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Minimum Stiffness and Optimal Position of an Intermediate Elastic Support to Maximize the Fundamental Frequency of a Vibrating Timoshenko Beam using Finite Element Method and Multi-Objective Genetic Algorithm

Hossein Ebrahimi1, Farshad Kakavand2, *, Hassan Seidi3
Department of Mechanical Engineering,
Takestan Branch, Islamic Azad University, Takestan, Iran
E-mail: Hossein_ebrahimi11@yahoo.com, F.kakavand@gmail.com, Hseidi@yahoo.com
*Corresponding author

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Abstract: This paper explores the optimal position and minimum stiffness of two intermediate supports to maximize the fundamental natural frequency of a vibrating cantilever Timoshenko beam with tip mass using Finite Element Method (FEM) and a multi-objective genetic algorithm (GA). After validating the results by comparison to previous works, the effects of the mass ratio and the position and stiffness of intermediate elastic support on the fundamental frequency are investigated. The numerical results demonstrated that as mass ratio increases, the optimal position moves toward the tip mass, and minimum stiffness increases. In many practical applications, it is not possible to place intermediate support in the optimal position; therefore, the minimum stiffness does not exist. In order to overcome this issue, a tolerance zone is considered, and design curves are proposed. The simultaneous optimization of the first and second natural frequencies of the beam with two intermediate supports was carried out using the genetic algorithm (GA) and the multi-objective GA. It was found that the optimization of the first and second natural frequencies did not require the two supports to have the same and high stiffness. The stiffness and optimal positions of the two supports differ at different mass ratios. Moreover, to optimize the first natural frequency, the second support should be stiffer, while the optimization of the second natural frequency requires the higher stiffness of the first support.

Keywords: Intermediate Support, Multi-Objective Genetic Algorithm (GA), Optimal Position and Minimum Stiffness, Timoshenko Beam

Biographical notes: Hossein Ebrahimi is PhD student of IAU, Takestan Branch, Iran in Mechanical Engineering. He received his MSc in Mechanical Engineering from IAU, Arak branch in 2011. **Farshad Kakavand** received his PhD in Mechanical Engineering from Sharif University of Technology in 2011. He is Assistant Professor at the department of engineering, IAU, Takestan Branch, Iran. His research interests include mechanical vibration, non-linear dynamics and optimization. **Hassan Seidi** received his PhD in Nanotechnology from National Academy of Science of Belarus in the State Scientific-Practical Materials Research Centre in 2014. He is Assistant Professor of Mechanical Engineering at IAU, Takestan Branch, Iran. His current research focuses on mechanical properties characterization of composite and Nano composites.

Research paper

1 INTRODUCTION

A beam with a tip mass at its free end is quite often applied in several engineering structures such as industrial mixers and robotic manipulator. Understanding of the modal characteristics of beam with tip mass is essential for avoiding resonance. By adding an intermediate support, we can improve its modal characteristics.

The supports situation of a structure plays a crucial role in the structural dynamic analysis and stability. Small change to the stiffness or position of an intermediate support can influence the natural frequencies and critical buckling load dramatically therefore, can significantly improve the structural performance significantly. From mathematical viewpoint, this strange sensitivity of natural frequencies and buckling loads is because of their eigenvalue nature. It is clear that the adding of new supports, changes the magnitude of deformations and structure deflections, but in this paper we are focused on the effect of an intermediate support on fundamental frequency of a beam.

Courant [1] stated that addition of n kinematical constraint to a system, will affect on the eigenvalues and eigenvectors of system as follows:

$$\lambda_{i+n} \leq \mu_i \leq \lambda_{i+n+1}$$

μ_i is the i-th eigenvalue of constrained system, and λ_i is the i-th eigenvalue of the unconstrained system. Suppose that the problem of investigating is modal analysis of a cantilever beam. Courant theorem state that: adding a rigid support (kinematic constants) to each location of the beam, will cause the first frequency of the constrained beam, to be a value between the first and second frequencies of unconstrained beam. From above inequality it is obvious that the maximum value of first natural frequency is the second natural frequency of unconstrained beam. Now the question is, if our goal is to increase the first natural frequency of the constrained beam, where we should place a rigid supports so that the first natural frequency of constrained beam reaches to its maximum value? Courant [1] showed that the optimal location is the node location of the second mode shape of unconstrained beam. For a beam with c-f boundary condition which has not tip mass the optimum position is x*= 0.7834L. That is, if we put a rigid support at this point, the first natural frequency of the constrained beam will be equal to the second natural frequency of unconstrained beam. The minimum stiffness of a support required to maximize the natural frequency is highly interested in engineering applications since producing a support with infinite stiffness is actually impossible. Akesson and Olhoff [2] showed that if the support stiffness be larger than a minimum value the maximizing of first natural frequency will be done and the support is not required be complete rigid. They calculated the minimum non-dimensional stiffness for a C-F beam as 267 numerically; Wang et.al [3] calculated this value 266.87 analytically. References [3] and [4] have examined the effect of an intermediate support on the natural frequency of an Euler-Bernoulli beam and have obtained the optimal support location for maximizing the natural frequency of the beam.

The effect of intermediate supports on critical buckling loads and dynamic response are studied in many published papers response of beam. Åkesson and Olhoff [5] studied on the effect of varying locations and stiffnesses of elastic supports on the frequencies of the maximum value of column buckling loads. Rao [6] presented the explicit and exact frequency and mode shape expressions for the clamped-clamped uniform beams with intermediate elastic support. Won and Park [7] presented a sensitivity analysis of eigenvalues to obtain its optimal support positions for a cantilever beam and a cantilever rectangular plate. Albaracin et al. [8] investigated the problem of a uniform beam with intermediate constraints and the ends are elastically restrained against rotation and translation. Zhu and Zhang [9] studied to maximize the natural frequency of structures and presented the support layout design that corresponds to optimization of boundary conditions. Wang studied on optimal design of structural support positions for minimizing maximal bending moment [10] and maximizing the natural frequency [11]. Friswell and Wang studied on the minimum support stiffness required to raise the fundamental natural frequency of plate structures [12] and Support position optimization with minimum stiffness for plate structures including support mass[13]. Kong [14] analyzed the vibration of plates with various boundary and internal support conditions to determine the optimal location and stiffness of discrete elastic supports in maximizing the fundamental frequency of both isotropic plates and composite plates. Wang et al [15] obtained the Minimum stiffness location of point support for control of fundamental natural frequency of rectangular plate. Aydin investigated on cantilever beams supported by optimal elastic springs to the reduction of dynamic deflections and accelerations [16] and the optimum distribution of elastic springs on which a cantilever beam is seated and minimization of the shear force on the support of the beam [17]. Roncevic et al. [18] studied on the frequency Equation and mode shapes of elastically supported Euler-Bernoulli beams. Abdullatif [19] analyzed effect of intermediate support on critical stability of a cantilever with non-conservative loading.

The most researches that have been carried out in this field have merely investigated the effect of an intermediate support on the dynamic specification of

types of beams without tip mass. Due to the importance of the dynamic response of the beams with intermediate supports, some studies are done on forced and natural vibrations on multi-span beams. Researchers studied the axial vibrations of multi-span beams with concentrated masses [20], the free vibration of multi-span beams with flexible constraints [21], the free vibration analysis of a uniform multi-span beam carrying multiple spring-mass systems [22], the free and forced vibration characteristics of a Bernoulli-Euler multi-span beam carrying a number of different concentrated elements [23], and the dynamic analysis of a multi-span beam subjected to moving loads [24].
There are many published works which studied the effect of concentrated mass on natural frequencies and mode shapes for various beam theories [25-30]. Studies on the optimization of fundamental frequencies by adding an intermediate support mostly adopted the Euler-Bernoulli beam theory. To investigate the effect of placing one and two middle elastic supports at any location in thick and thin beams with tip mass on the required stiffness and first natural frequency, the present study developed the finite element model of the Timoshenko beam. To the best of the authors' knowledge, this configuration has not been investigated.
This paper aims to study the frequency characteristics and optimization of the fundamental frequencies of a C-F Timoshenko beam with two intermediate elastic supports and tip mass. Motion Equations are derived using Hamilton's principle. The FEM is applied to free vibration. The validity and accuracy of the results are evaluated through comparison to previous works. The optimization of the fundamental frequency was carried out for a beam with in intermediate support, exploring the effects of the mass ratio and the position and stiffness of the intermediate elastic support on the fundamental frequency. The optimal positions of the additional supports are obtained. Since the addition of a support at a non-optimal position does not meet the minimum stiffness requirement, and it is impossible to add intermediate supports at the optimum position in many industrial applications due to geometric or process restrictions, a 5% tolerance zone is considered to determine the minimum stiffness. Through the minimum stiffness and optimum frequency at different mass ratios, the design curve is presented. Based on the revealed effects of the parameters, the simultaneous optimization of the first and second natural frequencies was carried out for the beam with two intermediate supports. Since it was difficult to implement two-variable optimization with several parameters through conventional algorithms, the GA and multi-objective GA were employed to optimize the fundamental frequency.

2 MATHEMATICAL FORMULATIONA

Consider a cantilever beam with tip mass. L, E, m, M_{tip} and I denote length, modulus of elasticity, mass density per unit length, tip mass and area moment of inertia respectively. As shown in "Fig. 1", an elastic intermediate supports with stiffness K_{is} are located at a distance x_i from clamp end.

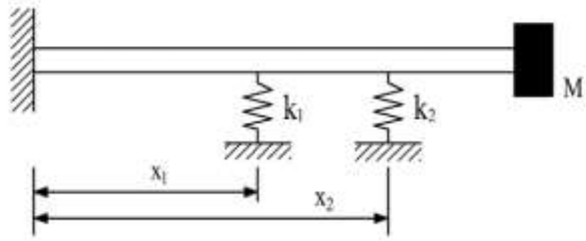

Fig. 1 Cantilever beam with two elastic intermediate supports and tip mass.

For Timoshenko beam Potential and Kinetic energy of the system can be expressed:

$$U=\frac{1}{2}\int_0^L\left[EI\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2+kGA\left(\frac{\partial W(x,t)}{\partial t}-\psi(x,t)\right)^2\right]dx \\ +\frac{1}{2}\int_0^L(k_1W^2(x,t)\delta(x-x_1)+k_2W^2(x,t)\delta(x-x_2))dx \tag{1}$$

$$T=\frac{1}{2}\int_0^L\left[\rho A\left(\frac{\partial W(x,t)}{\partial t}\right)^2+I\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2\right]dx \\ +\frac{1}{2}\int_0^L M_{tip}\left(\frac{\partial W(x,t)}{\partial t}\right)^2\delta(x-L)dx \tag{2}$$

Where, δ is the Dirac delta function and k is the sectional shear coefficient. In order to derive the governing Equations of motions and boundary conditions, Hamilton's principle is employed:

$$\int_{t_1}^{t_2}(\bar{\boldsymbol{\delta}}U-\bar{\boldsymbol{\delta}}T)dt=0 \tag{3}$$

Where, the virtual strain energy is ⍰⍰ and the virtual kinetic energy is ⍰⍰. Equations of motions can be obtained as:

$$\rho A\frac{\partial^2W(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left(kGA\left(\frac{\partial W(x,t)}{\partial x}-\psi(x,t)\right)\right) \\ +M_{tip}\delta(x-L)\frac{\partial^2W(x,t)}{\partial t^2}+k_1W(x,t)\delta(x-x_1) \\ +k_2W(x,t)\delta(x-x_2)=0 \tag{4}$$

$$EI\frac{\partial^2\psi(x,t)}{\partial x^2}+kGA\left(\frac{\partial W(x,t)}{\partial x}-\psi(x,t)\right)=I\frac{\partial^2\psi(x,t)}{\partial t^2} \tag{5}$$

Boundary conditions can be expressed as following Equations:

$$EI\,\frac{\partial\psi(x,t)}{\partial x}\delta\psi\Big|_0^L = 0$$
$$kGA\left(\frac{\partial W(x,t)}{\partial x} - \psi(x,t)\right)\delta W\Big|_0^L = 0 \tag{6}$$

For vibrating system as shown in “Fig. 1”, are obtained as:

$$W(x,t)|_{x=0} = 0$$
$$\psi(x,t)|_{x=0} = 0$$
$$kGA\left(\frac{\partial W(x,t)}{\partial x} - \psi(x,t)\right)\Big|_{x=L} = 0 \tag{7}$$
$$EI\,\frac{\partial\psi(x,t)}{\partial x}\Big|_{x=L} = 0$$

3 FINITE ELEMENT FORMULATION

Assuming the deformation vector and shape functions of the Timoshenko Beam as follows (Zohoor and Kakavand, [31]):

$$W = N_w q_e \tag{8}$$

$$\psi = N_\psi q_e \tag{9}$$

Where:

$$q_e = \{w_1, \psi_1, w_2, \psi_2\}^T \tag{10}$$

$$N_w = \frac{(h-x)}{h(\alpha h^2 - 12)}\Big[\alpha h^2 + \alpha hx - 2\alpha x^2 - 12, -x(-\alpha h^2 + \alpha hx + 6), \frac{-(x(2\alpha x^2 + 12) - 3\alpha hx^2)}{(h-x)}, -x(\alpha hx - 6)\Big] \tag{11}$$

$$N_\psi = \frac{(h-x)}{h(\alpha h^2 - 12)}\Big[-6\alpha x, \alpha h^2 - 3\alpha hx - 12, 6\alpha x, \frac{x(\alpha h^2 - 12)}{(h-x)} - 3\alpha hx\Big] \tag{12}$$

Where:

$$\alpha = \frac{kAG}{\mathrm{EI}}$$

Substituting “Eqs. (8 - 12)” in “Eqs. (1) and (2)”, after performing the conventional steps of variational calculus in “Eq. (3)”, one gets the Equation for a single finite element in the following form:

$$\mathrm{M_e}\ddot{\mathrm{q}}_\mathrm{e} + \mathrm{K_e}\mathrm{q_e} = 0 \tag{13}$$

Where:

$$m_e = \int_0^h \left(\rho A N_w N_w^T + \rho I N_\psi N_\psi^T\right) dx \tag{14}$$

$$k_e = \int_0^h \left(kAG\left(\frac{dN_w}{dx} - N_\psi\right)\left(\frac{dN_w}{dx} - N_\psi\right)^T + EI\,\frac{dN_\psi}{dx}\frac{dN_\psi^T}{dx}\right) dx \tag{15}$$

Obviously, the presence of end tip mass will change the mass matrix of the corresponding element which mass is connected to. The element matrix corresponding to the element which the mass is attached to, will be as follows:

$$m_e^n = m_e + M_{tip}[N_w N_w^T]|_{x=L} \tag{16}$$

Similarity, the presence of elastic supports will change the stiffness matrix of the corresponding element which spring is connected to. Assuming that the springs k_1 and k_2 are connected to the elements number r and s, respectively. Since the elastic support is not necessarily located at nodes, the element matrix corresponding to the element which the elastic support is attached to, will be as follows:

$$\mathrm{K_e^r} = \mathrm{K_e} + \mathrm{k_1}[N_w N_w^T]|_{x=x_1}$$
$$\mathrm{K_e^s} = \mathrm{K_e} + \mathrm{k_2}[N_w N_w^T]|_{x=x_2} \tag{17}$$

After the assembling process one gets the following expression:

$$\mathrm{M}\ddot{\mathrm{Q}} + \mathrm{KQ} = 0 \tag{18}$$

Q is global deformation matrix for Timoshenko model. M and K are global mass and stiffness matrix respectively. Finally, the corresponding eigenvalue problem is:

$$\left([\mathrm{K}] - \omega_i^2[\mathrm{M}]\right)\vec{X}_i = \vec{0} \tag{19}$$

Vector Xi represents the i-th mode shape, and the natural frequencies are the solution of following characteristic Equation:

$$\det([\mathrm{K}] - \omega_i^2[\mathrm{M}]) = 0 \tag{20}$$

Let us define the non-dimensional parameters as follow:

$$\mu = \frac{M_{tip}}{M_{beam}}$$

$$K_i = \frac{k_i EI}{L^3}$$
$$\zeta_i = \frac{x_i}{L}$$

4 VERIFY OF SOLUTION

The numerical results are compared with the previous works to demonstrate the accuracy of the present study. Firstly, the first natural frequencies are compared with results of Wang et al. [3]. The following material and properties of beam are used: E = 207 GPa, G = 70 GPa, ρ = 7800 kg/m^3, L = 1 m, d=0.2 m, k = 6/7.

The next, natural frequencies of beams are compared with results of Laura et al. [25]. The following material and properties for beam are used: E = 207 GPa, G = 70 GPa, ρ = 7834 kg/m^3, L = 1 m, d = 0.0254 m, k = 6/7.

To more verify the present results, the first three natural frequencies are compared with results of Hong et al. [32]. The following material and properties of beam are used: E = 190 GPa, G = 70 GPa, ρ = 7970 kg/m^3, L = 0.505 m, d = 0.254 m, k = 6/7.

As seen from “Table 1 – 3”, there are good agreement in results and the difference between the frequencies is very small. It should be noted that the Euler–Bernoulli theory has been used in the compared models. The Euler–Bernoulli beam theory is applicable to slender beams only. For moderately deep beams, this model underestimates deflection and overestimates natural frequency due to ignoring the transverse shear deformation effect. The Timoshenko beam theory has been proposed to overcome the limitations of the Euler–Bernoulli beam by accounting for the transverse shear deformation effect [33]. Therefore, the proposed accurate FE model is designed. Moreover, it was found that a rise in the mass ratio decreases the natural frequency, and the natural frequency is maximized when the intermediate support is positioned at the beam node.

Table 1 Comparison of the results: clamped-free

μ	ζ_1	K_1	$\beta_1 L = \left[\frac{\omega^2 \rho A L^4}{EI}\right]^{\frac{1}{4}}$	
			Present	Wang et al. [4]
0	0.7834	266.9	4.6786	4.6941
0	0.8000	200	4.4406	4.4469
0	0.8500	102	3.9127	3.9167

Table 2 Clamped-free with mass at free end

μ	ζ_1	K_1	Model	$\beta_1 L = \left[\frac{\omega^2 \rho A L^4}{EI}\right]^{\frac{1}{4}}$	
				Mode 1	Mode 2
0	0	0	Present	1.8738	4.6713
			Laura et al. [12]	1.8751	4.6941
0.6	0	0	Present	1.3751	4.0717
			Laura et al. [12]	1.3756	4.0866
1	0	0	Present	1.2474	4.0167
			Laura et al. [12]	1.2479	4.0311

Table 3 Comparison of the results: Clamped-free with an intermediate elastic support and mass at free end

μ	ζ_1	K_1		Frequency (Hz)	
				Mode 1	Mode 2
0.2	0.1	500	Present	16.87	119.31
			Hong et al. [23]	16.73	118.28
0.2	0.2	500	Present	19.71	143.02
			Hong et al. [23]	19.48	141.29

5 RESULTS AND DISCUSSION

Courant’s theorem states if a rigid support is placed in the location of the node of second mode of unconstrained beam, the first natural frequency of the constrained beam is equal to the second natural frequency of unconstrained beam. In order to verify the Courant theorem, we first obtain the first and second natural frequencies of unconstrained beam versus tip mass ratio.

Figure 2 shows the variation of the fundamental frequency of C–F beam with tip mass respect to μ ratio. It is observed that an increase in the value of mass ratio leads to a reduction of frequency. For μ=0, the results are completely coincident with Wang et al. [3]. It can be seen that as mass ratio increases the optimal position approaches toward the end of the beam. This is completely agreed with courant’s theorem. Optimum position was tabulated in “Table 4”.

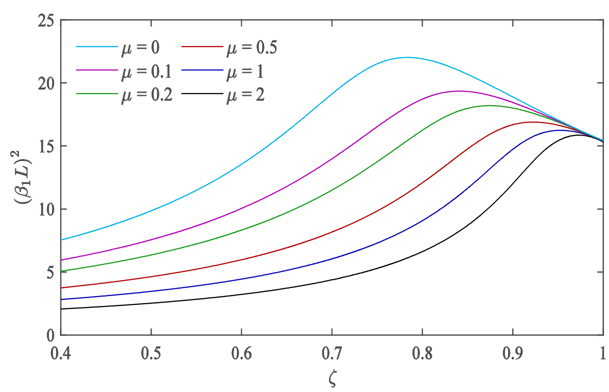

Fig. 2 Location First natural frequency of constrained beam with a rigid support.

Table 4 Raised frequency and minimum stiffness for different values to mass ratio

Mass ratio μ	Optimum position ζ_1	K_1	$\beta_1 L$
0	0.7833	267	4.6836
0.1	0.8409	233	4.4012
0.2	0.8743	235	4.2261
0.5	0.9225	284	4.1097
1	0.9526	394	4.0311
2	0.9733	628	3.9825

As mentioned before, adding a rigid intermediate support at the optimum position which are listed in "Table 4", for instance C-F beam with optimum position 0.7833, the fundamental frequency of constrained beam equals to second natural frequency of unconstrained beam, $\beta L=1.8754 \rightarrow 4.6836$. Akesson and Olhoff [5] have showed that for increasing natural frequency to its maximum level, it is not necessary to the support be rigid. They showed that if the stiffness of the support be larger than a "minimum stiffness" value the fundamental frequency will be maximized.
As "Fig. 3" depicts, for the ratio μ=0.5 and K_1=0 the natural frequency is equal to βL=1.4195. As the stiffness of the intermediate support increases, the natural frequency increases nonlinearly. For a critical value of stiffness which is called "minimum stiffness", K_1=284, the value of the natural frequency equals the second natural frequency of unconstrained beam, βL=4.10. We call this point knee point. After knee point, increasing of support stiffness does not have any effect on natural frequency, and it remains constant regardless of any change of stiffness.

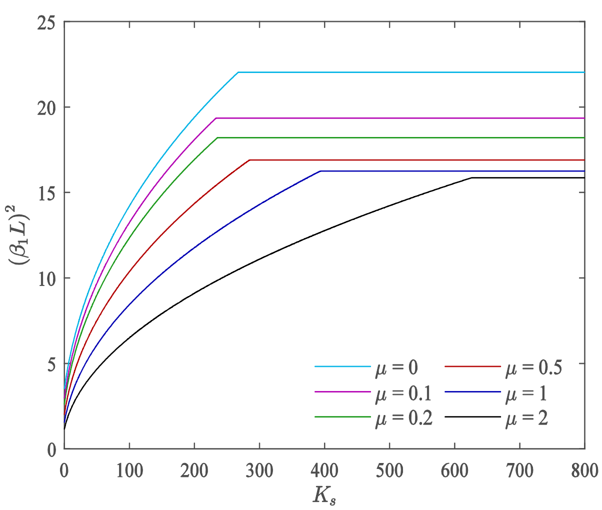

Fig. 3 Minimum stiffness of cantilever beam with tip mass.

In many practical problems, because of any geometric or process restraint it is not possible place additional support at optimum position. If we put a support in other points, the knee point in "Fig. 3" will not appear. In other words, as the stiffness increases the fundamental frequency increases asymptotically to its maximum value. Therefore, we need a criterion to defining minimum stiffness. We suggest a 5% tolerance zone about the maximum value which can be obtained by a rigid support at desired position of support.
Figure 4 shows the tolerance zone for a cantilever beam with μ=1 and ζ_1=0.5. Considering 5% tolerance, the minimum stiffness and natural frequency were obtained K_s =823.9 and $(\beta L)^2$ =3.2980, respectively. After this point, increasing stiffness from 823.9 to 1500 yields a slight increase in frequency about 0.2. As mentioned before, since the support position is not at optimum position, the stiffness curve has no knee point.

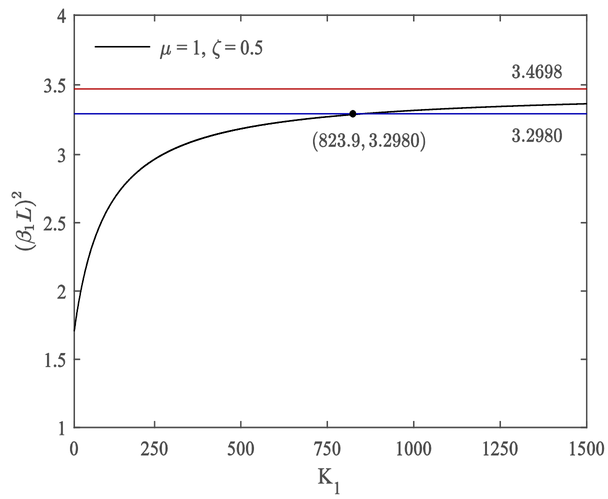

Fig. 4 Introducing tolerance zone.

Figure 5 depicts the raised fundamental frequency and minimum stiffness versus the location of support. We call this figure, design curve. We can extract the raised frequency and minimum support for an arbitrary position of support from this design curve.

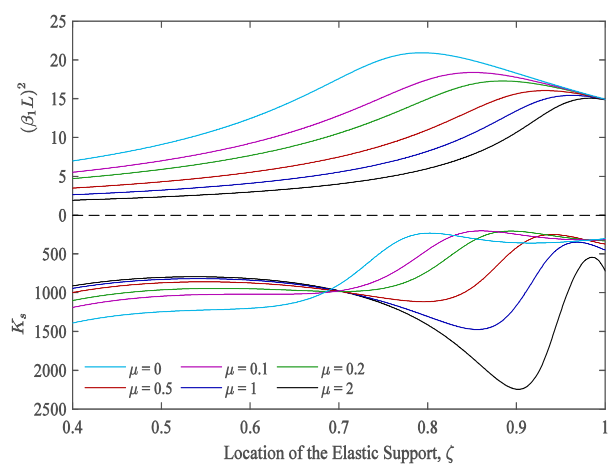

Fig. 5 Intermediate support design curve.

5 OPTIMIZATION THROUGH GA AND MULTI-OBJECTIVE GA

In the optimization of systems, the optimal solution can be found by investigating the results if the problem has one or two variables. For objective functions with three or more variables, however, it is very difficult, time-consuming, and sometimes impossible to optimize several parameters at the same time. Therefore, the advantages of the GA are exploited, calculating the optimal position of the intermediate support to maximize the first and second frequencies of the beam with tip mass and one and two intermediate supports. The objective function of the GA could be the first or second frequencies or a number of natural frequencies at the same time. Then, the optimal position is obtained using the multi-objective GA based on the Pareto front to maximize the second frequency and minimize the first frequency simultaneously.

5.1. GA

The GA begins with a random initial population. Then, the objective function is calculated for the initial population, sorting the results in descending order. Indeed, the points with the highest objective function values are desired. These desired points are employed as the parent population to generate the next population. The points with the maximum objective function values are referred to as elite to be used in the next round. The new population, known as children, is generated based on the parent population. This is carried out by either random changes in the parent population, which is known as mutation, or combining the characteristics of the parent population, which is referred to as crossover. These iterations continue until a discontinuance criterion is met**.**

To obtain the natural frequencies of the beam, FEM was employed to discretize the Equations. These Equations are a function of the geometric parameters and mechanical properties of the beam and the characteristics of the intermediate support, e.g., support resilience and position. Resilience was assumed to be high and constant. Therefore, the objective function is considered to be the first and second frequencies separately, obtaining the optimal position of the support. However, since more than one optimal support position could be found, the solution domain is divided into a number of sub-domains, sorting and comparing the maximum frequencies. The identification of more than one maximum natural frequency in a sub-domain would suggest that more than one optimal support positions exist for the natural frequency**.**

"Table 5" reports the optimal support positions to maximize the first frequency of the C-F beam with an intermediate support at different tip masses. As can be seen, the optimal position shifts toward the end of the beam as μ increases. The results are consistent with Wang et al. [4] at μ=0. Figure 6 show the variation of first and second frequencies of a cantilever beam for μ=0.5 various values of intermediate support stiffness. It is really surprising that before knee point, the second frequency remains constant and first frequency changes but after knee point the first frequency remains fixed and second frequency varies.

Table 5 Optimal support position to maximize the first natural frequency for one support

Mass Ratio, μ	Optimum position, ζ_1	$\beta_1 L$	$\beta_2 L$
0	0.7833	4.6836	7.0157
0.1	0.8409	4.4012	6.6776
0.2	0.8743	4.2261	6.6151
0.5	0.9225	4.1097	6.6753
1	0.9526	4.0311	6.8191
2	0.9733	3.9825	6.9361

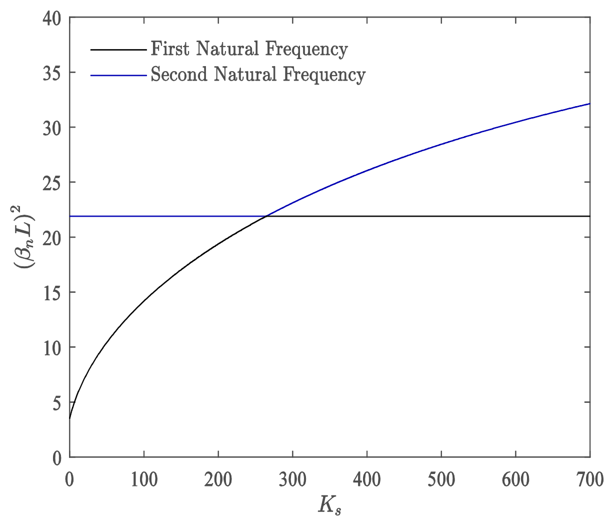

Fig. 6 First and Second Frequencies for μ=0.5 and rigid support at ζ_1=0.9225 for cantilever beam.

"Tables 6 and 7" provide the optimal positions of the first and second supports at different mass ratios to maximize the first and second natural frequencies under two intermediate supports, respectively. Tables show how two additional lateral supports be used for the beam, to increase its lowest eigenfrequency to the maximum possible value. For instance, for μ=0 and the optimum support positions ζ_1=0.5035 and ζ_2=0.8675, the first natural frequency (β_1L=1.8734) increase to the maximum value β_1L=7.8543. In addition, increases in the mass ratio decreases the natural frequency. Also, in order to maximize the second natural frequency, the optimal position approaches toward the end of the beam.

Table 6 Optimal support position to maximize the first natural frequency under two supports

Mass ratio, μ	Optimum support position, ζ_1	Optimum support position, ζ_2	β_1L	β_2L
0	0.5035	0.8675	7.8543	9.2898
0.1	0.5306	0.9208	7.4510	8.8882
0.2	0.5395	0.9433	7.3178	8.5024
0.5	0.5486	0.9695	7.1903	8.7235
1	0.5531	0.9800	7.1316	8.6753
2	0.5605	0.9810	7.0456	8.4764

Table 7 Optimal support position to maximize the second natural frequency under two supports

Mass ratio, μ	Optimum support position, ζ_1	Optimum support position, ζ_2	β_1L	β_2L
0	0.6443	0.9056	6.7587	10.9955
0.1	0.6728	0.9526	6.4637	10.5214
0.2	0.6802	0.9685	6.3867	10.4014
0.5	0.6875	0.9800	6.3182	10.2903
1	0.3840	0.6894	2.6533	10.2562
2	0.3850	0.6906	2.2583	10.2338

Figures 7 and 8 plot the first and second natural frequencies versus K_1 and K_2, respectively. As can be seen, the second support should be stiffer to optimize the first natural frequency, while the optimization of the second natural frequencies requires higher stiffness in the first support. For example, β_1L and β_1L are found to be 5.295 and 0.698 for K_1=250 and K_2=700, respectively, as shown in "Fig. 7".

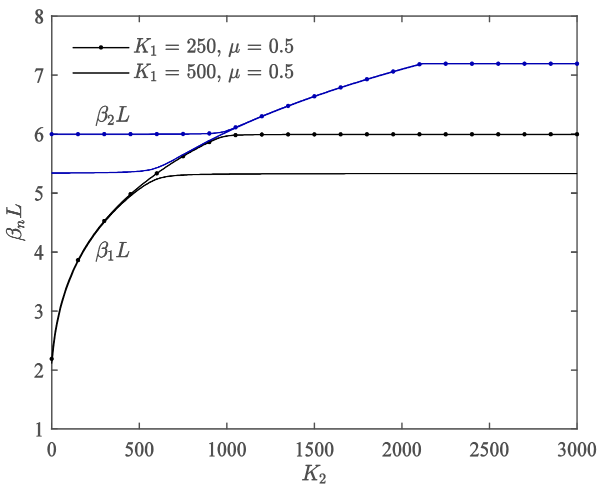

Fig. 7 First and Second Frequencies for μ=0.5, ζ_1=0.6875, K_1=250, 500 and different value of K_2.

The first natural frequency remains unchanged above this position, despite the increased stiffness of the second support. To maximize the second natural frequency, it is required that the minimum K_2 be 2100 and, consequently, β_2L be 7.19. For stiffness values of K_1=220 and K_2=250, the natural frequencies are obtained to be β_1L=4.313 and β_2L=5.243, as shown in "Fig. 8". Above this point, the first natural frequency remains unchanged. To maximize the second natural frequency, it is required that the minimum stiffness be K_1=1450 and, consequently, β_2L be 7.19.

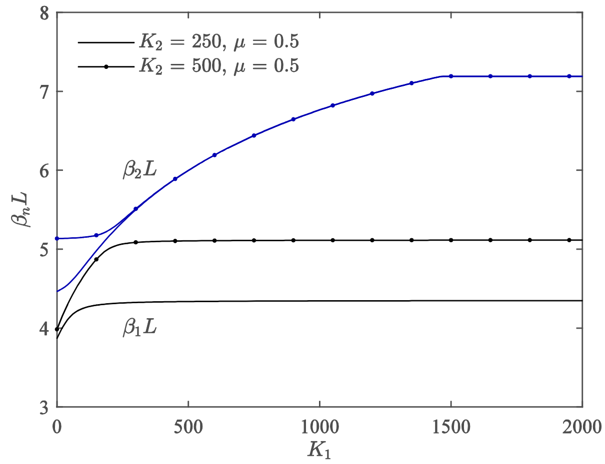

Fig. 8 First and Second Frequencies for μ=0.5, ζ_1=0.6875, K_2=250, 500 and different value of K_1.

5.2. Multi-Objective GA

The multi-objective GA is the same as the GA, except that more than one objective function is involved. The multi-objective GA utilizes a controlled elastic GA, which is a branch of the NSGA-II method. It prefers points with better fitness values or ranks of the objective functions. Also, the points that contribute to the diversity of the input population are preferred, even if they have lower ranks.

The concept of the dominancy of a point over its adjacent points is defined as:

1) The objective functions should not be lower at the dominant point than at the adjacent points.

2) At least one objective function should yield a higher value at the dominant point than at the adjacent points.

Then, the rank of the objective points is defined. Points of rank 1 are those over which no points are dominant. Points of rank 2 refer to points that are dominated by only points of rank 1. Overall, points of rank k are points that are dominated by points of rank k-1.

Points of lower ranks are more likely to be selected. They are used as the parent population to generate the new population. Dominant points are found, similarly to the GA, and the Pareto front is presented as the solution. Once the optimal solutions have been found, the corresponding optimal resilience values are

calculated. Eventually, the designer should select a point as the solution based on their intuition and priorities.
To optimize the fundamental frequency of the beam using the GA, the first and second frequencies are treated to be the objective function of the multi-objective GA.

$$\text{Objective Function} = \vec{F}(\omega_1, \omega_2) = [\omega_1, \omega_2]$$

The objective is to minimize the first frequency and maximize the second frequency. Since a variety of solutions are obtained from the Pareto front plot for the objective function and cannot be depicted, the points with the maximum difference between the first and second frequencies are selected. This selection was implemented due to specific design conditions, and other points in the Pareto front plot can also be acceptable solutions. For example, to increase the first frequency above a given value while maximizing the second frequency, different points are selected. "Table 8" represents the results based on the maximum difference between the first and second frequencies.

Table 8 Optimal support position to minimize the first natural frequency and maximize the second natural frequency by using two supports

Mass ratio, μ	Optimum support position, ζ_1	Optimum support position, ζ_2	$\beta_1 L$	$\beta_2 L$
0	0.4046	0.6078	4.2306	10.5128
0.1	0.4018	0.6645	3.9509	10.3586
0.2	0.4012	0.6755	3.6303	10.2922
0.5	0.4001	0.6834	3.0803	10.2328
1	0.4000	0.6867	2.6552	10.2029
2	0.4011	0.6789	2.2271	10.1642

Moreover, the Pareto points and Pareto front are represented for μ = 1 in "Figs. 9 and 10".

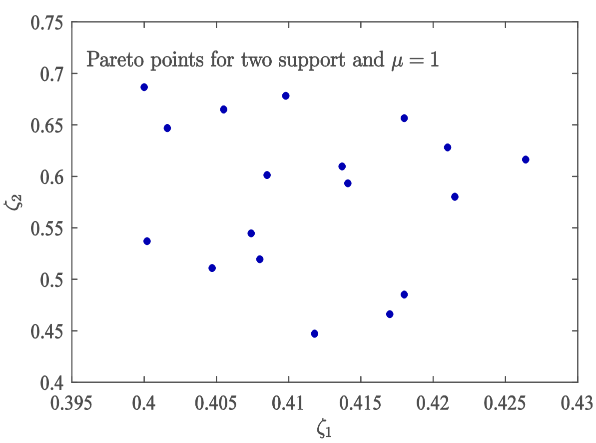

Fig. 9 Pareto points for two supports.

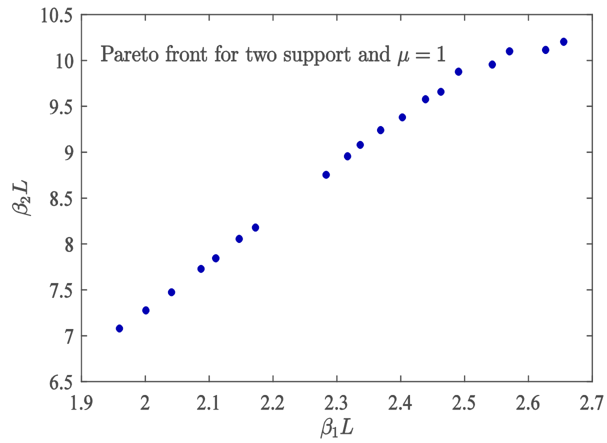

Fig. 10 Pareto front for two supports.

"Table 9" shows the results of the Pareto plots. Based on multi-objective GA, the points with the maximum difference between the first and second frequencies are selected. This difference is important for systems that operate between the first and second natural frequencies and must be far enough away from the natural frequencies.

Table 9 Values for charts pareto front and pareto points

Optimum support position, ζ_1	Optimum support position, ζ_2	$\beta_1 L$	$\beta_2 L$	$\beta_2 L - \beta_1 L$
0.4000	0.6867	2.6561	10.2029	7.5468
0.4002	0.537	2.1472	8.0555	5.9083
0.4016	0.6469	2.4906	9.8766	7.3860
0.4047	0.5109	2.0873	7.7291	5.6418
0.4055	0.666	2.5705	10.0992	7.5287
0.4074	0.5446	2.1726	8.1791	6.0065
0.4080	0.5195	2.1104	7.8450	5.7346
0.4085	0.6012	2.3367	9.0802	6.7435
0.4098	0.6782	2.6270	10.1161	7.4891
0.4118	0.4472	1.9597	7.0797	5.1200
0.4137	0.6097	2.3685	9.2401	6.8716
0.4141	0.5933	2.3166	8.9553	6.6387
0.417	0.4662	2.0011	7.2765	5.2754
0.418	0.6566	2.5434	9.9553	7.4119
0.418	0.4852	2.0411	7.4740	5.4329
0.421	0.6282	2.4389	9.5772	7.1383
0.4215	0.5802	2.2831	8.7530	6.4699
0.4264	0.6164	2.4026	9.3803	6.9777
0.4266	0.6335	2.4630	9.6581	7.1951

6 CONCLUSIONS

In this paper, the frequency characteristic and optimization of the fundamental frequencies of a C–F Timoshenko beam with two intermediate elastic supports and tip mass are studied. Using Courant's maximum-minimum theorem, an additional constraint was imposed on the beam. Motion Equations were derived from Hamilton's principle and solved using FEM. After validating the results through comparison to previous works, the effects of the mass ratio and the position and stiffness of intermediate elastic supports on the fundamental frequency were investigated. For a 5% tolerance zone to determine the minimum stiffness and obtain the minimum stiffness and optimal frequency at different mass ratios, the design curve was presented. Then, the simultaneous optimization of the first and second natural frequencies of the beam with two intermediate supports was carried out using the GA and multi-objective GA. It was found that the optimization of the first and second natural frequencies did not require the two supports to have the same and high stiffness, and the optimal stiffness and positions of the supports differed at different mass ratios.

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