Optimization of Location and Stiffness of an Intermediate Support to Maximize the First Natural Frequency of a Beam with Tip Mass-With Application

Hossein Ebrahimi, Farshad Kakavand*, Hassan Seidi
Department of Mechanical Engineering,
Takestan Branch, Islamic Azad University, Takestan, Iran
E-mail: Hossein_ebrahimi11@yahoo.com, F.kakavand@gmail.com, Hseidi@yahoo.com
*Corresponding author

Received: 22 June 2021, Revised: 23 October 2021, Accepted: 10 November 2021

Abstract: The optimal position and minimum stiffness of an intermediate support is implemented to maximize the fundamental natural frequency of a vibrating cantilever Euler-Bernoulli beam with tip mass. According to Courant's maximum-minimum theorem, maximum value of the first natural frequency of a beam with a single additional rigid internal support, is equal to the second natural frequency of the unsupported beam. In literature, for a cantilever beam without tip mass, the optimum position of intermediate support was reported as 0.7834L and minimum dimensionless stuffiness as 266.9. In this paper, the effect of tip mass ratio on optimum location and minimum stiffness is investigated. The Finite element method is employed. Cross sectional area is uniform and material is homogeneous and isotropic. Numerical results demonstrate that as tip mass ratio increases the optimal position moves toward the tip mass and minimum stiffness increases. For instance, for tip mass ratio 0.5, optimal position is 0.92L and minimum dimensionless stiffness is 284. Optimal position and minimum stiffness are presented for various range of mass ratio. In many applications, it is not possible to place intermediate support at optimal position; therefore, the minimum stiffness does not exist. In these cases, a tolerances zone is considered and related design curves are proposed. As a practical example, an agitator shaft is considered and end impeller is modeled as tip mass. The effectiveness of the proposed design curves in order to maximize natural frequency is shown. A design of an intermediate support is presented; in this example the fundamental frequency has increased as much as 300 percent without any change in shaft diameter.

Keywords: Euler-Bernoulli, Intermediate Support, Optimal Position and Minimum Stiffness

How to cite this paper: Hossein Ebrahimi, Farshad Kakavand, and Hassan Seidi, "Optimization of Location and Stiffness of an Intermediate Support to Maximize the First Natural Frequency of a Beam with Tip Mass-With Application ", Int J of Advanced Design and Manufacturing Technology, Vol. 15/No. 1, 2022, pp. 125–132.
DOI: 10.30495/admt.2022.1933790.1291.

Biographical notes: Hossein Ebrahimi is PhD student of IAU, Takestan Branch, Iran in Mechanical Engineering. He received his MSc in Mechanical Engineering from IAU, Arak branch, Iran in 2011. **Farshad Kakavand** received his PhD in Mechanical Engineering from Sharif University of Technology in 2011. He is Assistant Professor at the Department of Engineering, IAU, Takestan Branch, Iran. His research interests include mechanical vibration, non-linear dynamics and optimization. **Hassan Seidi** received his PhD in Nanotechnology from National Academy of Science of Belarus in the State Scientific-Practical Materials Research Centre in 2014. He is Assistant Professor of Mechanical Engineering at IAU, Takestan Branch, Iran. His current research focuses on mechanical properties characterization of composite and Nano composites.

1 INTRODUCTION

A beam with a tip mass at its free end is quite often applied in several engineering structures such as industrial mixers and robotic manipulator. Understanding of the modal characteristics of beam with tip mass is essential for avoiding resonance. By adding an intermediate support, we can improve its modal characteristics.

The supports situation of a structure plays a crucial role in the structural dynamic analysis and stability; therefore, close attention should be paid to their characteristics. Supports are not only expected to hold a structure firmly, it is well known that a small number of adjustments in supports positions can influence the natural frequencies and critical buckling load dramatically, therefore, improve the structural performance significantly. In addition, it is clear that adding new supports, changes the magnitude of deformations and structure deflections.

The most researches that have been carried out in this field have merely investigated the effect of an intermediate support on the dynamic specification of types of beams without tip mass [1-11]. Courant [1] showed that adding n kinematical constraint to a system, will affect on the eigenvalues and eigenvectors of system as follows:

$$\lambda_{i+n} \leq \mu_i \leq \lambda_{i+n+1}$$

μ_i is the i-th eigenvalue of constrained system, and λ_i is the i-th eigenvalue of the unconstrained system. Suppose that the problem of investigating is modal analysis of a cantilever beam. Courant theorem state that: adding a rigid support (kinematic constants) to each location of the beam, will cause the first frequency of the constrained beam, to be a value between the first and second frequencies of unconstrained beam. From above inequality it is obvious that the maximum value of first natural frequency is the second natural frequency of unconstrained beam. Now the question is, if our goal is to increase the first natural frequency of the constrained beam, where we should place a rigid supports so that the first natural frequency of constrained beam reaches to its maximum value? Courant [1] showed that the optimal location is the node location of the second mode shape of unconstrained beam. For a beam with c-f boundary condition which has not tip mass, the optimum position is x*= 0.7834L. That is, if we put a rigid support at this point, the first natural frequency of the constrained beam will be equal to the second natural frequency of unconstrained beam. In practical problem, adding a rigid support is not possible. Olhoff and Akesson [2] showed that if the support stiffness be larger than a minimum value, the maximizing of first natural frequency will be done and the support is not required be complete rigid. They calculated the minimum non-dimensional stiffness for a C-F beam as 267 numerically; Wang et.al [3] calculated this value 266.87 analytically. References [3-4] have examined the effect of an intermediate support on the natural frequency of an Euler-Bernoulli beam and have obtained the optimal support location for maximizing the natural frequency of the beam.

There are many published works which studied the effect of concentrated mass on natural frequencies and mode shapes for various beam theories [12-16]. Recently, most of the researches are about forced and natural vibration on multi-span beams. Researchers have studied the axial vibrations of multi-span beams with concentrated masses [17], the free vibration of multi-span beams with flexible constraints [18], the free vibration analysis of a uniform multi-span beam carrying multiple spring–mass systems [19], the free and forced vibration characteristics of Bernoulli-Euler multi-span beam carrying a number of various concentrated elements [20], and dynamic analysis of a multi-span beam subjected to a moving force [21]. Optimization of location and stiffness of an intermediate support to maximize the first natural frequency of a beam with tip mass has not been previously investigated.

The purpose of this paper is finding the optimal position of an intermediate elastic support and its minimum stiffness in order to maximize the fundamental frequency of a beam with tip mass. In many industrial applications, adding intermediate support at optimum position is not possible because of geometric or process restrictions. To overcome this problem a set of design curves are presented which is used in a practical numerical example.

This paper is organized as follows. In Section 2, we have modeled the problem using Euler-Bernoulli model. In Section 3, the finite element model is established. In Section 4, we obtained the optimal location of the additional support. Finally, in section 5 we solved a practical problem using design curve which we have presented.

2 ANALYTICAL MODEL

Consider a cantilever beam with tip mass. L, E, m, M_{tip} and I denote length, modulus of elasticity, mass density per unit length, tip mass and area moment of inertia, respectively. As shown in “Fig. 1”, an elastic intermediate support with stiffness K_0 is located at a distance bL from clamp end. Based on Euler-Bernoulli theory, the free vibration equation of motion for flexural vibration of beam is as follows [22]:

$$\mathrm{EI}\frac{\partial^4 \mathrm{W}(x^*,t)}{\partial x^4} + \mathrm{m}\frac{\partial^2 \mathrm{W}(x^*,t)}{\partial t^2} = 0 \tag{1}$$

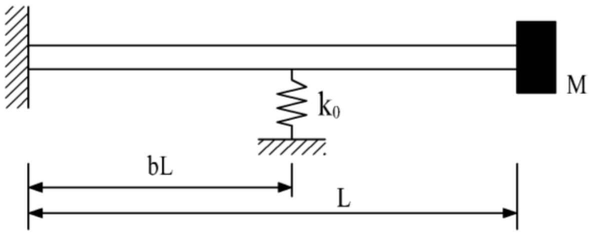

Fig. 1 Beam with end mass.

Using separation of variables technique, the corresponding eigenvalue problem can be formulated as:

$$Y''''(x^*) - \frac{m\omega^2}{EI} Y(x^*) = 0 \tag{2}$$

Where, Y(x) represents the mode shape function of beam. Introducing parameter β:

$$\beta^4 = \frac{m\omega^2}{EI} \tag{3}$$

From the solution of "Eq. (2)", the general solution of mode shapes is:

$$\begin{aligned} Y_1(x) &= C_1 \sin(\beta^* x) + C_2 \cos(\beta^* x) + C_3 \sinh(\beta^* x) \\ &\quad + C_4 \cosh(\beta^* x) \qquad 0 < x \le b \end{aligned} \tag{4}$$

$$\begin{aligned} Y_2(x) &= C_5 \sin(\beta^* x) + C_6 \cos(\beta^* x) + C_7 \sinh(\beta^* x) \\ &\quad + C_8 \cosh(\beta^* x) \qquad b \le x \le 1 \end{aligned} \tag{5}$$

Where, x=x*/L and β*= βL. We define the non-dimensional natural frequencies as β^{*2} throughout the paper either in Euler-Bernoulli model. To obtain constant coefficients C_1 to C_8, and natural frequencies, we have four boundary conditions at both ends of the beam and four continuity and jump conditions at junction point b. At clamped end, the deflection and the slope are zero. At the free end, the shear force and the bending moment are zero. As a result, the end boundary conditions of the beam will be as follows:

$$\begin{aligned} &Y_1(0) = Y_1'(0) = Y_2''(1) = 0 \\ &Y_2'''(1) + \frac{M_{tip}\omega^2}{EI} Y_2(1) = 0 \end{aligned} \tag{6}$$

At the point b, where the elastic support is located, the deflection, slope and bending moment are continuous and the shear force has a jump. Four boundary conditions at junction point are:

$$\begin{aligned} &Y_1(b) = Y_2(b), \quad Y_1'(b) = Y_2'(b) \\ &Y_1''(b) = Y_1''(b), \quad Y_1'''(b) - Y_2'''(b) = K_s Y_1(b) \end{aligned} \tag{7}$$

K_s is non-dimensionalized support stiffness:

$$K_s = \frac{K_0 L^3}{EI} \tag{8}$$

Appling eight boundary condition (6) and (7) to "Eqs. (4) and (5)" and establishing the corresponding characteristics equation, the natural frequencies and mode shapes will be found. It is clear that for known values of Ks and b, natural frequencies can be obtained from characteristic equation.

3 FINITE ELEMENT MODEL

Assuming the deformation vector and shape functions of Euler Bernoulli beam as follows [23]:

$$d_e = \{w_e, \theta_e, w_{e+1}, \theta_{e+1}\}^T \tag{9}$$

$$N_w = \begin{bmatrix} (x - x_{e+1})^2 (2x - 3x_e + x_{e+1}) / h_e^3 \\ (x - x_e)(x - x_{e+1})^2 / h_e^2 \\ -(x - x_e)^2 (2x + x_e - 3x_{e+1}) / h_e^3 \\ (x - x_e)^2 (x - x_{e+1}) / h_e^2 \end{bmatrix} \tag{10}$$

$$h_e = x_{e+1} - x_e \tag{11}$$

w_e, θ_e, w_{e+1} and θ_{e+1} are deflection and slope at node e and e+1, respectively. x_e is position of node e and x_{e+1} is position of node e+1. Expressing weak form of governing equations and imposing boundary condition, the finite element equations will be as follows:

$$M\ddot{d} + Kd = 0 \tag{12}$$

Where:

$$d = \{w_1, \theta_1, w_2, \theta_2, ..., w_{N+1}, \theta_{N+1}\}^T \tag{13}$$

$$M = \mathop{A}_{e=1}^{N} m_e, \qquad K = \mathop{A}_{e=1}^{N} k_e \tag{14}$$

Where d is global deformation matrix, and M, and K are global mass and stiffness matrix, respectively, and A is assembly operator. Elemental matrixes are:

$$k_e = EI \int_{x_e}^{x_{e+1}} \frac{d^2 N_w}{dx^2} \frac{d^2 N_w^T}{dx^2} dx \tag{15}$$

$$m_e = m \int_{x_e}^{x_{e+1}} N_w N_w^T dx \tag{16}$$

Obviously, the presence of elastic support will change the stiffness matrix of the corresponding element which spring is connected to. Since the elastic support is not necessarily located at nodes, the element matrix corresponding to the element which the elastic support is attached to, will be as follows:

$$K_e^{Spring} = K_0 N_w N_w^T \Big|_{x=bL} \tag{17}$$

Also tip mass affect on the elemental mass matrix of last element. It is clear that following matrix should be added to last elemental matrix.

$$m_N^{tip} = M_{tip} N_w N_w^T \Big|_{x=L} \tag{18}$$

Finally, the corresponding eigenvalue problem is:

$$\left([K] - \omega^2 [M]\right)\vec{X}_i = \vec{0} \tag{19}$$

Vector Xi represents the i-th mode shape, and the natural frequencies are the solution of following characteristic equation:

$$\det\left([K] - \omega^2 [M]\right) = 0 \tag{20}$$

Let us define the non-dimensional tip mass as follow:

$$\mu = \frac{M_{tip}}{M_{beam}} \tag{21}$$

4 OPTIMUM POSITION AND MINIMUM STIFFNESS

Courant's theorem states if a rigid support is placed in the location of the node of second mode of unconstrained beam, the first natural frequency of the constrained beam is equal to the second natural frequency of unconstrained beam. In order to verify the Courant theorem, we first obtain the first and second natural frequencies of unconstrained beam versus tip mass ratio. Figures 2 and 3 show the position of the node of second mode. It is known that increasing the tip mass decreases the natural frequencies and pushes the node toward to tip mass location.
Figure 4 depicts first natural frequency of constrained beam which was imposed by a rigid support at b. Figure was plotted for various tip mass ratio. For μ=0, the results are completely coincident with [22]. Furthermore, as mass ratio increases the optimal position approaches toward the end of the beam. This is completely agreed with courant's theorem. Optimum position was tabulated in "Table 1".

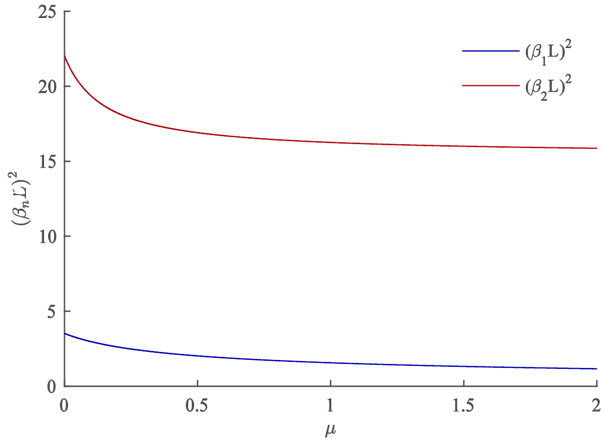

Fig. 2 First and second natural frequencies versus tip mass, without intermediate support.

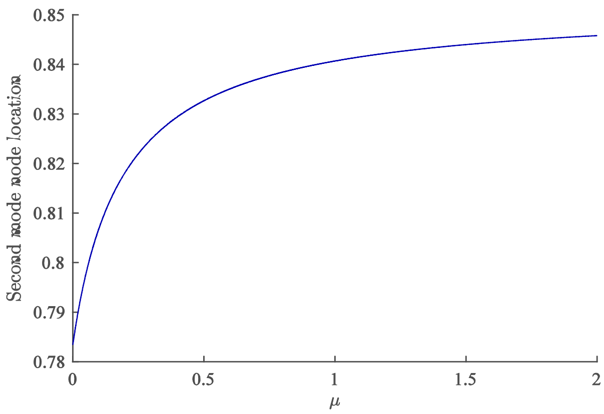

Fig. 3 Location of second mode node of unconstrained beam.

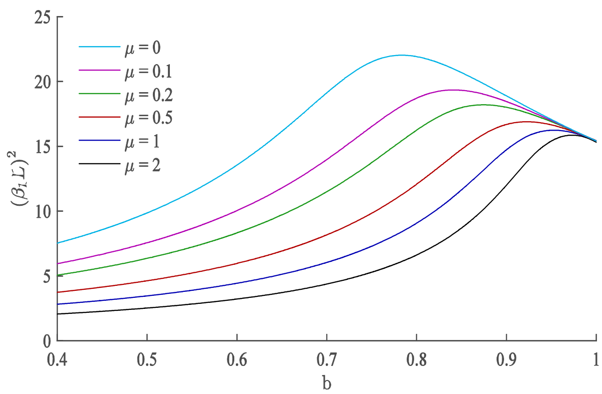

Fig. 4 First natural frequency of constrained beam with a rigid support.

Now, we place an elastic support at the optimum position. Figure 5 shows the variation of fundamental frequency versus support stiffness. As the stiffness of the intermediate support increases, the natural frequency increases nonlinearly.

Table 1 Raised frequency and minimum stiffness for different values to mass ratio

Mass ratio μ	Optimum position b	Minimum Stiffness Ks	Fundamental frequency β^{*2}
0	0.78	267	22.03
0.1	0.84	233	19.36
0.2	0.87	235	18.2
0.5	0.92	284	16.89
1	0.95	394	16.25
2	0.97	628	15.86

For a critical value of stiffness which is called "minimum stiffness" and tabulated in "Table 1", the value of the natural frequency equals the second natural frequency of unconstrained beam; we call this point "knee point". After knee point, increasing support stiffness does not have any effect on natural frequency, and it remains constant regardless of any change of stiffness. As it is clear from figure, there is knee point. Increasing stiffness above this point has no effect on fundamental frequency.

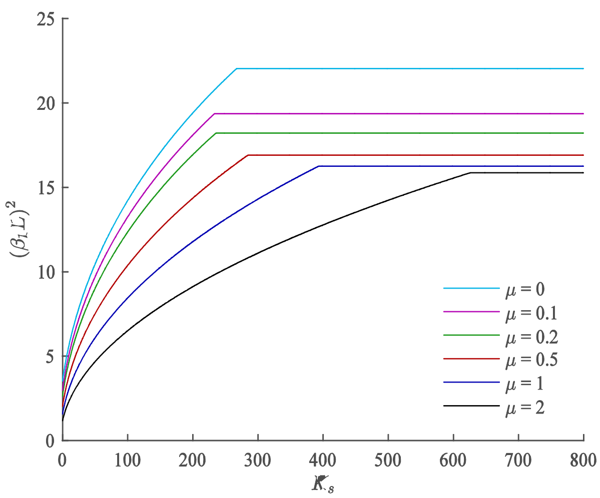

Fig. 5 Minimum stiffness of cantilever beam with tip mass.

In many practical problems because of any geometric or process restraint it is not possible to place additional support at optimum position. If we put a support in other points, the knee point in "Fig. 5" will not appear. In other words, as the stiffness increases the fundamental frequency increases asymptotically to its maximum value. Therefore, we need a criterion to define minimum stiffness. We suggest a 5% tolerance zone about the maximum value which can be obtained by a rigid support at desired position of support.

Figure 6 shows the tolerance zone for a cantilever beam with μ=1 and b=0.5. Considering 5% tolerance, the minimum stiffness and natural frequency were obtained K_s= 823.9 and $(\beta L)^2$ = 3.2980, respectively.

After this point, increasing stiffness from 823.9 to 1500 yields a slight increase in frequency about 0.2. As mentioned before since the support position is not at optimum position, the stiffness curve has no knee point.

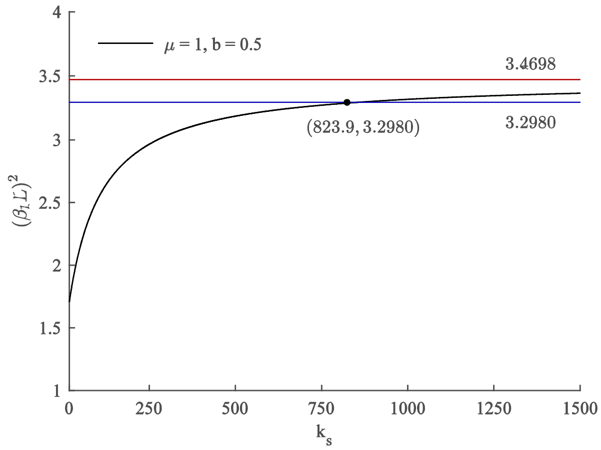

Fig. 6 Introducing tolerance zone.

Figure 7 depicts the raised fundamental frequency and minimum stiffness versus the location of support. We call this figure, design curve. We can extract the raised frequency and minimum support for an arbitrary position of support from this design curve.

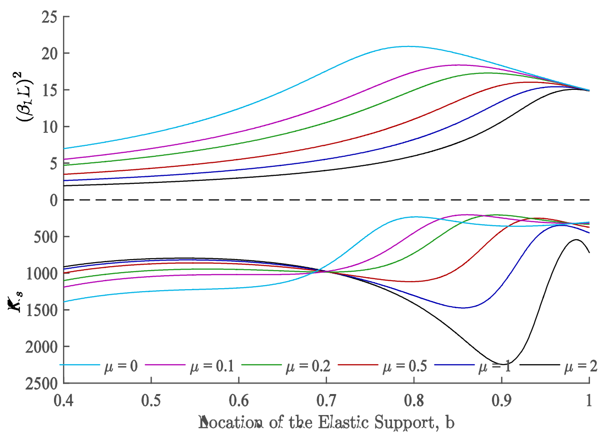

Fig. 7 Intermediate support design curve.

5 A PRACTICAL EXAMPLE

The industrial agitators are the most important equipment, which are used in many industries such as food, pharmaceutical, chemical, petrochemical and etc. Design of industrial mixers can be divided into two parts. The first part is process design and the second part is mechanical design. In process design, vessel dimensions, impeller types, rotating speed of shaft, material of mechanical parts and required power of motor will be selected according to media and chemical considerations. The main objects of mechanical design are design of vessel details, shaft, sealing system and bearing assembly. We focus on shaft design. Figure 8 shows a typical shaft-impeller system.

Fig. 8 A typical shaft-impeller system.

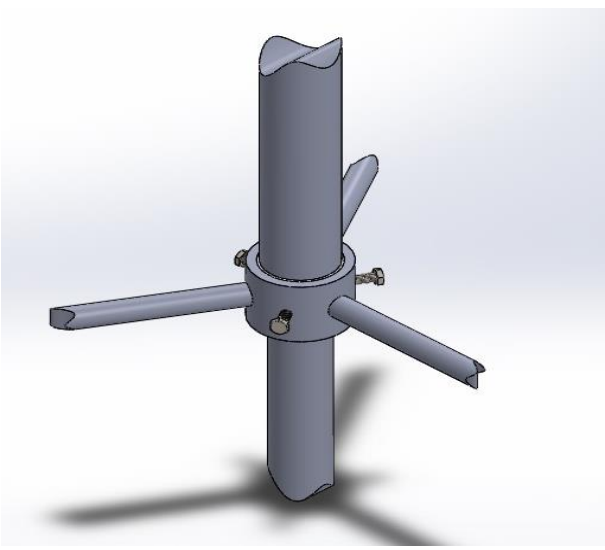

Fig. 9 Intermediate support detail.

It is obvious that the shaft diameter will affect on the size of mechanical seal and bearing assembly system. Increasing the shaft diameter can greatly increase construction costs. In designing of shaft, in addition to strength considerations such as yield and fatigue, the vibrational issues should be considered.

In many practical cases, the dominant phenomena to select appropriate shaft diameter is the resonance problem. This means that the required diameter to avoid resonance is greater than the required diameter for avoiding yield and Fatigue failure. In this paper, by adding an intermediate support, we have significantly reduced the required diameter of shaft to improve its fundamental frequency. Obviously, the intermediate support cannot be quite rigid. Therefore, the main purpose of this is to find the intermediate support position and its minimum stiffness so that fundamental frequency rises as high as possible. We consider the support configuration as following “Fig. 9”. It has three connecting bar and an appropriate bushing system.

We have modeled the shaft-impeller system as a cantilever beam, with tip mass. The moment of inertia of impeller has eliminated. We employed finite element method. Since the shaft is long enough, we use the Euler-Bernoulli model. As explained in the previous section, the intermediate support can be modeled as shown in “Fig. 10”.

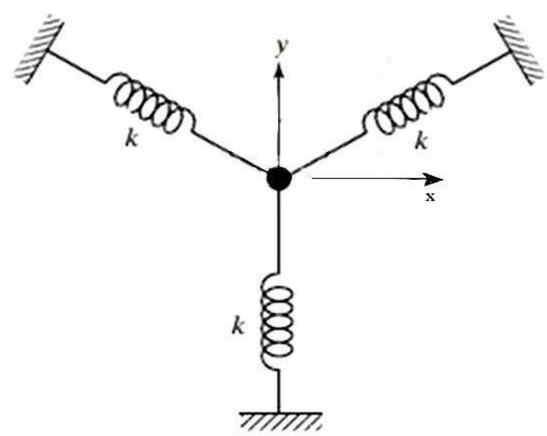

Fig. 10 Model of intermediate support.

If we use round bars as connecting bars, the relation between rod diameter and support stiffness can be written as follows:

$$\begin{aligned} & k_{eq} = 1.5k = K_0 \\ & k = \frac{EA}{L} = \frac{2E\left(\frac{\pi d_s^2}{4}\right)}{D_m} \\ & d_s = \sqrt{\frac{4K_0 D_m}{3\pi E}} \end{aligned} \tag{22}$$

The equivalent stiffness for above model is 1.5K. After determining minimum stiffness of support in next sections, we can evaluate the rod diameter using “Eq. (22)”. Consider a typical mixer which is a used in a pharmaceutical company. The agitator basic data are as shown in “Table 2”.

Table 2 Agitators Specifications

Tank diameter, m	2
Shaft length, mm	3700
Shaft speed, rpm	300
Electromotor power, kW	3
Modulus of elasticity, GPa	207
Density, kg.m^{-3}	7800
Allowable stress, MPa	120
Mass ratio	0.7

To avoid fatigue and yield failure, the required diameter of shaft is 47 mm. in this condition the design factor is 1.2. Using "Fig. 4" with mass ratio of 0.7, the non-dimensional fundamental frequency will be 1.79 then using "Eq. (3)" we will have:

$$\omega_1 = \frac{30}{\pi}\sqrt{\frac{m\beta^4}{EI}} = \frac{30}{\pi}\sqrt{\frac{(1.79/L)^2\rho A}{EI}} = 279 \quad \text{rpm}$$

It is too close to shaft speed. For smooth drive we prefer that fundamental frequency be at least 30% over than shaft speed i.e. 390 rpm. If we increase shaft diameter from 47 to 66 mm, the natural frequency will be 392 rpm and safety factor increases from 1.2 to 3.34; it means heavier shaft and larger mechanical seal and bearings which are undesired. Now let us add an intermediate support at the middle of shaft i.e. b=0.5; from design curve in "Fig. 7", we can find the raised non-dimensional fundamental frequency 4.1 and minimum stiffness $K_S = 780$. Therefore, natural frequency will increase to 899 rpm which is far enough from shaft speed.

$$K_0 = \frac{K_s EI}{L^3} = 2.967\times10^6 \quad N.m^{-1}$$

$$d_s = \sqrt{\frac{4K_0 D_m}{3\pi E}} = 3.4 \quad mm$$

The minimum diameter of connecting rod is 3.4 mm; therefore, we will choose 30 mm which is more practical.

6 CONCLUSIONS

In many of practical problems in agitator design, especially those have long shafts; the dominant phenomenon in shaft design is resonance avoiding design. In this paper, we presented a method for increasing natural frequency in order to distance it from shaft speed. Using the idea of Courant's maximum-minimum theorem, we imposed an additional constraint to beam. We have developed a design curve. We can obtain the raised frequency and minimum stiffness of intermediate support by various values of mass ratio and arbitrary location of intermediate support. A numeric practical example was solved. In this example, we raised the frequency to very high level without any change in shaft diameter.

REFERENCES

[1] Courant, R., Hilbert, D., Methods of Mathematical Physics, Interscience Publishers, New York, 1953.

[2] Olhoff, N., Akesson, B., Minimum Stiffness of Optimally Located Supports for Maximum Value of Column Buckling Loads, Structural Optimization, Vol. 3, No. 3, 1991, pp. 163-175, https://doi.org/10.1007/BF01743073.

[3] Wang, D., Friswell, M., and Lei, Y., Maximizing the Natural Frequency of a Beam with an Intermediate Elastic Support, Journal of Sound and Vibration, Vol. 291, No. 3, 2006, pp. 1229–1238, https://doi.org/10.1016/j.jsv.2005.06.028.

[4] Wang, C., Minimum Stiffness of an Internal Elastic Support to Maximize the Fundamental Frequency of a Vibrating Beam, Journal of Sound and Vibration, Vol. 259, No. 1, 2003, pp. 229–232, https://doi.org/10.1006/jsvi.2002.5100.

[5] Rao, C. K., Frequency Analysis of Clamped-Clamped Uniform Beams with Intermediate Elastic Support, Journal of Sound and Vibration, Vol. 133, No. 3, 1989, pp. 502–509, https://doi.org/10.1016/0022-460X(89)90615-9.

[6] Szelag, D., Mroz, Z., Optimal Design of Vibrating Beams with Unspecified Support Reactions, Computer Methods in Applied Mechanics and Engineering, Vol. 19, No. 3, 1979, pp. 333–349, https://doi.org/10.1016/0045-7825(79)90063-X.

[7] Albarracín, C. M., Zannier, L., and Grossi, R., Some Observations in the Dynamics of Beams with Intermediate Supports, Journal of Sound and Vibration, Vol. 271, No. 1-2, 2004, pp. 475–480, http://dx.doi.org/10.1016/S0022-460X(03)00631-X.

[8] Wang, D., Optimal Design of Structural Support Positions for Minimizing Maximal Bending Moment, Finite Elements in Analysis and Design, Vol. 43, No. 2, 2006, pp. 95–102, https://doi.org/10.1016/j.finel.2006.07.004.

[9] Motaghian, S., Mofid, M., and Alanjari, P., Exact Solution to Free Vibration of Beams Partially Supported by an Eastic Foundation, Scientia Iranica, Vol. 18, No. 4, 2011, pp. 861-866, https://doi.org/10.1016/j.scient.2011.07.013.

[10] Aydin, E., Minimum Dynamic Response of Cantilever Beams Supported by Optimal Elastic Springs, Structural Engineering and Mechanics, Vol. 51, No. 3, 2014, pp. 377–402, https://doi.org/10.12989/sem.2014.51.3.377.

[11] Goel, R., Vibrations of a Beam Carrying a Concentrated Mass, Journal of Applied Mechanics, Vol. 40, No. 3, 1973, pp. 821-832, https://doi.org/10.1115/1.3423102.

[12] Laura, P., Pombo, J., and Susemihl, E., A Note on the Vibrations of a Clamped-Free Beam with a Mass at the Free End, Journal of Sound and Vibration, Vol. 37, No. 2, 1974, pp. 161-168, https://doi.org/10.1016/S0022-460X(74)80325-1.

[13] Bruch, J., Mitchell, T., Vibrations of a Mass-Loaded Clamped-Free Timoshenko Beam, Journal of Sound and Vibration, Vol. 114, No. 2, 1987, pp. 341-345, https://doi.org/10.1016/S0022-460X(87)80158-X.

[14] Murgoze, M., On the Eigenfrequencies of a Cantilever Beam with Attached Tip Mass and a Spring-Mass System, Journal of Sound and Vibration, Vol. 190, No. 2, 1996, pp. 146-162, https://doi.org/10.1006/jsvi.1996.0053.

[15] Salarieh, H., Ghorashi, M., Free Vibration of Timoshenko Beam with Finite Mass Rigid Tip Load and Flexural–Torsional Coupling, International Journal of Mechanical Sciences, Vol. 48, No. 7, 2006, pp. 763-779, https://doi.org/10.1016/j.ijmecsci.2006.01.008.

[16] Magrab, E. B., Natural Frequencies and Mode Shapes of Timoshenko Beams with Attachments, Journal of Vibration and Control, Vol. 13, No. 7, 2007, pp. 905-934, https://doi.org/10.1177/1077546307078828.

[17] Kukla, S., Free Vibrations of Axially Loaded Beams with Concentrated Masses and Intermediate Elastic Supports, Journal of Sound and Vibration, Vol. 172, No. 4, 1994, pp. 449–458, https://doi.org/10.1006/jsvi.1994.1163.

[18] Lin, H. P., Chang, S., Free Vibration Analysis of Multi-Span Beams with Intermediate Flexible Constraints, Journal of Sound and Vibration, Vol. 281, No. 2, 2005, pp. 155–169, https://doi.org/10.1016/j.jsv.2004.01.010.

[19] Lin, H. Y., Tsai, Y. C., Free Vibration Analysis of a Uniform Multi-Span Beam Carrying Multiple Spring–Mass Systems, Journal of Sound and Vibration, Vol. 302, No. 3, 2007, pp. 442–456, https://doi.org/10.1016/j.jsv.2006.06.080.

[20] Lin, H. Y., Dynamic Analysis of a Multi-Span Uniform Beam Carrying a Number of Various Concentrated Elements, Journal of Sound and Vibration, Vol. 309, No. 1, 2008, pp. 262–275, https://doi.org/10.1016/j.jsv.2007.07.015.

[21] Kim, T., Lee, U., Dynamic Analysis of a Multi-Span Beam Subjected to a Moving Force Using the Frequency Domain Spectral Element Method, Computers and Structures, Vol. 192, 2017, pp. 181–195,https://doi.org/10.1016/j.compstruc.2017.07.028.

[22] Han, S. M., Benaroya, H., and Wei, T., Dynamics of Transversely Vibrating Beams Using Four Engineering Theories, Journal of Sound and Vibration, Vol. 225, No. 5, 1999, pp. 935–988, https://doi.org/10.1006/jsvi.1999.2257.

[23] Zohoor, H., Kakavand, F., Vibration of Euler–Bernoulli Beams in Large Overall Motion on Flying Support Using Finite Element Method, Scientia Iranica, Vol. 19, No. 4, 2012, pp. 1105–1116, https://doi.org/10.1016/j.scient.2012.06.019.