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Application of the Homotopy Perturbation Method to Solve Nonlinear Equations Arising in Oscillatory Systems

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Abstract: In this research, the application of the homotopy perturbation method to solve nonlinear Equations arising in oscillatory systems is investigated. In this way, the performance of the Homotopy Perturbation Method (HPM) is compared with the numerical methods to find the solutions of nonlinear Equations in the vibration field. To this end, the Duffing–Holmes oscillatory model with nonlinear terms is regarded and solved by the HPM method. In order to validate the obtained solution by the HPM, the answers are compared with those of numerical methods. The results clearly depict that the homotopy perturbation method, without needing to small parameters, could present the answers near to the exact solutions and also to the numerical one.

Keywords: Duffing-Holmes Model, Homotopy Perturbation Method, Nonlinear Equations, Oscillatory Systems

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Research paper

1 INTRODUCTION

There are many engineering problems, such as those appearing in mechanical vibrations which are nonlinear, therefore most of them are solved by using numerical approaches, while the others are solved via the analytical methods. In the numerical methods, stability and convergence should be considered to avoid divergent or inappropriate results. On the other hand, many different analytical methods have been recently introduced to eliminate these issues. In this way, Kumar [1] presented a comprehensive literature review on the application of the Rayleigh-Ritz method to analyze vibration, static, and buckling characteristics of beams, shells, and plates by employing different theories. The governing Equations were obtained using the Rayleigh-Ritz method to study the effects of constituent volume fractions, slenderness ratios, and the beam theories on the natural frequencies by Pradhan and Chakraverty [2]. Yserentant [3] displayed some new error estimates for the eigenvalues and Eigen functions obtained by the Rayleigh-Ritz method, the common variational method to solve Eigen problems. Natural frequencies of rectangular plates were obtained by employing a set of beam characteristic orthogonal polynomials in the Rayleigh-Ritz method by Bhat [4].

A new implementation was developed by Lu et al. [5] based on a modified variational principle in which the Lagrange multipliers were replaced at the outset by their physical meaning so that the discrete Equations were banded. Thomas et al. [6] examined the conservation law structure of the continuous Galerkin method for solving the scalar, advection-diffusion Equation as a model problem. Demkowicz and Gopalakrishnan [7] discussed the principles and methodology of the discontinuous Petrov Galerkin method with optimal test functions and provided a literature review on the subject. Thomas et al. [8] developed a computational formulation that combines the advantages of discontinuous Galerkin methods with the data structure of their continuous Galerkin counterparts. Wazwaz [10] proposed a powerful modification of the Adomian decomposition method, introduced in the 1970s to the 1990s by George Adomian [9], to accelerate the rapid convergence of the series solution. Moreover, a simple method to determine the rate of convergence of the Adomian decomposition method was introduced by Hosseini et al. [11]. In this way, the application of the Adomian method for solving fuzzy systems of linear Equations was considered by Allahviranloo [12].

The solution of an initial value problem of the parabolic type was discussed by Tatari et al. [13] to propose an alternative method of solution, one not based on finite difference or finite element or spectral methods. The Kantorovich theorem, or Newton–Kantorovich theorem, as a mathematical statement on the semi-local convergence of Newton's method, was first stated by Leonid Kantorovich to form the Banach fixed-point theorem [14]. A Taylor–Galerkin method was described to derive finite element schemes for the scalar convection Equation in one or more space dimensions based on the forward-time Taylor series expansions by Donea [15]. Several explicit Taylor-Galerkin-based time integration schemes were proposed for the solution of both linear and non-linear convection problems with the divergence-free velocity by Timmermans et al. [16]. Shafiee Sarvestany and Mahmoodabadi [17] investigated a novel combination of the firefly optimization algorithm and artificial bee colony for mathematical test functions and real-world problems. Mahmoodabadi, and Nemati [18] presented an optimum numerical method for analysis of nonlinear conductive heat transfer problems. Mahmoodabadi and Sadeghi Googhari [19] studied numerical solutions of the time-dependent Schrodinger Equation by the combination of the finite difference method and particle swarm optimization.

In this research work, the basic idea of the HPM is introduced, its application on the oscillatory Equations is studied, and a comparison with the exact solution is also made.

2 HOMOTOPY PERTURBATION METHOD (HPM)

To illustrate the basic ideas of the HPM, the following nonlinear differential Equation is considered.

$$A(y) - f(\rho) = 0, \qquad \rho \in \Omega, \tag{1}$$

With boundary conditions:

$$B(y, \partial y/\partial n) = 0, \qquad \rho \in \Gamma, \tag{2}$$

Where A denotes a general differential operator, B represents a boundary operator, $f(\rho)$ signifies a known analytical function, and Γ is the boundary of domain Ω. Operator A can be generally divided into two linear (l) and nonlinear (N) parts. Therefore, “Eq. (1)” can be rewritten as follows:

$$L(y) + N(y) - f(\rho) = 0. \tag{3}$$

Hence, Homotopy function is constructed as follows:

$$H(\rho, v) = L(v) - L(y_0) + pL(y_0) + p[N(v) - f(\rho)] = 0, \tag{4}$$

Where ρ denoted the homotopy parameter.

According to the homotopy perturbation idea, the approximate solution of “Eq. (4)” can be expressed as a series of the powers of ρ, i.e:

$$y = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots, \quad (5)$$

3 APPLICATIONS OF THE HPM FOR DUFFING HOLMES OSCILLATOR

Consider the Duffing-Holmes oscillator Equation with the following state-space configuration:

$$\dot{y}_1(t) = y_2(t). \quad (6)$$

$$\dot{y}_2(t) = y_1(t) - 0.25y_2(t) - y_1^3 \quad (7)$$

By regarding:

$$y = y_1$$

And

$$\dot{y} = y_2$$

Then,

$$\ddot{y} + 0.25\dot{y} - y + y^3 = 0. \quad (8)$$

Therefore, the homotopy function is defined as follows:

$$H\,(p\,,v) = \ddot{v} + 0.25\,\dot{v} - v - L(y_0) + pL(y_0) + pv^3 = 0. \quad (9)$$

By employing $v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots$, and boundary conditions $y(0) = 0.2$ and $\dot{y}(0) = 0.2$, the following relations are obtained:

$$H(p\,,v) = (p^0\ddot{v}_0 + p^1\ddot{v}_1 + p^2\ddot{v}_2 + \cdots) + 0.25(p^0\dot{v}_0 + p^1\dot{v}_1 + p^2\dot{v}_2 + \cdots) - (p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots) + p^1[(p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots)^3] = 0;$$
$$v_0(0) = 0.2, \qquad \dot{v}_0(0) = 0.2, \quad (10)$$

By arranging the terms of the above Equation with respect to the power of parameter p, the following differential Equations are reached:

$$p^0:\ \ddot{v}_0 + 0.25\dot{v}_0 - v_0 = 0.$$
$$v_0(0) = 0.2, \qquad \dot{v}_0(0) = 0.2\,. \quad (11)$$

$$p^1:\ \ddot{v}_1 + 0.25\dot{v}_1 - v_1 + (v_0)^3 = 0.$$
$$v_1(0) = 0, \qquad \dot{v}_1\,(0) = 0\,. \quad (12)$$

$$p^2:\ \ddot{v}_2 + 0.25\dot{v}_2 - v_2 + (3{v_0}^2 v_1) = 0. \qquad v_2(0) = 0, \qquad \dot{v}_2\,(0) = 0\,. \quad (13)$$

By utilizing the Laplace transformation on Equation (11), we have:

$$[s\hat{}2\, v_0(s) - sv_0(0) - v_0(0)] + 0.25[sv_0(s) - v_0(0)] - v_0(s) = 0. \quad (14)$$

Then,

$$V_0(s)(s^2 + 0.25s - 1) = 0. \quad (15)$$

Hence,

$$s_1 = 0.882, \quad s_2 = -1.132\,. \quad (16)$$

Therefore:

$$v_0 = \alpha e^{(-1.132t)} + \beta e^{(0.882t)},$$
$$v_0(0) = 0.2, \quad \dot{v}_0(0) = 0.2\,. \quad (17)$$

By employing the initial conditions, parameters α and β would be computed as follows:

$$\begin{cases} \alpha + \beta = 0.2 \\ -1.132\alpha + 0.882\beta = 0.2 \end{cases} \longrightarrow$$
$$\begin{cases} 1.132\alpha + 1.132\beta = 0.2264 \\ -1.132\alpha + 0.882\beta = 0.2 \end{cases} \quad (18)$$

By substituting $\alpha = -0.01$ and $\beta = 0.21$ for α and β into Equation (12), the first term of the solution could be introduced as follows:

$$v_0 = -0.01e^{(-1.132t)} + 0.21e^{(0.882t)}. \quad (19)$$

If Equation (12) is rewritten as follows:

$$v_1 = \ddot{v}_1 + 0.25\dot{v}_1 - v_1 + [(v_0)].^3 \quad (20)$$

If

$$[v_0]^3 = (-1\times 10^{-6}e^{-3.396t}) + (0.0093e^{2.646t}) + (6.3\times 10^{-5}e^{-1.382t}) + (-0.0013e^{0.628t}) \quad (21)$$

Then, particular solution z is formulated as follows:

$$z = a\,e^{-3.396t} + be^{2.646t} + ce^{-1.382t} + de^{0.632t} \quad (22)$$

With

$$\dot{z} = -3.396\,ae^{-3.396t} + 2.646\,be^{2.646t} - 1.382\,ce^{-1.382t} + 0.632\,de^{0.632t} \quad (23)$$

And

$$\ddot{z} = 11.5328\, ae^{-3.396t} + 7.0013\, be^{2.646t} + 1.9099\, ce^{-1.382t} + 0.3994\, de^{0.632t} \quad (24)$$

Therefore,

$$9.6828a + 6.6628b + 0.5644c - 0.4426d + [(-1 \times 10^{-6} e^{-3.396t}) + (0.0093 e^{2.646t}) + (6.3 \times 10^{-5} e^{-1.382t}) + (-0.0013 e^{0.628t})] = 0. \quad (25)$$

The following algebraic Equations will determine unknown parameters a, b, c, and d.

$$\begin{aligned} &9.6838a - 1 \times 10^{-6} = 0. \Rightarrow a = 1.0324 \times 10^{-7} \\ &6.6628b + 0.0093 = 0. \Rightarrow b = -0.0014 \\ &0.5644c + 6.3 \times 10^{-5} = 0. \Rightarrow c = -1.1162 \times 10^{-4} \\ &-0.4426d - 0.0013 = 0. \Rightarrow d = -0.0029 \end{aligned} \quad (26)$$

Finally, the particular solution is rewritten as follows.

$$z = 1.0324 \times 10^{-7} e^{-3.396t} - 0.0014 e^{2.646t} - 1.1262 \times 10^{-4} e^{-1.382t} - 0.0029 e^{0.628t} \quad (27)$$

By applying the general solution, the total relation for v_1 could be introduced as follows:

$$v_1 = \alpha e^{-1.132t} + \beta e^{0.882t} + (1.0327 \times 10^{-7} e^{-3.396t} - 0.0014 e^{2.464t} - 1.1262 \times 10^{-4} e^{-1.382t} - 0.0029 e^{0.628t} \quad (28)$$

Where, constant parameters α and β would be calculated by the following initial conditions:

$$v_1(0) = 0\,, \dot{v}_1 = 0. \quad (29)$$

$$v_1 = -0.0008\, e^{-1.132t} + 0.0058 e^{0.882t} + 1.0327 \times 10^{-7} e^{-3.396t} - 0.0014 e^{2.464t} - 1.1262 \times 10^{-4} e^{-1.382t} - 0.0029 e^{0.628t} \quad (30)$$

Regarding the homotopy solution, the answer to the problem would be defined as follows:

$$y = \lim_{p \to 1} (\, v_0 + p v_1 + p^2 v_2 + \cdots) \quad (31)$$

Finally,

$$y(t) = -0.01 e^{(-1.132t)} + 0.21 e^{(0.882t)} - 0.0008\, e^{-1.132t} + 0.0058 e^{0.882t} + 1.0327 \times 10^{-7} e^{-3.396t} - 0.0014 e^{2.464t} - 1.1262 \times 10^{-4} e^{-1.382t} - 0.0029 e^{0.628t} \quad (32)$$

In order to validate the obtained solution by the HPM, the answers related to interval time [0, 10] (s) are compared with those of the fourth-order Runge-Kutta Method (RKM) in "Fig. 1". Although the results have a good agreement at the initial times, the differences could be obviously seen at the bigger times.

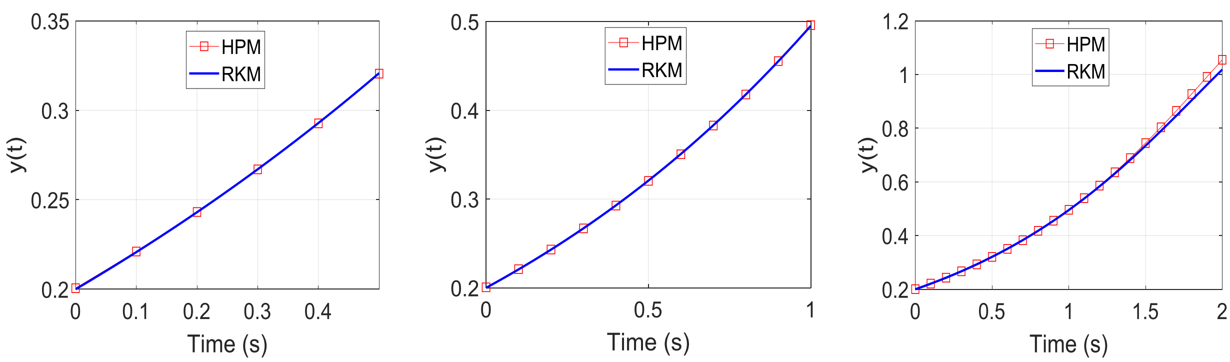

Fig. 1 Solutions found by the homotopy perturbation method and forth-order Runge-Kutta approach for different time intervals.

4 CONCLUSIONS

This research study implemented the homotopy perturbation method to analytically solve the nonlinear Dofing-Holmes Equation related to oscillatory dynamical systems. A closed mathematical formulation was determined to calculate the unknown parameter of the Equation at each time. The validations were performed through comparisons of the results with the numerical ones. The accuracy of the HPM was challenged by comparing the found results with those of the fourth-order Runge-Kutta technique.

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