of the Fourier transform is

$$F\left[\frac{\partial f(x,y,t)}{\partial x}\right] = iqF\left[f(x,y,t)\right]$$
$$F\left[\frac{\partial^2 f(x,y,t)}{\partial x^2}\right] = -q^2F\left[f(x,y,t)\right]$$

Taking the Fourier transform of both sides of Eqs. (16)-(19), we get

$$-\beta^2 q^2\bar{u}^* + \bar{u}^*_{,yy} + iq(\beta^2-1)\bar{v}^*_{,y} - iq\beta^2\bar{\theta}^* = \beta^2 s^2\bar{u}^* \tag{20}$$

$$iq(\beta^2-1)\bar{u}^*_{,y} + \beta^2\bar{v}^*_{,yy} - q^2\bar{v}^* - \beta^2\bar{\theta}^*_{,y} = \beta^2 s^2\bar{v}^* \tag{21}$$

$$\bar{\sigma}^*_{xx} = iq\beta^2\bar{u}^* + (\beta^2-2)\bar{v}^*_{,y} - \beta^2\bar{\theta}^* \tag{22a}$$

$$\bar{\sigma}_{xy}^{*} = \bar{u}_{,y}^{*} + iq\bar{v}^{*} \tag{22b}$$

$$\bar{\sigma}_{yy}^{*} = \beta^2\bar{v}_{,y}^{*} + iq(\beta^2 - 2)\bar{u}^{*} - \beta^2\bar{\theta}^{*} \tag{22c}$$

$$-q^2\bar{\theta}^{*} + \bar{\theta}_{,yy}^{*} = P(\bar{\theta}^{*} + \varepsilon\bar{e}^{*}) \tag{23}$$

5 STATE – SPACE FORMULATION

We take the quantities e, θ, De and $D\theta$ as state variables in the physical domain. In the transformed domain, the state space variables are taken as $\bar{e}^{*}, \bar{\theta}^{*}, D\bar{e}^{*}$ and $D\bar{\theta}^{*}$
where

$$\bar{e}^{*} = iq\bar{u}^{*} + D\bar{v}^{*} \text{ and } D \equiv \frac{d}{dy}. \tag{24}$$

Eliminating $\bar{u}^{*}$ and $\bar{v}^{*}$ between Eqs. (20), (21) and (23) with the help of Eq. (24), we obtain the following equation:

$$D^2\bar{e}^{*} = (s^2 + q^2 + P\varepsilon)\bar{e}^{*} + P\bar{\theta}^{*} \tag{25}$$

$$D^2\bar{\theta}^{*} = P\varepsilon\bar{e}^{*} + (q^2 + P)\bar{\theta}^{*} \tag{26}$$

Eqs. (25) and (26) can be written in matrix form as follows:

$$\frac{d\tilde{v}(q,y,s)}{dy} = \tilde{A}(q,s)\tilde{v}(q,y,s) \tag{27}$$

where $\tilde{A}(q,s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ s^2 + q^2 + P\varepsilon & P & 0 & 0 \\ P\varepsilon & q^2 + P & 0 & 0 \end{bmatrix}$, $\tilde{v}(q,y,s) = \begin{bmatrix} \bar{e}^{*}(q,y,s) \\ \bar{\theta}^{*}(q,y,s) \\ D\bar{e}^{*}(q,y,s) \\ D\bar{\theta}^{*}(q,y,s) \end{bmatrix}$.

The formal solution of system (27) can be written in the form

$$\tilde{v}(q,y,s) = \exp(\tilde{A}(q,s)y)\tilde{v}(q,y_0,s) \tag{28}$$

where y_0 denotes any arbitrarily chosen initial value for y. The characteristic equation for the matrix $\tilde{A}(q,s)$ is

$$k^4 - \left(s^2 + 2q^2 + P\varepsilon + P\right)k^2 + q^4 + q^2\left(s^2 + \varepsilon P + P\right) + Ps^2 = 0 \tag{29}$$

The roots of the Eq. (29) satisfy the relations

$$k_1^2 + k_2^2 = s^2 + 2q^2 + P\varepsilon + P \tag{30a}$$

$$k_1^2 k_2^2 = q^4 + q^2\left(s^2 + P\varepsilon + P\right) + Ps^2 \tag{30b}$$

Using Cayley-Hamilton theorem, the finite series representing can be truncated to the following form:

$$\exp(\tilde{A}(q,s)y) = \tilde{L}(q,y,s) = b_0\tilde{I} + b_1\tilde{A} + b_2\tilde{A}^2 + b_3\tilde{A}^3 \tag{31}$$

where $\tilde{I}$ is the unit matrix of order 4 and $b_0,....,b_3$ are some parameters depending on y, q and s. By Cayley-Hamilton theorem, the characteristic roots $\pm k_1$ and $\pm k_2$ of the matrix $\tilde{A}$ must satisfy the equations

$$\begin{aligned}
\exp(k_1 y) &= b_0 + b_1 k_1 + b_2 k_1^2 + b_3 k_1^3 \\
\exp(-k_1 y) &= b_0 - b_1 k_1 + b_2 k_1^2 - b_3 k_1^3 \\
\exp(k_2 y) &= b_0 + b_1 k_2 + b_2 k_2^2 + b_3 k_2^3 \\
\exp(-k_2 y) &= b_0 - b_1 k_2 + b_2 k_2^2 - b_3 k_2^3
\end{aligned}$$

The solution of the above system is given by

$$\begin{aligned}
b_0 &= \frac{1}{k_1^2 - k_2^2}\left[k_1^2\cosh(k_2 y) - k_2^2\cosh(k_1 y)\right] \\
b_1 &= \frac{1}{k_1^2 - k_2^2}\left[\frac{k_1^2}{k_2}\sinh(k_2 y) - \frac{k_2^2}{k_1}\sinh(k_1 y)\right] \\
b_2 &= \frac{1}{k_1^2 - k_2^2}\left[\cosh(k_1 y) - \cosh(k_2 y)\right] \\
b_3 &= \frac{1}{k_1^2 - k_2^2}\left[\frac{1}{k_1}\sinh(k_1 y) - \frac{1}{k_2}\sinh(k_2 y)\right]
\end{aligned} \tag{32}$$

Substituting the expressions (32) into (31) and computing $\tilde{A}^2$ and $\tilde{A}^3$, we obtain after repeated use of Eqs. (30a) and (30b), the elements $l_{ij}\,(i,j = 1,2,3,4)$ of the matrix $\tilde{L}(q,y,s)$ as:

$$\begin{aligned}
l_{11} &= \frac{1}{k_1^2 - k_2^2}\left[\left(k_1^2 - q^2 - P\right)\cosh(k_1 y) - \left(k_2^2 - q^2 - P\right)\cosh(k_2 y)\right] \\
l_{12} &= \frac{P}{k_1^2 - k_2^2}\left[\cosh\left(k_1 y\right) - \cosh\left(k_2 y\right)\right] \\
l_{13} &= \frac{1}{k_1^2 - k_2^2}\left[\frac{(k_1^2 - q^2 - P)}{k_1}\sinh(k_1 y) - \frac{(k_2^2 - q^2 - P)}{k_2}\sinh(k_2 y)\right] \\
l_{14} &= \frac{P}{k_1^2 - k_2^2}\left[\frac{1}{k_1}\sinh(k_1 y) - \frac{1}{k_2}\sinh(k_2 y)\right] \\
l_{21} &= \frac{P\varepsilon}{k_1^2 - k_2^2}\left[\cosh(k_1 y) - \cosh(k_2 y)\right] \\
l_{22} &= \frac{1}{k_1^2 - k_2^2}\left[(k_1^2 - q^2 - P)\cosh(k_2 y) - (k_2^2 - q^2 - P)\cosh(k_1 y)\right] \\
l_{23} &= \frac{P\varepsilon}{k_1^2 - k_2^2}\left[\frac{1}{k_1}\sinh(k_1 y) - \frac{1}{k_2}\sinh(k_2 y)\right] \\
l_{24} &= \frac{1}{k_1^2 - k_2^2}\left[\frac{(k_1^2 - q^2 - P)}{k_2}\sinh(k_2 y) - \frac{(k_2^2 - q^2 - P)}{k_1}\sinh(k_1 y)\right]
\end{aligned} \tag{33}$$

$$\begin{aligned}
l_{31} &= \frac{1}{k_1^2-k_2^2}\left(\left[k_1(k_1^2-q^2-P)\right]\sinh(k_1y)-\left[k_2(k_2^2-q^2-P)\right]\sinh(k_2y)]\right)\\
l_{32} &= \frac{P}{k_1^2-k_2^2}\left[k_1\sinh(k_1y)-k_2\sinh(k_2y)\right]\\
l_{33} &= \frac{1}{k_1^2-k_2^2}\left[(k_1^2-q^2-P)\cosh(k_1y)-(k_2^2-q^2-P)\cosh(k_2y)\right]\\
l_{34} &= \frac{P}{k_1^2-k_2^2}\left[\cosh(k_1y)-\cosh(k_2y)\right]\\
l_{41} &= \frac{P\varepsilon}{k_1^2-k_2^2}\left[k_1\sinh(k_1y)-k_2\sinh(k_2y)\right]\\
l_{42} &= \frac{1}{k_1^2-k_2^2}\left(\left[k_1(q^2+P-k_2^2)\right]\sinh(k_1y)-\left[k_2(q^2+P-k_1^2)\right]\sinh(k_2y)]\right)\\
l_{43} &= \frac{\varepsilon P}{k_1^2-k_2^2}\left[\cosh(k_1y)-\cosh(k_2y)\right]\\
l_{44} &= \frac{1}{k_1^2-k_2^2}\left[\left(k_1^2-q^2-P\right)\cosh(k_2y)-\left(k_2^2-q^2-P\right)\cosh(k_1y)\right]
\end{aligned} \tag{33}$$

We have repeatedly used Eqs. (30a) and (30b) in order to write (33) in the simplest possible form.

6 APPLICATION

We consider the problem of a thick plate of finite height $2h$ and of infinite extent with heating on a part of the surface. The initial state of plate is assumed to be quiescent. Choosing the $y-$axis perpendicular to the surface of the plate with the origin coinciding with the middle plate, the region Ω under consideration becomes

$$\Omega=\{(x,y,z):-\infty<x<\infty,-h<y<h,-\infty<z<\infty\}.$$

The surfaces of the plate are taken to be traction free. The both surfaces of the plate are heated by time varying heat sources. The boundary conditions of the problem in the transformed one, thus

$$\bar{\sigma}_{xy}^{*}=0 \text{ on } y=\pm h \tag{34}$$

$$\bar{\sigma}_{yy}^{*}=0 \text{ on } y=\pm h \tag{35}$$

$$\bar{q}_{y}^{*}+h\bar{\theta}^{*}=\bar{r}^{*}(q,s) \text{ on } y=\pm h \tag{36}$$

where q_y denotes the normal component of the heat flux vector, h is Biot's number and $r(x,t)$ represents the intensity of the applied heat sources.

In order to obtain the heat flux component, we use Laplace transform of Eq. (3) in which we have $\bar{q}_y=-\frac{1}{\chi(s)}\frac{d\bar{\theta}}{dy}$. Taking Fourier transform, this reduces to $\bar{q}_y^{*}=-\frac{1}{\chi}\frac{d\bar{\theta}^{*}}{dy}$. Using the heat conduction, condition (36) reduces to

$$\pm D\bar{\theta}^{*}=\chi(h\bar{\theta}^{*}-\bar{r}^{*}) \text{ on } y=\pm h \tag{37a}$$

The temperature and the dilatation are given from Eq. (28) as:

$$\bar{\theta}^*(q,y,s) = l_{21}e_0 + l_{22}\theta_0 + l_{23}e_0' + l_{24}\theta_0'$$
$$\bar{e}^*(q,y,s) = l_{11}e_0 + l_{12}\theta_0 + l_{13}e_0' + l_{14}\theta_0'$$

where

$$e_0 = \bar{e}^*\left(q,y_0,s\right), \theta_o = \bar{\theta}^*\left(q,y_0,s\right), e_0' = D\bar{e}^*\left(q,y_0,s\right), \theta_0' = D\bar{e}^*\left(q,y_0,s\right)$$

The solution of the problem given by Eq. (28) with y_0 chosen as zero for convenience. Thus two components of the initial state vector $\tilde{v}_0 = \tilde{v}\left(q,0,s\right)$ are known as:

$$D\bar{e}^*\left(q,0,s\right) = 0, D\bar{\theta}^*\left(q,0,s\right) = 0 \tag{37b}$$

The remaining two components $\left(\bar{e}^*\left(q,0,s\right), \bar{\theta}^*\left(q,0,s\right)\right)$ are obtained from the boundary conditions (34)-(36). Applying Eq. (28) with $y = \pm h, y_0 = 0$ and using Eq. (37b) we arrive at

$$\bar{\theta}^*(q,y,s) = l_{21}e_0 + l_{22}\theta_0$$
$$\bar{e}^*(q,y,s) = l_{11}e_0 + l_{12}\theta_0$$

In case of symmetry the above equations reduce to

$$\bar{\theta}^*(q,y,s) = \frac{1}{k_1^2 - k_2^2}\sum_{i=1}^{2}(-1)^{i-1}\left[\varepsilon P e_0 - \left(q^2 + s^2 + P\varepsilon - k_i^2\right)\theta_0\right]\cosh(k_i y) \tag{38}$$

$$\bar{e}^*(q,y,s) = \frac{1}{k_1^2 - k_2^2}\sum_{i=1}^{2}(-1)^{i-1}\left[\left(k_i^2 - q^2 - P\right)e_0 + P\theta_0\right]\cosh(k_i y) \tag{39}$$

Substituting from Eq. (24) into Eq. (20), we obtain

$$(D^2 - k_3^2)\bar{u}^* = iq[\beta^2\bar{\theta}^* - (\beta^2 - 1)\bar{e}^*] \tag{40}$$

where $k_3^2 = \beta^2 s^2 + q^2$.

Substituting from Eqs. (38) and (39) into the right hand side of Eq. (40) and solving the resulting differential equation, we get

$$\bar{u}^* = C\cosh(k_3 y) + \frac{iq}{k_1^2 - k_2^2}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{k_i^2 - k_3^2}\{[\beta^2 P\varepsilon + (\beta^2 - 1)(q^2 + P - k_i^2)]e_0 - [P(\beta^2 - 1) + \beta^2(q^2 + s^2 + P\varepsilon - k_i^2)]\theta_0\}\cosh(k_i y) \tag{41}$$

Substituting (41) into (24) and integrating the resulting equation, we get

$$\bar{v}^* = \frac{-iqC}{k_3}\sinh(k_3 y) + \frac{1}{k_1^2 - k_2^2}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{k_i^2 - k_3^2}\{(k_i^2 - q^2 - P)e_0 + P\theta_0\}\sinh(k_i y) + \frac{q^2}{k_1^2 - k_2^2}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{k_i^2}\{[\beta^2 P\varepsilon + (\beta^2 - 1)(q^2 + P - k_i^2)]e_0 + [P(\beta^2 - 1) + \beta^2(q^2 + s^2 + P\varepsilon) - k_i^2]\theta_0\}\sinh(k_i y) \tag{42}$$

The stress components can be obtained by substituting from the above equations into Eq. (22a)-(22c). The above approach gives the solution of the problem in the transformed domain in terms of the constants C, e_0 and θ_0 which can be obtained from the boundary conditions of the articulate problem under consideration.

$$\begin{aligned}
\bar{\sigma}^*_{xx} &= iq\beta^2 C\cosh(k_3 y) - \frac{q^2\beta^2}{k_1^2-k_2^2}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{(k_i^2-k_3^2)}\{[\beta^2 P\varepsilon+(\beta^2-1)(q^2+P-k_i^2)]e_0-[P(\beta^2-1)+ \\
&\beta^2(q^2+s^2+P\varepsilon-k_i^2)]\theta_0\}\cosh(k_i y) - iq(\beta^2-2)C\cosh(k_3 y)+\frac{(\beta^2-2)}{(k_1^2-k_2^2)}\sum_{i=1}^{2}(-1)^{i-1}\{(k_i^2-q^2-P)e_0 \\
&+P\theta_0\}\cosh(k_i y)+\frac{q^2(\beta^2-2)}{(k_1^2-k_2^2)}\sum_{i=1}^{2}(-1)^{i-1}\{[P\varepsilon\beta^2+(\beta^2-1)(q^2+P-k_i^2)]e_0-[P(\beta^2-1)+\beta^2(q^2+s^2) \\
&+P\varepsilon-k_i^2]\theta_0\}\cosh(k_i y)-\frac{\beta^2}{(k_i^2-k^2)}\sum_{i=1}^{2}(-1)^{i-1}\{P\varepsilon e_0-[q^2+s^2+P\varepsilon-k_i^2]\theta_0\}\cosh(k_i y)
\end{aligned}\tag{43}$$

$$\begin{aligned}
\bar{\sigma}^*_{xy} &= \frac{(q^2+k_3^2)}{k_3}C\sinh(k_3 y)+\frac{iq}{k_1^2-k_2^2}\sum_{i=1}^{2}(-1)^{i-1}\{\frac{k_i}{(k_i^2-k_3^2)}[\beta^2 P\varepsilon+(\beta^2-1)(q^2+p-k_i^2)]+\frac{1}{k_i}[(k_i^2-q^2-P) \\
&+q^2(\beta^2 P\varepsilon+(\beta^2-1)(q^2+P-k_i^2)]\}e_0\sinh(k_i y)-\frac{iq}{(k_1^2-k_2^2)}\sum_{i=1}^{2}(-1)^{i-1}\{\frac{k_i}{(k_i^2-k_3^2)}[P(\beta^2-1)+\beta^2(q^2+s^2+P\varepsilon-k_i^2] \\
&-\frac{1}{k_i}[P-q^2\left\{P(\beta^2-1)+\beta^2(q^2+s^2+P\varepsilon)-k_i^2\right\}]\}\theta_0\sinh(k_i y)
\end{aligned}\tag{44}$$

$$\begin{aligned}
\bar{\sigma}^*_{yy} &= -2iq\beta^2 C\cosh(k_3 y)-\frac{1}{k_1^2-k_2^2}\sum_{i=1}^{2}(-1)^{i-1}\{[\frac{q^2(\beta^2-2)_i}{(k_i^2-k_3^2)}[\beta^2 P\varepsilon+(\beta^2-1)(q^2+P-k_i^2)]-\beta^2(k_i^2-q^2-P)+\beta^2 P\varepsilon \\
&-\beta^2 q^2[\beta^2 P\varepsilon+(\beta^2-1)(q^2+P-k_i^2)]]e_0-[\frac{q^2(\beta^2-2)}{(k_i^2-k_3^2)}[P(\beta^2-1)+\beta^2(q^2+s^2+P\varepsilon-k_i^2]+\beta^2 P+\beta^2[q^2P(\beta^2-1)+(q^2-1) \\
&(q^2+s^2+P\varepsilon-k_i^2)]]\theta_0\}\cosh\left(k_i y\right)
\end{aligned}\tag{45}$$

where

$$e_0=\frac{-\chi\bar{r}^*\left(q,s\right)}{A_1}\left[\begin{array}{l}1+A_2[-2iqA_3\cosh(k_3 h)-\frac{A_5}{k_3}(k_3^2+q^2)\sinh(k_3 h)]\times\{A_1[-2iqA_4\cosh(k_3 h) \\ -\frac{A_6}{k_3}(k_3^2+q^2)\sinh(k_3 h)]+A_2[\frac{A_5}{k_3}(k_3^2+q^2)\sinh(k_3 h)+2iqA_3\cosh(k_3 h)]\}^{-1}\end{array}\right],$$

$$\begin{aligned}
\theta_0 &= \chi\bar{r}^*\left(q,s\right)[-2iqA_3\cosh(k_3 h)-\frac{A_5}{k_3}(k_3^2+q^2)\sinh(k_3 h)]\times\{A_1[-2iqA_4\cosh(k_3 h)-\frac{A_6}{k_3}(k_3^2+q^2) \\
&\sinh(k_3 h)]+A_2[\frac{A_5}{k_3}(k_3^2+q^2)\sinh(k_3 h)+2iqAl3\cosh(k_3 h)]\}^{-1},
\end{aligned}$$

$$\begin{aligned}
C &= \frac{\chi\bar{r}^*\left(q,s\right)k_3}{A_1(k_3^2+q^2)\sinh(k_3 h)}[A_3+[-2iqA_3\cosh(k_3 h)-\frac{A_5}{k_3}(k_3^2+q^2)\sinh(k_3 h)]\left(A_3A_2-A_4A_1\right)\times\{A_1 \\
&[-2iqA_4\cosh(k_3 h)-\frac{A_6}{k_3}(k_3^2+q^2)\sinh(k_3 h)]+A_2[\frac{A_5}{k_3}(k_3^2+q^2)\sinh(k_3 h)-2iqA_3\cosh(k_3 h)]\}^{-1}],
\end{aligned}$$

In which

$$\begin{aligned}
A_1 &= \frac{\varepsilon P}{k_1^2-k_2^2}\sum_{i=1}^{2}(-1)^{i-1}\left[k_i\sinh(k_i h)-h(1+P)\cosh(k_i h)\right], \\
A_2 &= \frac{1}{k_1^2-k_2^2}\sum_{i=1}^{2}(-1)^{i-1}(\varepsilon P+q^2+s^2-k_i^2)[h(1+\varepsilon P)\cosh(k_i h)-k_i\sinh(k_i h)],
\end{aligned}\tag{46}$$

$$A_3=\frac{iq}{k_1^2-k_2^2}\sum_{i=1}^{2}(-1)^{i-1}\left\{\frac{(k_i^2-q^2)}{k_i(k_i^2-k_3^2)}[\beta^2P\varepsilon+(\beta^2-1)(q^2+P-k_i^2)]-\frac{1}{k_i}(q^2+P-k_i^2)\right\}\sinh(k_ih),$$
$$A_4=\frac{iq}{k_1^2-k_2^2}\sum_{i=1}^{2}(-1)^{i-1}\left\{\frac{(k_i^2-q^2)}{k_i^2(k_1^2-k_3^2)}[-P(\beta^2-1)-\beta^2(q^2+P-k_i^2)]+\frac{P}{k_i}\right\}\sinh(k_ih),$$
$$A_5=\frac{2q^2}{k_1^2-k_2^2}\sum_{i=1}^{2}(-1)^{i-1}\left\{\frac{1}{(k_i^2-k_3^2)}[\beta^2P\varepsilon+(\beta^2-1)(q^2+P-k_i^2)]+\beta^2(k_i^2-q^2-P-P\varepsilon)\right\}\sinh(k_ih),$$
$$A_6=\frac{2q^2}{k_1^2-k_2^2}\sum_{i=1}^{2}(-1)^{i-1}\left\{\frac{-1}{(k_i^2-k_3^2)}[P\left(\beta^2-1\right)\left(1+P\right)+\beta^2\left(q^2+P-k_i^2\right)]+\beta^2(P+P\varepsilon+q^2+s^2-k_i^2)\right\}\sinh(k_ih). \quad (46)$$

This completes the solution of the problem in the transformed domain.

7 INVERSION OF THE TRANSFORMS

In order to obtain to obtain the solution of the problem in the physical domain, we have to invert the iterated transforms in (41)-(45). These expressions can be formally expressed as function of y and the parameters of the Fourier and Laplace transforms of q and s, of the form $\bar{f}^*(q,y,s)$.

First, we invert the Fourier transform using the inversion formula. This gives the Laplace transform expression $\bar{f}(q,y,s)$ of the function $f(q,y,t)$ as:

$$\bar{f}(q,y,s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iqx}f(q,y,s)dq=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}(\cos qxf_e+i\sin qxf_0)dq \quad (47)$$

where f_e and f_0 denote the even and odd parts of the function $\bar{f}^*(q,y,s)$ respectively.

We shall now outline the method used to invert the Laplace transforms in the above equations. Let $\bar{f}(s)$ be the Laplace transform of a function $f(t)$. The inversion formula for Laplace transforms can be written as [27]:

$$f(t)=\frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty}e^{ity}\bar{f}(c+iy)dy$$

where c is an arbitrary real number greater than all of the real parts of the singularities of $\bar{f}(s)$.

Expanding the function $h(t)=\exp(-ct)f(t)$ in a Fourier series in the interval [0,2L], we obtain the approximate formula:

$$f(t)\approx f_N(t)=\frac{1}{2}c_0+\sum_{k=1}^{N}c_k,\ \text{for}\ \ 0\le t\le 2 \quad (48)$$

where

$$c_k=\frac{e^{ct}}{L}\mathrm{Re}\left[e^{\frac{i\pi t}{L}}\bar{f}\left(c+\frac{ik\pi}{L}\right)\right] \quad (49)$$

Two methods are used to reduce the total error. First, the 'Korrektur' method is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and therefore to accelerate convergence. The Korrektur method uses the following formula to evaluate the function $f(t)$:

$$f(t)=f_{Nk}(t)=f_N(t)-e^{-2cL}f_{N'}(2L+t) \tag{50}$$

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in Eq. (48). Let N be an odd natural number and let $s_m=\sum_{k=1}^{n}c_k$ be the sequence of partial sums of (48). We define the ε-sequence by $\varepsilon_{0,m}=0,\ \varepsilon_{1,m}=s_m, m=1,2,3,.........$ and $\varepsilon_{n+1},m=\varepsilon_{n-1,m+1}+\dfrac{1}{\left(\varepsilon_{n,m+1}-\varepsilon_{n,m}\right)}, n,m=1,2,3,....$

It can be shown from Honig and Hirdes [27] that the sequence $\varepsilon_{1,1},\varepsilon_{3,1},....,\varepsilon_{N,1},........$converges to $f(t)-\dfrac{c_0}{2}$ faster than the sequence of partial sums.

8 NUMERICAL DISCUSSION

In order to illustrate the above results graphically the source $r(x,t)$ was taken in the following form:

$$r(x,t)=H(x-|a|)H(t)\exp(-dt),$$

where a and d are fixed constants and H denotes Heaviside's unit step function. This represents a localized heat source acting in the region $-a\le x\le a$ starting at $t=0$ with a value of unity and exponentially decaying in time. The double transform of $r(x,t)$ is given by $\bar{r}^*(q,s)=\sqrt{\dfrac{2}{\pi}}\dfrac{\sin(qa)[1+iq\pi\delta(q)]}{q(s+d)}$, where $\delta(q)$ denotes the Dirac delta function.

The data values of copper material are as follows:

$$\lambda=7.76\times10^{10}\,\text{N/m}^2,\ \mu=3.86\times10^{10}\,\text{N/m}^2,\ \alpha_t=1.78\times10^{-5}\,\text{K}^{-1},\ \rho=8954\text{Kg/m}^3, C_e=383.1\text{m}^2/\text{K}$$
$$K=386\text{W/mK}, K^*=124\text{W/mKs}, T_0=293\text{K}.$$

The numerical inversion technique was used to invert the iterated transforms in Eq. (49) giving the temperature and (41)-(45) giving the displacement components u,v and the stress components $\sigma_{xx},\sigma_{xy},\sigma_{yy}$ on the plate $y=1$. For the numerical purpose we take $\tau_q=2\times10^{-7}s,\ \tau_T=1.5\times10^{-7}s,\ \tau_\nu=1\times10^{-8}s,\ k(t-\xi)=1,\ a=2\text{m},\ h=d=1\text{m}$, time delay $\omega_1=0.002s, \omega_2=0.0015s$ and $\omega_3=0.001s$.

In Fig. 1, comparison of u with respect to x for different models of thermoelasticity is presented. It is observed that for $-2\le x\le 0$, u increases with the increase of x and for $0\le x\le 2$, u decreases with the increase of x. The value of the displacement for LS model is maximum and for TPL model is minimum and the displacement for DPL model lies in between the displacements for other two models.

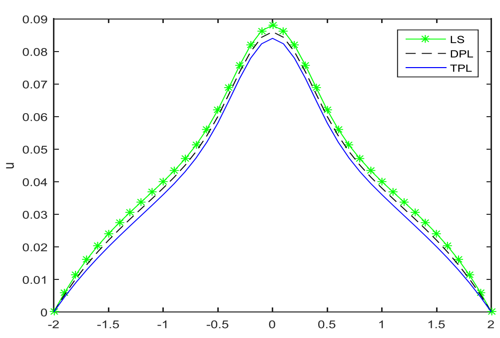

Fig.1
Comparison of u with respect to x for different models.

In Fig. 2, it is observed that u decreases with the increase of time delay. In Fig. 3, it is noticed that u for $k(t-\xi)=1$ is maximum and for $k(t-\xi)=\left(1-\frac{t-\xi}{\omega}\right)^2$ is minimum. The value of displacement for kernel $k(t-\xi)=\left(1-\frac{t-\xi}{\omega}\right)$ lies in between the displacements for other two kernels.

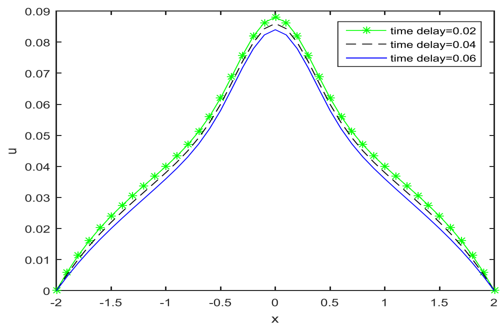

Fig.2
Comparison of u with respect to x for different time delay.

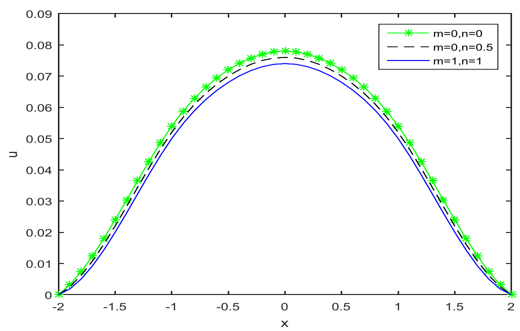

Fig.3
Comparison of u with respect to x for different kernel.

In Fig. 4, comparison of v with respect to x for different models of thermoelasticity is presented. It is observed that for $-2 \le x \le 0$, v increases with the increase of x and for $0 \le x \le 2$, v decreases with the increase of x. The value of the displacement for LS model is maximum and for TPL model is minimum and the displacement for DPL model lies in between the displacements for other two models.

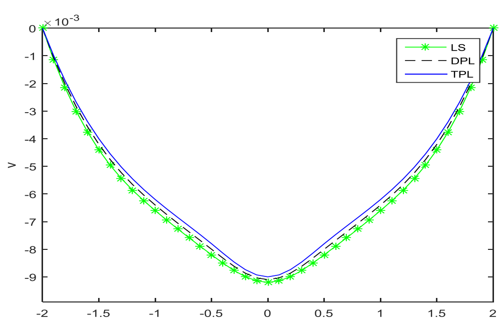

Fig.4
Comparison of v with respect to x for different models.

In Fig. 5, it is observed that v decreases with the increase of time delay. In Fig. 6, it is noticed that v for $k(t-\xi)=1$ is maximum and for $k(t-\xi)=\left(1-\frac{t-\xi}{\omega}\right)^2$ is minimum. The value of displacement for kernel $k(t-\xi)=\left(1-\frac{t-\xi}{\omega}\right)$ lies in between the displacements for other two kernels.

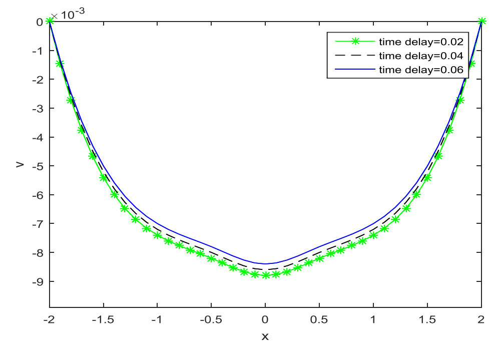

Fig.5
Comparison of v with respect to x for different time delay.

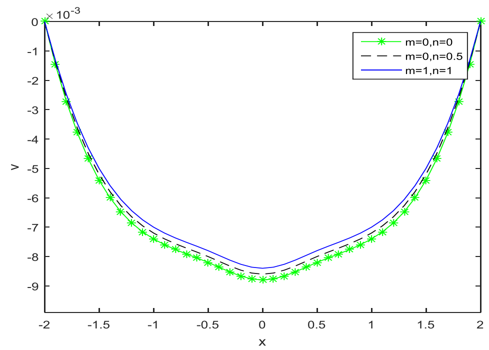

Fig.6
Comparison of v with respect to x for different time delay.

In Fig. 7, comparison of σ_{xx} with respect to x for different models of thermoelasticity is presented. It is observed that for $-2 \le x \le 0$, σ_{xx} increases with the increase of x and for $0 \le x \le 2$, σ_{xx} decreases with the increase of x. The value of stress for LS model is maximum and for TPL model is minimum and the stress for DPL model lies in between the displacements for other two models.

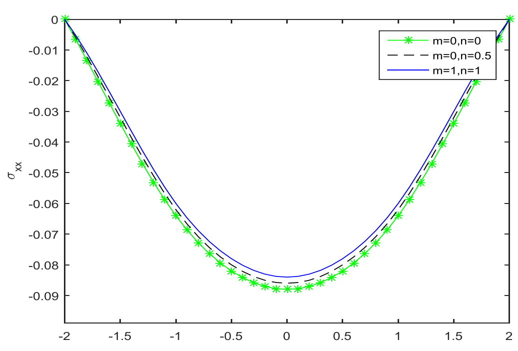

Fig.7
Comparison of σ_{xx} with respect to x for different kernels.

In Fig. 8, it is found that σ_{xx} decreases with the increase of time delay. In Fig. 9, it is noticed that σ_{xx} for $k(t-\xi)=1$ is maximum and for $k(t-\xi)=\left(1-\frac{t-\xi}{\omega}\right)^2$ is minimum. The value of stress for kernel $k(t-\xi)=\left(1-\frac{t-\xi}{\omega}\right)$ lies in between the displacements for other two kernels.

In Fig.10, comparison of temperature with respect to x for different models of thermoelasticity is presented. It is observed that for $-2 \le x \le 0$, temperature increases with the increase of x and for $0 \le x \le 2$, temperature decreases with the increase of x. The value of the temperature for LS model is maximum and for TPL model is minimum and the temperature for DPL model lies in between the temperatures for other two models.

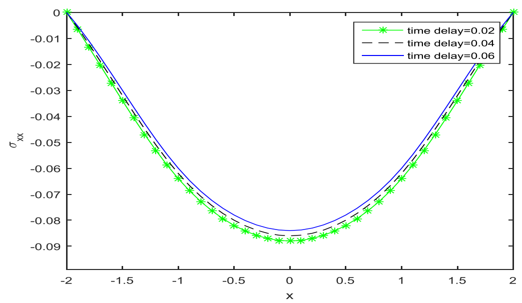

Fig.8
Comparison of σ_{xx} with respect to x for different time delay.

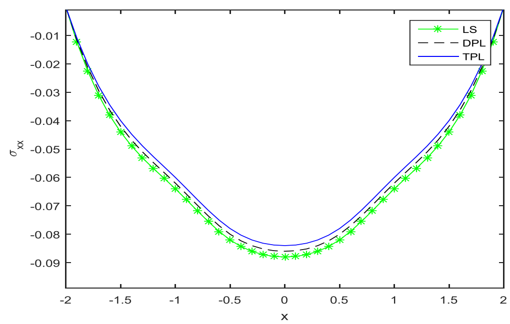

Fig.9
Comparison of σ_{xx} with respect to x for different kernels.

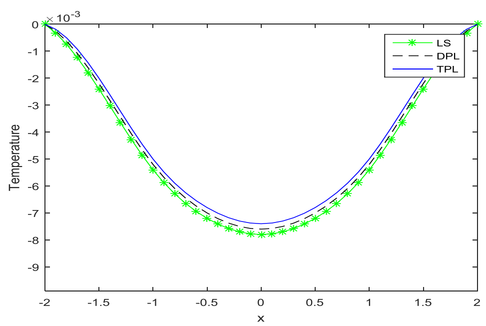

Fig.10
Comparison of temperature with respect to x for different models.

In Fig. 11, it is observed that temperature decreases with the increase of time delay. In Fig. 12, it is noticed that temperature for $k(t-\xi)=1$ is maximum and for $k(t-\xi)=\left(1-\frac{t-\xi}{\omega}\right)^2$ is minimum. The value of temperature for kernel $k(t-\xi)=\left(1-\frac{t-\xi}{\omega}\right)$ lies in between the temperatures for other two kernels.

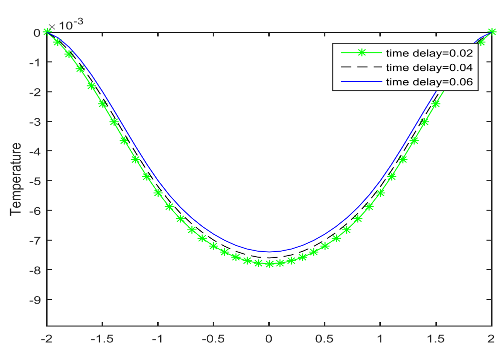

Fig.11
Comparison of temperature with respect to x for different time delay.

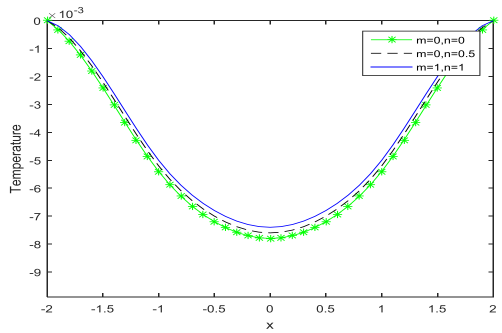

Fig.12
Comparison of temperature with respect to x for different kernels.

9 CONCLUSIONS

The main goal of this work is to introduce a unified generalized model for Fourier law with memory-dependent derivative for the three-phase-lag model of thermoelasticity by using the definition for reflecting the memory effect. This newly proposed model is unique in the form, its physical meaning is apparent and it provides more approaches to capture material's practical response.

From the theoretical and numerical discussion we can conclude the following points:

i. The kernel function and time delay of memory-dependent derivatives can be chosen freely according to the necessity of applications.
ii. The value of considered parameters is maximum for LS model and minimum for TPL model.
iii. The value of the considered parameters increase with the decrease of time delay.
iv. There is significant change in the considered parameters for different kernels.

The method used in the present article is applicable to a wide range of problems in thermodynamics and fluid dynamics when the governing equations are coupled. It is promising that MDD will also make sense in other generalized thermoelasticity theories, diffusion theories and in describing material's viscoelasticity.

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