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Dispersion of SH-Wave in a Heterogeneous Orthotropic Layer Sandwiched Between an Inhomogeneous Semi-Infinite Medium and a Heterogeneous Elastic Half-Space

R.M. Prasad [1,*], S. Kundu [2]

[1]*Department of Mathematics, S. N. Sinha College, Tekari Magadh University, Bodh-Gaya, India*
[2]*Department of Mathematics & Computing, Indian Institute of Technology (Indian School of Mines), Dhanbad, India*

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ABSTRACT
The aim of this paper is to investigates the existence of the dispersion of SH-wave in a heterogeneous orthotropic layer lying over a heterogeneous elastic half-space and underlying an inhomogeneous semi-infinite medium. Hyperbolic variation in upper semi-infinite associated with directional rigidities and density has been considered while linear variation in the intermediate layer associated with initial stress, density, shear moduli and lower half-space associated with rigidity and density has been considered. The dispersion equation of SH-wave has been obtained in a closed form by using variable separation method. The effects of inhomogeneities of the assumed media are illustrated by figures using MATLAB programming. The Earth's composition is heterogeneous that incorporates extremely hard layers. The propagation of SH-wave across crustal layer of the Earth very much depends upon heterogeneity and orthotropic properties. In fact, the observation reveals that the phase velocity of SH-wave is directly proportionate to inhomogeneity parameter, orthotropic parameter and heterogeneity parameter. That means as inhomogeneity parameter and heterogeneity orthotropic parameter increases, the phase velocity of SH-wave increases proportionately. Moreover, the obtained dispersion equation of SH-wave coincides with the classical result of Love wave as initial stress, inhomogeneities, and the upper semi-infinite medium is neglected. This analysis may be helpful to expound the nature of the dispersion of seismic waves in elastic media.

Keywords : Inhomogeneity; Orthotropic medium; Heterogeneous half-space; SH-wave.

1 INTRODUCTION

CRUST and mantle of the Earth are inhomogeneous, it is interesting to study the dispersion of SH-wave in a heterogeneous orthotropic layer sandwiched between an inhomogeneous semi-infinite medium and a heterogeneous elastic half-space. The earth is usually considered in both theory and practical application to be isotropic, or at most, to be composed of isotropic layers. Sufficiently exact studies (Shearer [1]), however, often

[*]Corresponding author.
E-mail address: *ratanmaniprasad@gmail.com* (R.M.Prasad).

suggest the presence of anisotropy. SH-waves, those are shear waves and are polarized in this kind of way, in order that its particles motion and direction of dispersion are contained in a horizontal plane. The analysis of dispersion of surface waves are often to explain the properties and structure of the interior of the Earth. Several problems in seismology can be solved by way of representing the Earth as a layered medium with the certain thickness and mechanical properties. In fact, the study of SH-waves in heterogeneous, homogeneous, and different layered media has been of imperative interest to theoretical seismologists and geologists until recently. Dispersion of seismic waves in layered media, due to its essential utility in Geophysics, Seismology, and Applied Mathematics, has the subject of numerous investigators. On the basis of dispersion of wave, Boit [2] delivered the mechanics of incremental deformation. Afterward, Gubbins's [3] book gave a concept of seismology and plate tectonics and Ewing et al. [4] pointed out the elastic wave in layered media. The dispersion of SH-waves has been studied by means of many authors with assuming different forms of irregularities at the interface. Son and Kang [5] analyzed the propagation of shear wave in a layered poroelastic structure. Kalyani et al. [6] formulated Finite difference modeling of seismic wave propagation in monoclinic media. Generation of SH-type waves due to shearing stress discontinuity in a sandy layer overlying an isotropic and inhomogeneous elastic half-space has been studied by Pal and Mandal [7]. Chattopadhyay et al. [8] examined the propagation of SH-waves in an irregular monoclinic crustal layer. Also, propagation of SH-waves in an irregular non-homogeneous monoclinic crustal layer over a semi-infinite monoclinic medium was examined by Chattopadhyay et al [9]. Ding and Darvinski [10] discussed scattering of SH-waves in multi-layered media with irregular interfaces. Reflection and refraction of SH-waves at a corrugated interface between two laterally and vertically heterogeneous viscoelastic solid half-space was pointed by Kaur et al. [11]. The work has been done on modeling of SH-wave propagation in an irregularly layered medium application to seismic profiles near a dome by Campillo [12]. Chaudhary et al. [13] gave a thought regarding reflection/transmission of plane SH wave through a self-reinforced elastic layer between two half-spaces. SH wave propagation in a cylindrically layered piezoelectric structure with initial stress was detailed by Du et al. [14]. Heterogeneities exist at all levels within the crust of the earth and its resolution is of primary significance in seismology. It is well-known fact that the crust of the earth consists of a huge amount of sand. Weiskopf [15] talked about that because of mutual slippage of the granular particles in soil, the resistance of shear diminished so that the shear modulus of elasticity of a dry sandy body turns into small in comparison to that of a solid material. Abd-Alla et al. [16] analyzed the propagation of Love waves in a non-homogeneous orthotropic magneto-elastic layer under initial stress overlying a semi-infinite medium. Singh and Kumar [17] mulled reflection and refraction of plane waves at an interface between micropolar elastic solid and viscoelastic solid. Propagation of Rayleigh waves in the generalized magneto-thermoelastic orthotropic material under initial stress and gravity field was focused on by Abd-Alla et al. [18]. A study on torsional surface waves in inhomogeneous elastic media was given by Vardoulakis [19]. Tomar and Kaur [20] presented Reflection and transmission of SH-waves at a corrugated interface between anisotropic elastic and viscoelastic solid half-spaces. A model for spherical SH wave propagation in self-reinforced linearly elastic media was given by Chattopadhyay and Michel [21]. Chattopadhyay examined the propagation of SH-waves in an irregular non-homogeneous monoclinic crustal layer over a semi-infinite monoclinic medium. Torsional surface wave dispersion in pre-stressed dry sandy layer over a gravitating anisotropic porous half-space was studied by Prasad and Kundu [22]. In this way, the investigation of dispersion of SH-wave in the pre-stressed heterogeneous orthotropic layer has gained prime enthusiasm to seismologist and geologist because of its conceivable application in the field of construction modeling.

In this paper, an analytical approach has been used to investigate the dispersion of SH-wave a heterogeneous orthotropic layer sandwiched between an inhomogeneous semi-infinite medium and a heterogeneous elastic half-space. Here, the inhomogeneities in the upper semi-infinite medium and lower semi-infinite medium have been taken as $N = N_1 \cosh \alpha z$, $L = L_1 \cosh \alpha z$, $\rho = \rho_1 \cosh \alpha z$ and $\mu = \mu_3 \left(1 + az\right)$, $\rho = \rho_3 \left(1 + bz\right)$ respectively. Also, the heterogeneity in intermediate layer has been taken as $p = p_2 \left(1 + \cos \beta z\right)$, $\rho = \rho_2 \left(1 + \cos \beta z\right)$, $Q_i = a_i \left(1 + \cos \beta z\right)$. Wherein α , *a, b* and β are constants and having dimension that is inverse of the length.

2 FORMULATION OF PROBLEM

We consider an initially stressed heterogeneous orthotropic layer $\left(M_2\right)$ of finite thickness H lying over a heterogeneous elastic semi-infinite $\left(M_3\right)$ and underlying an inhomogeneous anisotropic semi-infinite medium $\left(M_1\right)$. The origin O is considered at the interface of an initially stressed heterogeneous orthotropic layer and a heterogeneous elastic half-space. The *z*-axis is considered as positively in downwards and the *x*-axis is taken along the direction of SH-wave propagation. The existence geometry of the problem is depicted in Fig. 1.

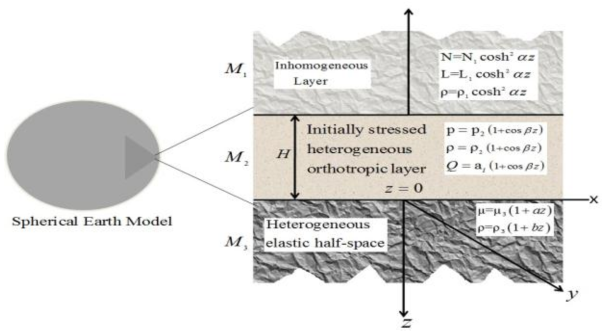

Fig.1
Geometry of problem.

3 DYNAMIC OF HETEROGENEOUS ORTHOTROPIC LAYER

The upper layer is considered under initial compressive stress p along x-axis. Then the dynamical equations of motion in this medium under initial stress in the absence of body forces are (Biot, [2])

$$\left.\begin{aligned}
&\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{12}}{\partial y}+\frac{\partial\sigma_{13}}{\partial z}-p\left(\frac{\partial w_z}{\partial y}-\frac{\partial w_y}{\partial z}\right)=\rho\frac{\partial^2 u_2}{\partial t^2}\\
&\frac{\partial\sigma_{21}}{\partial x}+\frac{\partial\sigma_{22}}{\partial y}+\frac{\partial\sigma_{23}}{\partial z}-p\left(\frac{\partial w_z}{\partial x}\right)=\rho\frac{\partial^2 v_2}{\partial t^2}\\
&\frac{\partial\sigma_{31}}{\partial x}+\frac{\partial\sigma_{32}}{\partial y}+\frac{\partial\sigma_{33}}{\partial z}-p\left(\frac{\partial w_z}{\partial x}\right)=\rho\frac{\partial^2 w_2}{\partial t^2}
\end{aligned}\right\}, \tag{1}$$

where u_2, v_2 and w_2 are the displacement components along *x, y, z* direction. Here σ_{ij} are the incremental stress components and ρ is the density of the material in this medium. The rotational components w_x, w_y and w_z are given by:

$$\left.\begin{aligned}
w_x&=\frac{1}{2}\left(\frac{\partial w_2}{\partial y}-\frac{\partial v_2}{\partial z}\right)\\
w_y&=\frac{1}{2}\left(\frac{\partial u_2}{\partial z}-\frac{\partial w_2}{\partial x}\right)\\
w_z&=\frac{1}{2}\left(\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}\right)
\end{aligned}\right\}. \tag{2}$$

The relations between the strain and incremental stress components are

$$\left.\begin{aligned}
\sigma_{12}&=B_{11}e_{xx}+B_{12}e_{yy}+B_{13}e_{zz}\\
\sigma_{12}&=2Q_3e_{xy}\\
\sigma_{22}&=B_{21}e_{xx}+B_{22}e_{yy}+B_{23}e_{zz}\\
\sigma_{23}&=2Q_1e_{yz}\\
\sigma_{33}&=B_{31}e_{xx}+B_{32}e_{yy}+B_{33}e_{zz}\\
\sigma_{31}&=2Q_2e_{zz}
\end{aligned}\right\}, \tag{3}$$

where B_{ij} and Q_i are the incremental normal elastic coefficients and shear moduli respectively. For the dispersion of SH-wave along *x*-axis, we have

$$u_2 = 0, \quad v_2 = v_2(x, y, z), \quad w_2 = 0. \tag{4}$$

The strain-displacement relations are

$$\left.\begin{aligned} e_{xy} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right) \\ e_{yz} &= \frac{1}{2}\left(\frac{\partial v_2}{\partial z} + \frac{\partial w_2}{\partial y}\right) \\ e_{xy} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\right) \end{aligned}\right\}. \tag{5}$$

Using the relations (2), (3), (4) and (5), the equation of motion given by (1) becomes

$$Q_1 \frac{\partial^2 v_2}{\partial z^2} + \left(Q_3 - \frac{p}{2}\right)\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial Q_3}{\partial x}\frac{\partial v_2}{\partial x} + \frac{\partial Q_1}{\partial z}\frac{\partial v_2}{\partial z} = \rho \frac{\partial^2 v_2}{\partial t^2}. \tag{6}$$

For the wave propagating along x-direction, we assume

$$v_2(x, zt) = V_2(z)e^{ik(x-ct)}, \tag{7}$$

wherein $c = \left(\frac{\omega}{k}\right)$ is the velocity of simple harmonic waves of wavelength $\frac{2\pi}{k}$ traveling forward, ω is the angular frequency and *k* is the wave number. On substituting (7) into (6), we acquire

$$\frac{d^2V_2}{dz^2} + \frac{1}{Q_1}\frac{\partial Q_1}{\partial z}\frac{dV_2}{dz} + \frac{ik}{Q_1}\frac{\partial Q_1}{\partial x}V_2 + \frac{k^2}{Q_1}\left(\rho c^2 + \frac{p}{2} - Q_3\right)V_2 = 0. \tag{8}$$

The variations in pre-stress, density and shear moduli are taken as:

$$\left.\begin{aligned} p &= p_2(1+\cos\beta z) \\ \rho &= \rho_2(1+\cos\beta z) \\ Q_i &= a_i(1+\cos\beta z) \end{aligned}\right\}. \tag{9}$$

The usage of (9), (8) changes to

$$\frac{d^2V_2}{dz^2} + \frac{\beta\sin\beta z}{(1+\cos\beta z)}\frac{dV_2}{dz} + k^2\left\{\frac{c^2}{c_1^2} + \frac{p_2 - 2a_3}{2a_1}\right\}V_2 = 0, \tag{10}$$

where $c_2^2 = \frac{a_1}{\rho_2}$ is the shear wave velocity in the heterogeneous orthotropic layer. Presently, substituting $V_2(z) = \frac{\psi(z)}{\sqrt{(1+\cos\beta z)}}$ in mathematical statement (10), we have

$$\frac{d^2\psi(z)}{dz^2}+m_2^2\psi(z)=0, \tag{11}$$

where $m_2^2=k^2\left[\frac{\beta^2}{4k^2}+\left\{\frac{c^2}{c_2^2}+\frac{p_2-2a_3}{2a_1}\right\}\right]$. Arrangement of mathematical statement (11) is acquired by

$$\psi(z)=S_3\cos m_2z+S_4\sin m_2z, \tag{12}$$

where S_3 and S_4 are arbitrary constants. In this manner, the displacement of heterogeneous orthotropic layer under pre-stress is given by

$$v_2(x,z,t)=\frac{S_3cosm_2z+S_4\sin m_2z}{\sqrt{(1+\cos\beta z)}}e^{ik(x-ct)}. \tag{13}$$

4 DYNAMIC OF INHOMOGENEOUS ANISOTROPIC LAYER

The upper semi-infinite medium (M_1) is considered as inhomogeneous anisotropic medium. Let *v, u*, and *w* be the displacement components in the *x*-, *y*- and *z*-direction, respectively. Beginning from the general equation of motion and the usage of SH-wave conditions, i.e., $u=0$, $w=0$, and $v=v_1(x,z,t)$, the only *y* component equation of motion in the absence of body force may be written as (Biot, [2])

$$N\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial}{\partial z}\left(L\frac{\partial v_1}{\partial z}\right)=\rho\frac{\partial^2 v_1}{\partial t^2}. \tag{14}$$

For a wave propagating alongside the *x*-heading, we assume

$$v_1(x,z,t)=V_1(z)e^{ik(x-ct)}, \tag{15}$$

Utilizing (15), (14) takes the form as:

$$\frac{d^2V_1}{dz^2}+\frac{1}{L}\frac{dL}{dz}\frac{dV_1}{dz}+\frac{k^2}{L}\left(c^2\rho-N\right)V_1=0. \tag{16}$$

Subsequent to putting, $V_1=\frac{V_1^{'}}{\sqrt{L}}$ in (16), we obtain

$$\frac{d^2V_1^{'}}{dz^2}-\frac{1}{2L}\frac{d^2L}{dz^2}V_1^{'}+\frac{1}{4L}\left(\frac{dL}{dz}\right)^2V_1^{'}+\frac{k^2}{L}\left(c^2\rho-N\right)V_1^{'}=0. \tag{17}$$

The variations in rigidities and density are taken as:

$$\left.\begin{aligned}N&=N_1\cosh\alpha z\\ L&=L_1\cosh\alpha z\\ \rho&=\rho_1\cosh\alpha z\end{aligned}\right\}, \tag{18}$$

where α is a constant.

The usage of (18), (17) changes to

$$\frac{d^2V_1^{'}}{dz^2} - m_1^2 V_1^{'} = 0, \tag{19}$$

where,

$$m_1^2 = k^2\left[\frac{\alpha^2}{k^2} + \frac{N_1}{L_1}\left(1 - \frac{c^2}{c_1^2}\right)\right], \tag{20}$$

where $c_1^2 = \dfrac{N_1}{\rho_1}$ is the velocity of the shear wave in the upper semi-infinite medium. The solution to (19) can be assumed as:

$$V_1^{'} = S_1 e^{m_1 z} + S_2 e^{-m_1 z}. \tag{21}$$

As a consequence, the solution for the inhomogeneous, anisotropic upper semi-infinite medium will be of the form $v_1(x,z,t) = V_1(z)e^{ik(x-ct)} = \dfrac{V_1^{'}}{\sqrt{L}} e^{ik(x-ct)}$. As a result, the solution for the non-homogeneous, anisotropic lower semi-infinite medium will be of the form

$$v_1(x,z,t) = \frac{S_1 e^{m_1 z}}{\sqrt{L_1}\cosh\alpha z} e^{ik(x-ct)}. \tag{22}$$

5 DYNAMIC OF HETEROGENEOUS ELASTIC LAYER

The lower semi infinite medium (M_3) is considered as heterogeneous elastic half-space. Let the displacement component of this medium be u_3, v_3, and w_3. Then the equation of motion corresponding to the displacement due to SH-waves can be written as (Biot, [2])

$$\frac{\partial \tau_{21}}{\partial x} + \frac{\partial \tau_{23}}{\partial z} = \frac{\partial^2}{\partial t^2}(\rho v_3) \tag{23}$$

where τ_{ij} are the incremental stress components in the half-space and ρ is the density of the material of the half-space. Now, the in-homogeneity in medium is considered as:

$$\left.\begin{aligned} \mu &= \mu_3(1+az) \\ \rho &= \rho_3(1+bz) \end{aligned}\right\}, \tag{24}$$

where μ_3 and ρ_3 are the values of μ and ρ at z = 0, and a, b are constants. Therefore, for the SH-waves, propagating along the x-direction having the displacement of the particles along the y-direction will produce only the e_{12} and e_{23} strain components and the other strain components will be zero. Hence, the stress-strain relations give

$$\tau_{21} = 2\mu_3(1+az)e_{12}, \quad \tau_{23} = 2\mu_3(1+az)e_{23}. \tag{25}$$

Now, using Eqs. (24) and (25), the equation of motion (23) can be written as:

$$\frac{\partial^2 v_3}{\partial x^2}+\frac{\partial^2 v_3}{\partial z^2}+\left(\frac{a}{1+az}\right)\frac{\partial v_3}{\partial z}=\frac{\rho_3\left(1+bz\right)}{\mu_3\left(1+az\right)}\left(\frac{\partial^2 v_3}{\partial t^2}\right). \tag{26}$$

Let us consider the solution of Eq. (26) be

$$v_3=V_3\left(z\right)e^{ik\left(x-ct\right)}, \tag{27}$$

Then the Eq. (26) can be written as:

$$\frac{dV_3}{dz^2}+\left(\frac{a}{1+az}\right)\frac{dV_3}{dz}+\left[\frac{\rho_3\left(1+bz\right)}{\mu_3\left(1+az\right)}\left(ck\right)^2-k^2\right]V_3=0. \tag{28}$$

Now, let us put $V_3\left(z\right)=\frac{\phi\left(z\right)}{\sqrt{\left(1+az\right)}}$ in (28) we obtain

$$\phi''\left(z\right)+\left[\frac{a^2}{4\left(1+az\right)^2}-k^2\left\{1-\frac{c^2\left(1+bz\right)}{c_3^2\left(1+az\right)}\right\}\right]\phi\left(z\right)=0, \tag{29}$$

where $c_3^2=\frac{\mu_3}{\rho_3}$ is shear wave velocity. Now, we put $\gamma=\sqrt{\left[1-\frac{c^2b}{c_3^2a}\right]}$, $\eta=\frac{2\gamma k\left(1+az\right)}{a}$, $\omega=ck$.

In Eq. (26), we get

$$\frac{d^2\phi}{d\eta^2}+\left[\frac{N}{2\eta}+\frac{1}{4\eta^2}-\frac{1}{4}\right]\phi\left(\eta\right)=0, \tag{30}$$

where $N=\frac{\omega^2\left(a-b\right)}{c_3^2a^2\gamma k}$.

The solution of Eq. (30) satisfying the condition $\lim_{z\to\infty}V_3\left(z\right)\to 0$ that is $\lim_{\eta\to\infty}\left(\eta\right)\to 0$ may be taken as:

$$\phi\left(\eta\right)=S_5W_{\frac{N}{2},0}\left(\eta\right), \tag{31}$$

where $W_{\frac{N}{2},0}\left(\eta\right)$ is the Whittaker's function [23] and S_5 is an arbitrary constant.

Hence, the displacement component $v_3\left(z\right)$ in the inhomogeneous medium is given by

$$v_3\left(z\right)=\frac{S_5W_{\frac{N}{2},0}\left(\eta\right)}{\sqrt{\left(1+az\right)}}e^{ik\left(x-ct\right)}. \tag{32}$$

Expanding Whittaker's function up to linear terms, Eq. (32) reduces to

$$v_3\left(z\right)=S_5e^{\frac{-\gamma kv_3\left(1+az\right)}{a}}\left(\sqrt{\frac{2\gamma k}{a}}\right)\left[1+\left(1-z\right)\frac{\gamma k}{a}\left(1+az\right)\right]e^{ik\left(x-ct\right)}. \tag{33}$$

6 BOUNDARY CONDITIONS

The following boundary conditions must be satisfied:

1) At the interface of the layer and the upper half-space, the displacement component and stress component are continuous, i.e.,

$$v_1 = v_2 \qquad at \quad z = -H. \tag{34}$$

$$L_1 \frac{\partial v_1}{\partial z} = a_1 \frac{\partial v_2}{\partial z} \qquad at \quad z = -H. \tag{35}$$

2) At the interface of the layer and the lower half-space, the displacement component and the stress component are continuous, i.e.,

$$v_2 = v_3 \qquad at \quad z = 0 \tag{36}$$

$$a_1 \frac{\partial v_2}{\partial z} = \mu_3 \frac{\partial v_3}{\partial z} \qquad at \quad z = 0. \tag{37}$$

Using Eqs. (22), (13) and (33) in the above four boundary conditions, the following equations have been obtained

$$S_1 e^{m_1 H} \sqrt{1-\cos\beta H} + S_3 \cos m_2 H \cosh\alpha H \sqrt{L_1} + S_4 \sin m_2 H \cosh\alpha H \sqrt{L_1} = 0. \tag{38}$$

$$\begin{aligned} & S_1 2\sqrt{L_1}\left(1+\cos\beta H\right)^{3/2}\left\{m_1 e^{-m_1 H}\cosh\alpha H - \alpha e^{-m_1 H}\sinh\alpha H\right\} \\ & -S_3\left\{2a_1\cosh^2\alpha H m_2 \sin m_2 H\left(1+\cos\beta H\right) - \cos m_2 H\,\beta\sin\beta H a_1\cosh^2\alpha H m_2\right\} \\ & -S_4\left\{2a_1\cosh^2\alpha H m_2 \cos m_2 H\left(1+\cos\beta H\right) + \sin m_2 H\,\beta\sin\beta H a_1\cosh^2\alpha H m_2\right\} = 0. \end{aligned} \tag{39}$$

$$S_3 - S_5 q_1 = 0. \tag{40}$$

where, $q_1 = \left[e^{\frac{-\gamma k v_3 (1+\alpha z)}{a}} \left(\sqrt{\frac{2\gamma k}{a}}\right)\left[1+\left(1-z\right)\frac{\gamma k}{a}\left(1+\alpha z\right)\right] e^{ik(x-ct)}\right]_{z=0}$.

$$S_4 a_1 m_2 - S_5 \mu_3 q_2 = 0. \tag{41}$$

where, $q_2 = \frac{\partial}{\partial z}\left[e^{\frac{-\gamma k v_3 (1+\alpha z)}{a}} \left(\sqrt{\frac{2\gamma k}{a}}\right)\left[1+\left(1-z\right)\frac{\gamma k}{a}\left(1+\alpha z\right)\right] e^{ik(x-ct)}\right]_{z=0}$.

Eliminating the arbitrary constants S_1, S_3, S_4 and S_5 from the above four equations we obtain:

$$\tan\left[\sqrt{\frac{\beta^2}{4k^2} + \left\{\frac{c^2}{c_2^2} + \frac{p_2 - 2a_3}{2a_1}\right\}}\,\right] kH = \frac{E_1 - E_2 - E_3 + E_4}{E_5 + E_6 + E_7 - E_8} \tag{42}$$

where

$$\left.\begin{aligned}
E_1 &= a_1^2\left[\sqrt{\frac{\beta^2}{4k^2}+\left\{\frac{c^2}{c_2^2}+\frac{p_2-2a_3}{2a_1}\right\}}\right]\left(\frac{\beta}{k}\right)\sin\frac{\beta}{k}kH\,\cosh\frac{\alpha}{k}kH\\
E_2 &= 2\mu_3\left(\frac{q_2}{q_1k}\right)a_1\left[\sqrt{\frac{\beta^2}{4k^2}+\left\{\frac{c^2}{c_2^2}+\frac{p_2-2a_3}{2a_1}\right\}}\right]\cosh\frac{\alpha}{k}kH\left(1+\cos\frac{\beta}{k}kH\right)\\
E_3 &= 2L_1a_1\left[\sqrt{\frac{\alpha^2}{k^2}+\frac{N_1}{L_1}\left(1-\frac{c^2}{c_1^2}\right)}\right]\left[\sqrt{\frac{\beta^2}{4k^2}+\left\{\frac{c^2}{c_2^2}+\frac{p_2-2a_3}{2a_1}\right\}}\right]\cosh\frac{\alpha}{k}kH\left(1+\cos\frac{\beta}{k}kH\right)\\
E_4 &= 2L_1a_1\left[\sqrt{\frac{\beta^2}{4k^2}+\left\{\frac{c^2}{c_2^2}+\frac{p_2-2a_3}{2a_1}\right\}}\right]\left(\frac{\alpha}{k}\right)\sinh\frac{\alpha}{k}kH\\
E_5 &= 2a_1^2\left[\frac{\beta^2}{4k^2}+\left\{\frac{c^2}{c_2^2}+\frac{p_2-2a_3}{2a_1}\right\}\right]\cosh\frac{\alpha}{k}kH\left(1+\cos\frac{\beta}{k}kH\right)\\
E_6 &= \mu_3\left(\frac{q_2}{q_1k}\right)a_1\left(\frac{\beta}{k}\right)\sin\frac{\beta}{k}kH\,\cosh\frac{\alpha}{k}kH\\
E_7 &= 2L_1\left[\sqrt{\frac{\alpha^2}{k^2}+\frac{N_1}{L_1}\left(1-\frac{c^2}{c_1^2}\right)}\right]\mu_3\left(\frac{q_2}{q_1k}\right)\cosh\frac{\alpha}{k}kH\left(1+\cos\frac{\beta}{k}kH\right)\\
E_8 &= 2L_1\left(\frac{\alpha}{k}\right)\mu_3\left(\frac{q_2}{q_1k}\right)\sinh\frac{\alpha}{k}kH
\end{aligned}\right\}. \tag{43}$$

Therefore, Eq.(42) is required dispersion equation of SH-wave in the heterogeneous orthotropic layer sandwiched between an inhomogeneous anisotropic semi-infinite medium and a heterogeneous elastic half-space.

7 PARTICULAR CASES

Case (i): When the intermediary heterogeneous orthotropic layer is homogeneous that is $\beta=0$ then the dispersion Eq. (42) reduces to

$$\tan M_2kH=\frac{-4\mu_3\left(\frac{q_2}{q_1k}\right)a_1M_2\cosh\frac{\alpha}{k}kH-4L_1a_1M_1M_2+2L_1a_1M_2\left(\frac{\alpha}{k}\right)\sinh\frac{\alpha}{k}kH}{4a_1^2\left[\frac{c^2}{c_2^2}+\frac{p_2-2a_3}{2a_1}\right]\cosh\frac{\alpha}{k}kH+4L_1M_1\mu_3\left(\frac{q_2}{q_1k}\right)\cosh\frac{\alpha}{k}kH-2L_1\left(\frac{\alpha}{k}\right)\mu_3\left(\frac{q_2}{q_1k}\right)\sinh\frac{\alpha}{k}kH}$$

where, $M_1=\left[\sqrt{\frac{\alpha^2}{k^2}+\frac{N_1}{L_1}\left(1-\frac{c^2}{c_1^2}\right)}\right]$, $M_2=\left[\sqrt{\frac{c^2}{c_2^2}+\frac{p_2-2a_3}{2a_1}}\right]$

Case (ii): When the upper semi infinite medium and intermediate layer is homogeneous that is ($\alpha=0$, $\beta=0$), then the (42) becomes

$$\tan M_2kH=\frac{-\mu_3\left(\frac{q_2}{q_1k}\right)a_1M_2\cosh\frac{\alpha}{k}kH-L_1a_1\sqrt{\frac{N_1}{L_1}\left(1-\frac{c^2}{c_1^2}\right)}M_2}{a_1^2\left[\frac{c^2}{c_2^2}+\frac{p_2-2a_3}{2a_1}\right]\cosh\frac{\alpha}{k}kH+L_1\sqrt{\frac{N_1}{L_1}\left(1-\frac{c^2}{c_1^2}\right)}\mu_3\left(\frac{q_2}{q_1k}\right)\cosh\frac{\alpha}{k}kH}$$

where, $M_2 = \left[\sqrt{\frac{c^2}{c_2^2} + \frac{p_2 - 2a_3}{2a_1}} \right]$

Case (iii): When the heterogeneous orthotropic medium is initially stress free, homogeneous, isotropic with rigidity μ_1 laying over a homogeneous half-space with rigidity μ_3 and underlying a homogeneous semi-infinite medium with rigidity μ_1 that is ($\beta = 0$, $\alpha = 0$, $a = b = 0$, $p_2 = 0$, $a_1 = a_3 = \mu_2$, $L_1 = N_1 = \mu_1$) the (42) reduces to

$$\tan\left[kH\sqrt{\frac{c^2}{c_2^2} - 1} \right] = \frac{\mu_3\mu_2\sqrt{\left[1 - \frac{c^2}{c_3^2}\right]}\sqrt{\frac{c^2}{c_2^2} - 1} - \mu_1\mu_2\sqrt{1 - \frac{c^2}{c_1^2}}\sqrt{\frac{c^2}{c_2^2} - 1}}{\mu_2^2\left(\frac{c^2}{c_2^2} - 1\right) - \mu_1\mu_3\sqrt{1 - \frac{c^2}{c_1^2}}\sqrt{\left[1 - \frac{c^2}{c_3^2}\right]}}$$

Case (iv): In the absence of the upper half-space, case (iii): reduces to

$$\tan\left[kH\sqrt{\frac{c^2}{c_2^2} - 1} \right] = \frac{\mu_3}{\mu_2}\frac{\sqrt{\left[1 - \frac{c^2}{c_3^2}\right]}}{\sqrt{\left[\frac{c^2}{c_2^2} - 1\right]}}$$

which is the classical dispersion relation of Love wave in a homogeneous layer over an homogeneous half-space.

8 NUMERICAL CALCULATIONS AND DISCUSSION

The phase velocity c/c_2 of the SH-wave in a heterogeneous orthotropic layer lying over a heterogeneous elastic half-space and underlying an inhomogeneous semi-infinite medium has been calculated numerically from the dispersion Eq.(42). The dispersion relation relates the phase velocity to the wave number and medium characteristics. For this, we take some numerical data from Gubbins [3] for the upper inhomogeneous semi infinite medium, intermediate heterogeneous orthotropic medium, and the lower heterogeneous elastic half-space.

For upper inhomogeneous semi infinite medium,

$$N_1 = 6.34\times10^{10}\,N/m^2$$
$$L_1 = 3.99\times10^{10}\,N/m^2$$
$$\rho_1 = 3364\,Kg/m^3$$

For the intermediate heterogeneous orthotropic medium,

$$a_1 = 5.82\times10^{10}\,N/m^2$$
$$a_3 = 3.99\times10^{10}\,N/m^2$$
$$\rho_2 = 4500\,Kg/m^3$$

For the lower heterogeneous elastic half-space,

$$\mu_3 = 6.34\times10^{10}\,N/m^2$$
$$\rho_3 = 3364\,Kg/m^3$$

Fig. 2-7 show the variation of dimensionless phase velocity c/c_2 against dimensionless wave number kH by using aforementioned numerical data.

Fig.2 presents the variation of dimensionless phase velocity c/c_2 of SH-wave against non-dimension wave number kH for different value of inhomogeneity parameter α/k associated with directional rigidities and density of the upper inhomogeneous semi-infinite medium. The dispersion curves 1, curve 2, curve 3 and curve 4 have been plotted by taking the value of α/k as 0.2, 0.4, 0.6 and 0.8 respectively. The values of β/k , a/k , b/k and p_2/k have been taken as 0.2, 0.2, 0.3 and 0.3 respectively for all the dispersion curves. It is inferred that phase velocity of SH-wave increases for increases of inhomogeneity factor α/k .

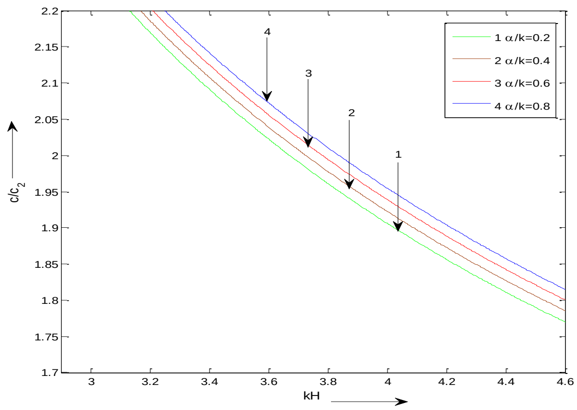

Fig.2
Variation of dimensionless phase velocity c/c_2 with non-dimensional wave number kH for different values of inhomogeneity parameter α/k in inhomogeneous medium.

In Fig.3 study has been made to get the effect of heterogeneity parameter β/k associated with the shear moduli, density and initial stress of the heterogeneous orthotropic medium. The values of α/k , a/k , b/k and p_2/k have been taken as 0.2, 0.1, 0.6 and 0.4 respectively for all the dispersion curves. Here the values of β/k have considered as 0.1, 0.2, 0.3 and 0.4 for dispersion curves 1, curve 2, curve 3 and curve 4 respectively. From the dispersion curves, it has been investigated that as the heterogeneity of the heterogeneous orthotropic medium increases, the phase velocity of SH-wave also increases.

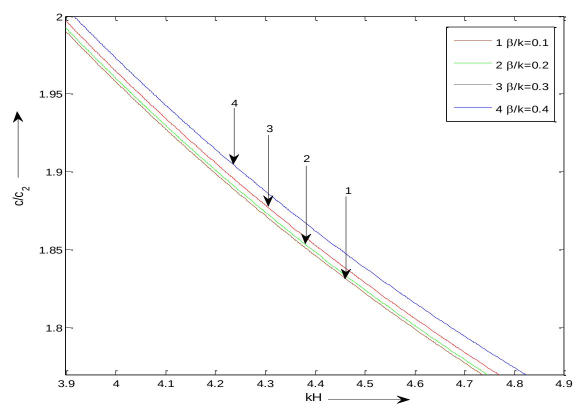

Fig.3
Comparison of dimensionless phase velocity c/c_2 against dimensionless wave number kH for various values of inhomogeneity parameter β/k in heterogeneous orthotropic layer.

Fig. 4 depicts the variation of non-dimensional phase velocity c/c_2 with dimensionless wave number kH for numerous values of heterogeneity factor a/k associated with rigidity of heterogeneous elastic half-space. The dispersion curves 1, curve 2, curve 3 and curve 4 have been plotted by taking the value of a/k as 0.20, 0.22, 0.24 and 0.26 respectively. The values of α/k , β/k , b/k and p_2/k have been considered as 0.2, 0.6, 0.6, and 0.5 respectively for all the dispersion curves. It has been observed that as increases in non-dimensional parameter a/k in the lower heterogeneous elastic half-space increases the phase velocity of SH-wave.

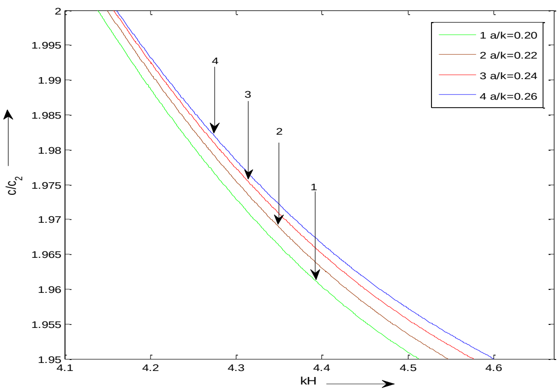

Fig.4
Variation of dimensionless phase velocity c/c_2 as a function of dimensionless wave number kH for different values of heterogeneity parameter a/k in heterogeneous elastic lower half-space.

Fig. 5 manifests the effect of heterogeneity factor b/k associated with the density of heterogeneous elastic half-space. The values of α/k , β/k , a/k and p_2/k have been taken as 0.2, 0.6, 0.1 and 0.5 respectively for all the dispersion curves. Here the values of b/k have considered as 0.6, 0.7, 0.8 and 0.9 for dispersion curves 1, curve 2, curve 3 and curve 4 respectively. The curves confirm that as heterogeneity parameter b/k increases the phase velocity c/c_2 decreases for a fixed value of kH. Thereby reflects the fact that phase velocity of SH-wave is inversely proportional to the heterogeneous elastic half-space.

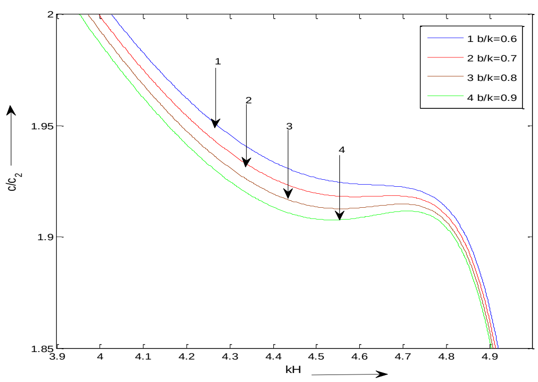

Fig.5
Comparison of dimensionless phase velocity c/c_2 with dimensionless wave number kH for various values of heterogeneity parameter b/k in heterogeneous elastic lower half-space.

In Fig.6, an analysis has been made to get the impact of inhomogeneity parameter α/k which associated with directional rigidities, density of the inhomogeneous medium and β/k which correspond to shear moduli, density and initial stress of the heterogeneous orthotropic medium. The values of a/k , b/k , and p_2/k have been taken as 0.1, 0.15, and 0.2 respectively for all the dispersion curves. For the solid curves 1,2,3 and 4, the values of α/k have been consider as 0.25, 0.45, 0.65, 0.85 respectively and the value of β/k have been taken as 0.2. While for the dotted curves, the value of α/k has been taken as 0.01 and the values of β/k have been taken as 0.25, 0.45, 0.65, 0.85 respectively. The result reveals an increase in the phase velocity c/c_2 of SH-wave with an increase in inhomogeneity parameter α/k and heterogeneity parameter β/k . Finally, it leads to the fact that the phase velocity of SH-wave increases proportionately.

In Fig.7 study has been made to get the effect of heterogeneity factor a/k associated with rigidity of heterogeneous elastic half-space and the effect of heterogeneity factor b/k associated with the density of heterogeneous elastic half-space. The values of α/k , β/k , and p_2/k have been taken as 0.8, 0.8, and 0.2 respectively for all the dispersion curves. For the solid curves 1, 2, 3 and 4, the values of a/k have been consider as 0.05, 0.10, 0.15, 0.20 respectively and the value of b/k has been taken as 0.2. While for the dotted curves, the value of a/k has been taken as 0.001and the values of b/k have been taken as 0.05, 0.10, 0.15, 0.20 respectively. It has been observed that the increment in heterogeneity parameter a/k leads to an increment in the phase velocity of SH-wave. And, the phase velocity c/c_2 of SH-wave diminishes as the heterogeneity parameter b/k increases.

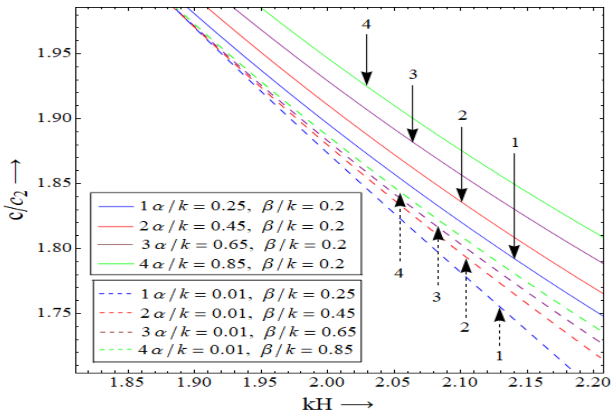

Fig.6
Comparison of dimensionless phase velocity c/c_2 against dimensionless wave number kH for various values of inhomogeneity parameter α/k in inhomogeneous medium and β/k in heterogeneous orthotropic layer.

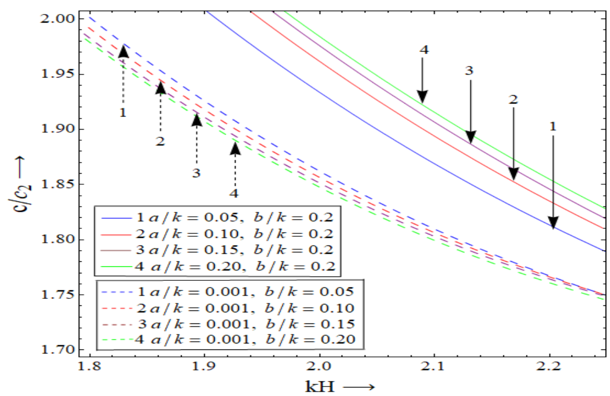

Fig.7
Comparison of dimensionless phase velocity c/c_2 against dimensionless wave number kH for various values of heterogeneity parameter a/k and b/k in heterogeneous elastic lower half-space.

9 CONCLUSIONS

In this paper, we studied about the dispersion of SH-wave in a heterogeneous orthotropic layer sandwiched between an inhomogeneous semi-infinite medium and a heterogeneous elastic half-space. The solutions for displacement in the inhomogeneous layer, heterogeneous orthotropic layer, and heterogeneous elastic half-space have been derived separately in closed form. The analysis of frequency equation bears out the notable effect of inhomogeneity of the upper and Lower semi-infinite medium, and pre-stress, heterogeneity of intermediate layer on the dispersion of SH-wave. Moreover, the numerical results for the dispersion relation are performed and the effects of inhomogeneities are studied graphically with the help of MATLAB software. From the aforementioned figures, the results can be summarized as follow:

- Dimensionless phase velocity c/c_2 of SH-wave decreases with increases of non-dimensional wave number kH in all figures under assumed condition.
- The effect of dimensionless inhomogeneity parameter α/k associated with directional rigidities of the upper semi-infinite medium is very prominent in the dispersion of SH waves. The dispersion curves confirm that as inhomogeneity parameter increases the phase velocity increases for the fixed value of kH.
- The dispersion curves confirm that as heterogeneity parameter β/k increases the phase velocity c/c_2 of SH-wave increases for a fixed value of kH. This leads to the fact that shear moduli, density, and pre-stress of the heterogeneous orthotropic medium are directly proportional to the phase velocity of the SH-wave.
- Dimensionless phase velocity c/c_2 of SH-wave increments as the heterogeneity parameter a/k of heterogeneous elastic half-space increments. Thereby reflects the fact that rigidity of the heterogeneous elastic half-space is directly proportional to the phase velocity of the SH-wave.
- The phase velocity of SH wave diminishes as the heterogeneity parameter b/k increases. This leads to the fact that density of the heterogeneous elastic half-space is inversely proportional to the phase velocity of the SH-wave.

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