Analytical Solution for Buckling of Composite Sandwich Truncated Conical Shells subject to Combined External Pressure and Axial Compression Load

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Abstract: The present study deals with the buckling analysis laminated composite truncated conical sandwich shells with flexible core subject to combined axial compressive load and external pressure. The higher order governing equations of motion are presented for conical composite sandwich shells. They are derived from the Hamilton principle. Then, by use of Improved Higher-order Sandwich Shell Theory, the base solutions of the governing equations are obtained in the form of power series via general recursive relations. The first order shear deformation theory is used for the face sheets and a 3D-elasticity solution of weak core is employed for the flexible core. By application of various boundary conditions such as clamped and simply-supported edges, the natural frequencies of the conical composite sandwich shell are obtained. The obtained results are compared with the numerical results from FEM analysis and good agreements are reached. An extensive parametric study is also conducted to investigate the effect of total thickness to radius ratio on the buckling load.

Keywords: Buckling, Composite, Sandwich Truncated Conical Shell, Combined Load

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1 INTRODUCTION

Shell structures are widely used in engineering fields because of their strength characteristics. Conical shells, specifically circular cylindrical shells and annular plates, play an important role in many industrial fields. Typical sandwich structure consists of two thin face sheets embedding thick and soft core. The most modern aerospace structures are constructed of a composite sandwich laminate, usually carbon/epoxy face sheets over a honeycomb or foam core. Due to their outstanding flexural stiffness-to-weight ratio compared to other constructions, sandwich structures are widely used in many fields of technology, such as aerospace, aeronautic, marine, vehicles and etc. In many applications the primary concern is the stability of the structure and analytical solution is necessary to predict the critical buckling load.

The buckling of circular sandwich shells subject to various types of combined loading is of current interest to engineers engaged in mechanical engineering practice. It is of great technical importance to clarify the buckling behavior of cylindrical and conical composite sandwich shells under combined external pressure and axial compression. The buckling of isotropic conical shells under various loading conditions has been studied by many researchers. For example, Seide [1] proposed a formula for buckling of isotropic conical shell which is independent of boundary conditions and best fits the behavior of long shells. Singer [2] presented a solution for the buckling of conical shells under external pressure. The stability of simply supported isotropic conical shells under axial load for four different sets of in-plane boundary conditions using Donnell-type theory were investigated by Baruch et al. [3]. Singer [4], [5] analyzed the buckling of orthotropic conical shells. Weigarten and Seide [6], [7] studied the stability of conical shells under axial compression and external pressure.

Then, laminated composite materials have found extensive industrial applications during the past decades. Using Donnell-type shell theory, Tong and Wang [8] proposed a power series based solution for buckling analysis of laminated conical shells under axial compressive load and external pressure. Eight first-order differential equations were obtained for linear buckling analysis of laminated conical shells. These equations were solved by using the numerical integration technique and the multi-segment method. By analyzing the buckling of a series of conical shells, the effects of boundary conditions, elastic coefficients and the stretching-bending coupling, on the buckling loads and circumferential wavenumbers, were investigated.

Sofiyev [9] found a theoretical formula for the buckling of an orthotropic conical shell with continuously varying thickness subjected to a time dependent external pressure. Donnell-type shell theory was assumed in his work and Galerkin method and variational technique were applied to obtain the solution. Li [10] considered the stability of composite stiffened shell under axial compression load. Static, free vibration and buckling analysis of laminated conical shell using finite element method based on higher order shear deformation theory was carried out by Pinto Correia et al., [11]. Goldfeld et al., [12] did a work that deals with the optimization of laminate configuration of filament-wound laminated conical shells for the maximum buckling load. Due to the complex nature of the problem a preliminary investigation was made into the characteristic behavior of the buckling load with respect to the volume as a function of the ply orientation. Patel et al., [13] studied post buckling characteristics of angle-ply laminated conical shells subjected to torsion, external pressure, axial compression, and thermal loading using the finite element approach.

Wu and colleagues [14] studied 3D solution laminated conical shells subjected to axisymmetric loads using the method of perturbation. Liang et al., [15] explored the feasibility of using the transfer matrix method to analyze a composite laminated conical plate shell. Shadmehri et al., [16] proposed a semi-analytical approach to obtain the linear buckling response of composite conical shells under axial compression load. A first order shear deformation shell theory along with linear strain–displacement relations was assumed. Parametric study was performed by finding the effect of cone angle and fiber orientation on the critical buckling load of the conical composite shells. Lavasani [17] presented a simple and exact procedure using Donnell-type shell theory for linear buckling analysis of functionally graded conical shells under axial compressive loads and external pressure. Sofiyev [18] discussed the buckling analysis of composite orthotropic truncated conical shells under a combined axial compression and external pressure and resting on a Pasternak foundation. The governing equations had been obtained and solved by applying the Superposition and Galerkin methods. The novelty of present work was to achieve closed-form solutions for critical combined loads.

Xie Xiang et al., [19] focused on the free vibration analysis of composite laminated conical, cylindrical shells and annular plates with various boundary conditions based on the first order shear deformation theory, using the Haar wavelet discretization method.

Only few studies targeting the buckling behavior of sandwich cylindrical shells have been hitherto published. For instance, Neves et al. [20] used quasi-3d higher-order shear deformation theory (HOSDT) and mesh less technique to the static, free vibration and buckling analysis of isotropic and sandwich FG plates. Sofiyev [21] discussed the vibration and buckling of

sandwich cylindrical shells covered by different types of coatings and subjected to a uniform hydrostatic pressure using first order shear deformation theory (FOSDT).
Rahmani [22] studied vibration of the composite sandwich cylindrical shell with flexible core by using a higher order sandwich panel theory. The formulation used the classical shell theory for the face sheets and an elasticity theory for the core and included derivation of the governing equations along with the appropriate boundary conditions. Then, an improved higher order sandwich plate theory, applying the first-order shear deformation theory for the face sheets and three-dimensional elasticity theory for the soft core, was introduced by Malekzadeh et al., [23].
The analysis determined the damped natural frequencies, loss factors, and local and global mode shapes of plates. Biglari [24] proposed a refined higher order sandwich panel theory to investigate doubly-curved sandwich panels with flexible core. In this theory, equations of motion were formulated based on displacements and transverse stresses at the interfaces of the core. Parametric study was also included to investigate the effects of radius of curvature, thickness and flexibility of core. Lopatin and Morozov [25] presented an approximate analytical solution of the buckling problem formulated for a composite sandwich cylindrical shell under uniform external lateral pressure. Both ends of the shell of finite length were fully clamped.
The problem was solved using the Galerkin method. The analytical formula for the critical load had been obtained and verified by comparison with a finite-element solution.
The work done in [8], [16-25] does not regard the analysis of composite sandwich conical shells. Herein, we concentrate on such buckling analysis of composite sandwich truncated conical shells under combined external pressure and axial compression load. Therefore, there is no research on the analysis of buckling of composite sandwich conical shell with transversely compliant core under combined load. In this study, a higher order sandwich shell theory is used for analyzing the buckling of composite sandwich conical shell, which is paramount to study of sandwich structures with flexible core.
Herein, the first order shear deformation theory is used for the face sheets and a 3D-elasticity solution of weak core is employed for the flexible core, whose material properties follow a power-low function. Furthermore, the combined loading is applied by simultaneous axial compressive load and external pressure, and equations of motion are derived by utilizing Hamilton principle. To the best knowledge of the researcher, it is the first time that the higher order governing equations of motion are presented for conical composite sandwich shells.

2 GOVERNING EQUATIONS

Consider a circular truncated composite sandwich conical shell as shown in Figure 1. Let R_1 and R_2 indicate radii of the inner and the outer face sheets in the middle of cone length in the normal to middle axis, respectively. α denotes the semi-vertex angle of the cone and L is the cone length along its generator.

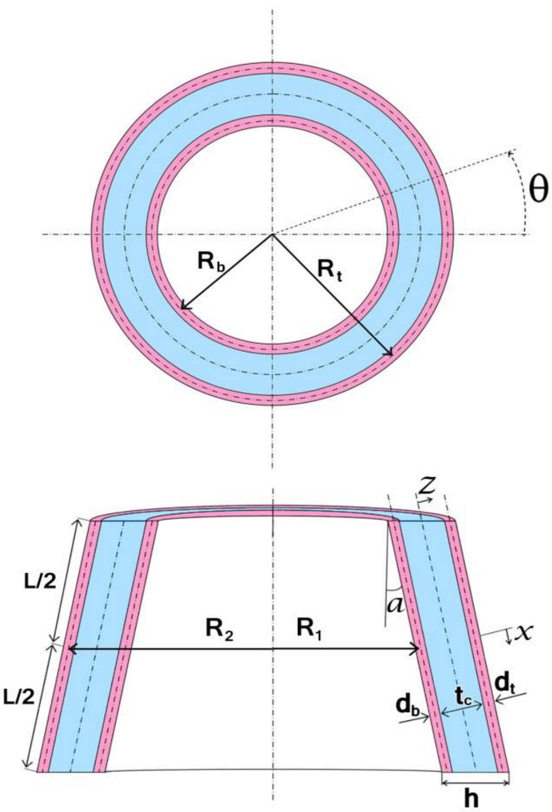

Fig.1 Geometry of the truncated conical composite sandwich shell.

The displacements of any arbitrary point on the middle surface of the shell are denoted by u, v and w in the x, θ and z directions, respectively. In terms of these variables the cone's radius at any point along its length may be expressed as:

$$R_t = R_2 + x\,\sin\alpha = r_t\cos\alpha$$
$$R_b = R_1 + x\,\sin\alpha = r_b\cos\alpha$$
$$r_{tc} = r_t - \frac{d_t}{2}$$
$$r_{bc} = r_b + \frac{d_b}{2} \quad (1)$$

The displacement components for face sheets in the longitudinal direction x, circumferential θ and the thickness z based on first-order shear deformation theory (FSDT) are considered as follows:

$$\bar{u}_i(x,\theta,z) = u_i(x,\theta) + z_i\psi_{xi}$$
$$_i(x,\theta,z) = v_i(x,\theta) + z_i\psi_{\theta i} \quad (2)$$
$$_i(x,\theta,z) = w_i(x,\theta)$$

Where u_i, v_i and w_i are displacement components in the middle surface of face sheets, ψ_{xi} and $\psi_{\theta i}$ are the rotations in the face sheets.
The kinematic equations for the strains in the face sheets are as follow:

$$\varepsilon_{xxi} = \varepsilon_{xx0i} + z_i k_{xxi}$$
$$\varepsilon_{\theta\theta i} = \varepsilon_{\theta\theta 0i} + z_i k_{\theta\theta i}$$
$$\gamma_{x\theta i} = \gamma_{x\theta 0i} + z_i k_{x\theta i}$$
$$\varepsilon_{\theta zi} = \psi_{\theta i} + \frac{w_{i,\theta} - v_i\cos\alpha}{R_i}$$
$$\varepsilon_{xzi} = \psi_{xi} + w_{i,x}$$
$$\varepsilon_{zzi} = 0 \quad (3)$$

The following strain-displacement relations and curvature on the mid-surface face sheets are obtained

$$\varepsilon_{xx0i} = u_{i,x} + \frac{1}{2}w_{i,x}^2$$
$$\varepsilon_{\theta\theta 0i} = \frac{v_{i,\theta} + u_i\sin\alpha + w_i\cos\alpha}{R_i} + \frac{1}{2}\left(\frac{W_{i,\theta}}{R_i}\right)^2$$
$$\gamma_{x\theta 0i} = v_{i,x} + \frac{u_{i,\theta} - v_i\sin\alpha}{R_i} + \frac{w_{i,x}\,w_{i,\theta}}{R_i} \quad (4)$$
$$k_{xxi} = \psi_{xi,x}$$
$$k_{\theta\theta i} = \frac{\psi_{\theta i,\theta} + \psi_{xi}\sin\alpha}{R_i}$$
$$k_{x\theta i} = \psi_{\theta i,x} + \frac{\psi_{xi,\theta} - \psi_{\theta i}\sin\alpha}{R_i}$$

Where $(W_j)_{,x}$ and $(\psi_i)_{,\theta}$ symbols represent the differentiations with respect to x and θ variables, respectively.
The kinematic relations used for the core, assuming small linear deformations, take the following form:

$$\gamma_{\theta rc} = v_{c,r} + \frac{w_{c,\theta} - v_c\cos\alpha}{R}, \qquad \varepsilon_{rrc} = w_{c,r}$$
$$\gamma_{xrc} = w_{c,x} + u_{c,r} \quad (5)$$

In this equations, E_c is elastic modulus of core and G_{xc}, $G_{\theta c}$ are shear modulus at x and θ axis directions, respectively.
The compatibility conditions, which ensure perfect bonding between the faces and the core at the interface layers for both the outer and inner faces, are:

$$\bar{u}_c\ (r=r_{ic}) = \bar{u}_{ci} = u_i + (-1)^k\frac{d_i}{2}\psi_{xi}$$
$$\bar{v}_c\ (r=r_{ic}) = \bar{v}_{ci} = v_i + (-1)^k\frac{d_i}{2}\psi_{\theta i} \quad (6)$$
$$\bar{w}_c\ (r=r_{ic}) = \bar{w}_{ci} = w_i$$

Where d_i (i = t,b) are the thickness of the outer and the inner face sheets, respectively, and k = 1 when i = t, and k = 0 when i = b. The constitutive equations for each face sheet based on classical lamination theory are given by the following stress resultant–displacement relations [24]:

$$\begin{Bmatrix} N_x^i \\ N_\varphi^i \\ N_{x\varphi}^i \end{Bmatrix} = \begin{bmatrix} A_{11i} & A_{12i} & A_{16i} \\ A_{12i} & A_{22i} & A_{26i} \\ A_{16i} & A_{26i} & A_{66i} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx0,i} \\ \varepsilon_{\varphi\varphi 0,i} \\ \gamma_{x\varphi 0,i} \end{Bmatrix} + \begin{bmatrix} B_{11i} & B_{12i} & B_{16i} \\ B_{12i} & B_{22i} & B_{26i} \\ B_{16i} & B_{26i} & B_{66i} \end{bmatrix} \begin{Bmatrix} K_{xx0,i} \\ K_{\varphi\varphi 0,i} \\ K_{x\varphi 0,i} \end{Bmatrix}$$
$$\begin{Bmatrix} M_x^i \\ M_\varphi^i \\ M_{x\varphi}^i \end{Bmatrix} = \begin{bmatrix} B_{11i} & B_{12i} & B_{16i} \\ B_{12i} & B_{22i} & B_{26i} \\ B_{16i} & B_{26i} & B_{66i} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx0,i} \\ \varepsilon_{\varphi\varphi 0,i} \\ \gamma_{x\varphi 0,i} \end{Bmatrix} + \begin{bmatrix} D_{11i} & D_{12i} & D_{16i} \\ D_{12i} & D_{22i} & D_{26i} \\ D_{16i} & D_{26i} & D_{66i} \end{bmatrix} \begin{Bmatrix} K_{xx0,i} \\ K_{\varphi\varphi 0,i} \\ K_{x\varphi 0,i} \end{Bmatrix} \quad (7)$$

Where N_{ikl} and M_{ikl} are stress and moment resultants in the outer and inner face sheets, respectively, A_{mn}, B_{mn} and D_{mn} are extensional, coupling and bending rigidities, respectively, and $\varepsilon_{kl0,i}$ and $k_{kl0,i}$ are the mid-plane strains and the curvatures of the shell. The core material is assumed to behave as a linear elastic and specially (transversely) orthotropic solid material with the following constitutive relations:

$$\sigma_{rrc} = E_c\varepsilon_{rrc}$$
$$\tau_{xrc} = G_{xc}\gamma_{xrc} \quad (8)$$
$$\tau_{\theta rc} = G_{\theta c}\gamma_{\theta rc}$$

In this paper a particular set of laminated cones, namely, regularly anti-symmetric cross-plied cones, are numerically studied. The coefficients in the constitutive Eqs. (7) for this lamination are

$$A_{11i} = A_{22i} = \frac{1}{2}(Q_{11i} + Q_{22i})h, A_{12i} = Q_{12i}h,\ A_{66i} = Q_{66i}h$$
$$B_{11i} = -B_{22i} = \pm\frac{1}{4N}(Q_{11i} - Q_{22i})h^2, B_{12i} = 0$$
$$D_{11i} = D_{22i} = \frac{1}{24}(Q_{11i} + Q_{22i})h^3, D_{12i} = \frac{1}{12}Q_{12i}h^3$$
$$D_{66i} = \frac{1}{12}Q_{66i}h^3,$$
$$A_{k6i} = B_{k6i} = D_{k6i} = 0 \quad where \quad k = 1,2 \quad (9)$$

To derivate the governing equations of motion and the boundary conditions by the Hamilton principle, the energy functional of the conical shell must be obtained that is defined as:

$$\delta(U+V)=0 \tag{10}$$

Where U is the potential energy of the body, V is the potential of external loads. The first variation of the potential energy can be expressed as:

$$\delta U=\iiint_{V_t}(\sigma_{xxt}\delta\varepsilon_{xxt}+\delta_{\theta\theta t}\delta\varepsilon_{\theta\theta t}+\tau_{x\theta t}\delta\gamma_{x\theta t}+\tau_{xzt}\delta\gamma_{xzt}+\tau_{\theta zt}\delta\gamma_{\theta zt})dV_t+\iiint_{V_b}(\sigma_{xxb}\delta\varepsilon_{xxb}+\sigma_{\theta\theta b}\delta\varepsilon_{\theta\theta b}+\tau_{x\theta b}\delta\gamma_{x\theta b}+\tau_{xzb}\delta\gamma_{xzb}+\tau_{\theta zb}\delta\gamma_{\theta zb})dV_b+\iiint_{V_c}(\tau_{xrc}\delta\varepsilon_{xrc}+\tau_{\theta rc}\delta\varepsilon_{\theta rc}+\sigma_{rrc}\delta\varepsilon_{rrc})dV_c \tag{11}$$

Where σ_{xxi}, $\sigma_{\theta\theta i}$, $\tau_{x\theta i}$, $\gamma_{x\theta i}$, $\varepsilon_{\theta\theta i}$, ε_{xxi} $(i=t,b)$ are the in-plane normal and shear stresses and strains in the outer and inner face sheets, respectively:
σ_{rrc}, $\tau_{\theta rc}$, ε_{rrc}, $\gamma_{\theta rc}$, γ_{xrc}, γ_{xrc} are the radial and shear stresses and strains in the core, respectively, and V_t, V_b, V_c are the appropriate volumes of the outer and inner face sheets and the core, respectively. The in-plane stress components containing $\tau_{x\theta c}$, $\sigma_{\theta\theta c}$, σ_{xxc} in core are disregarded.

$$\delta V=-\int_{-l/2}^{l/2}\int_\theta(q_r^t R_t\delta w_t+q_r^b R_b\delta w_b+q_x^t R_t\delta u_t+q_x^b R_b\delta u_b+q_\theta^t R_t\delta v_t+q_\theta^b R_b\delta v_b)\,dx\,d\theta \tag{12}$$

Where u_i, v_i, w_i (i = t,b) are displacements components of the face sheets and q_r^i, q_θ^i, q_x^i are force components on the outer and inner face sheets.
Hence, by using from the obtained relations and applying Hamilton principle, equilibrium equations of conical sandwich shell are derived as follows:

$$-N_x^t\sin\alpha-R_tN_{x,x}^t-N_{x\theta,\theta}^t+N_\theta^t\sin\alpha+\tau_{xrc}(r=r_{tc})r_{tc}\cos\alpha+q_x^tR_t=0 \tag{13}$$

$$-N_x^b\sin\alpha-R_bN_{x,x}^b-N_{x\theta,\theta}^b+N_\theta^b\sin\alpha-\tau_{xrc}(r=r_{bc})r_{bc}\cos\alpha+q_x^bR_b=0 \tag{14}$$

$$2N_{x\theta}^t\sin\alpha-R_tN_{x\theta,x}^t-N_{\theta,\theta}^t-Q_{\theta zt}\cos\alpha+\tau_{\theta rc}(r=r_{tc})r_{tc}\cos\alpha+q_\theta^tR_t=0 \tag{15}$$

$$2N_{x\theta}^b\sin\alpha-R_bN_{x\theta,x}^b-N_{\theta,\theta}^b+Q_{\theta zb}\cos\alpha+\tau_{\theta rc}(r=r_{bc})r_{bc}\cos\alpha+q_\theta^bR_b=0 \tag{16}$$

$$N_\theta^t\cos\alpha-Q_{xzt}\sin\alpha-R_tQ_{xzt,x}-Q_{\theta zt,\theta}-\left[\left(R_tN_x^tw_{t,x}+N_{x\theta}^tw_{t,\theta}\right)_{,x}+\left(\frac{1}{R_t}N_\theta^tw_{t,\theta}+N_{x\theta}^tw_{t,x}\right)_{,\theta}\right]+\sigma_{rrc}(r=r_{tc})r_{tc}\cos\alpha+q_r^tR_t=0 \tag{17}$$

$$N_\theta^b\cos\alpha-Q_{xzb}\sin\alpha-R_bQ_{xzb,x}-Q_{\theta zb,\theta}-\left[\left(R_bN_x^bw_{b,x}+N_{x\theta}^bw_{b,\theta}\right)_{,x}+\left(\frac{1}{R_b}N_{x\theta}^bw_{b,x}+N_{x\theta}^bw_{b,x}\right)_{,\theta}\right]-\sigma_{rrc}(r=r_{bc})r_{bc}\cos\alpha+q_r^bR_b=0 \tag{18}$$

$$-R_tQ_{xzt}-M_x^t\sin\alpha-R_tM_{x,x}^t+M_\theta^t\sin\alpha-M_{x\theta,\theta}^t-r_{tc}\tau_{xrc}(r=r_{tc})\cos\alpha\frac{d_t}{2}=0 \tag{19}$$

$$R_bQ_{xzb}-M_x^b\sin\alpha-R_bM_{x,x}^b+M_\theta^b\sin\alpha-M_{x\theta,x}^b-r_{bc}\tau_{xrc}(r=r_{bc})\cos\alpha\frac{d_b}{2}=0 \tag{20}$$

$$R_tQ_{\theta zt}-M_{\theta,\theta}^t-2M_{x\theta}^t\sin\alpha-R_tM_{x\theta,x}^t-r_{tc}\tau_{\theta rc}(r=r_{tc})\cos\alpha\frac{d_t}{2}=0 \tag{21}$$

$$R_bQ_{\theta zb}-M_{\theta,\theta}^b-2M_{x\theta}^b\sin\alpha-R_bM_{x\theta,x}^b-r_{bc}\tau_{\theta rc}(r=r_{bc})\cos\alpha\frac{d_b}{2}=0 \tag{22}$$

$$\tau_{\theta rc}\cos\alpha+(r\tau_{\theta rc}\cos\alpha)_{,r}=0 \tag{23}$$

$$(r\tau_{xrc}\cos\alpha)_{,r}=0 \tag{24}$$

$$(r\tau_{xrc}\cos\alpha)_{,x}+\tau_{\theta rc,\theta}+(r\sigma_{rrc}\cos\alpha)_{,r}=0 \tag{25}$$

Where N_{kl}^i and $M_{kl}^i(k=x,\theta),(i=t,b)$ are the force and moment resultants respectively. Q_{kzi} is shear forces out of plane per unit length that they are defined as:

$$(N_X^i\ ,\ N_\theta^i\ ,\ N_{x\theta}^i)=\int_{-d_i/2}^{d_i/2}(\sigma_{xxi},\sigma_{\theta\theta i},\tau_{x\theta i})\,dz_i$$
$$(M_x^i\ ,\ M_\theta^i\ ,\ M_{x\theta}^i)=\int_{-d_i/2}^{d_i/2}(\sigma_{xxi},\sigma_{\theta\theta i},\tau_{x\theta i})\,z_idz_i \tag{26}$$

$$Q_{xzi}=\int_{-d_i/2}^{d_i/2}\tau_{xzi}\,dz$$
$$Q_{\theta zi}=\int_{-d_i/2}^{d_i/2}\tau_{\theta zi}\,dz \tag{27}$$

In addition to extraction equilibrium equations using the energy method, boundary conditions also are obtained for the truncated conical composite sandwich shells.
The corresponding boundary conditions for the clamped and simply-supported edges are considered as Clamped-Clamped (C-C):

$$u_i = 0, v_i = 0, w_i = 0, w_i' = 0, \psi_{xi} = 0, \psi_{\theta i} = 0 \quad (28)$$

When $x = \pm\frac{L}{2}$ (i=t,b)

Simply-Supported (S-S):

$$N_x^i = 0, v_i = 0, w_i = 0, M_x^i = 0, M_{x\theta}^i = 0, \tau_{xrc} = 0 \quad (29)$$

when $x = \pm\frac{L}{2}$ (i=t,b)

The obtained equations reduce to the governing equations of cylindrical sandwich shells if the assumption of α = 0 is applied. Evidently, the higher order governing equations for a conical shell are too complex and obtaining the exact analytical solutions sounds impossible. In the following section the solution procedure of the power series is presented to obtain an exact solution to the problem.

3 SOLUTION METHODOLOGY

In order to solve the equations of conical composite sandwich shell with a transversely flexible core, first values of stress and moment resultants are deployed in the Eqs. (13)-(25) according to face sheets displacement components in the longitudinal, circumferential and radial directions. Then the system of partial differential equations of motion in a way that follows the structure is obtained. Thus the equations in terms of the values of an unknown problem involve displacement of longitudinal, circumferential and radial middle plane procedures and component rotation procedures and shear stresses of core, τ_θ, τ_x, $\Psi_{\theta i}$, Ψ_{xi}, w_i, v_i, $u_i (i = t, b)$ offered. As seen from Eqs. (30)-(39), the equations of motion are formulated in terms of the following twelve unknowns: The circumferential and longitudinal displacements of the outer and the inner face sheets (ui, vi , i = t,b), the radial deflections of the outer and the inner face sheets (wi, i = t, b), rotation components of the outer and the inner face sheets in the longitudinal and circumferential direction $(\Psi_{\theta i}, \Psi_{xi}, i = t, b)$ and the two radial core shear stresses (τ_x, τ_θ), respectively

$$\begin{aligned}
&-A_{11t}\left[u_{t,x}\sin\alpha + R_t u_{t,xx}\right] - A_{12t}\left[v_{t,x\theta} + w_{t,x}\cos\alpha\right] - A_{16t}\left[R_t v_{t,xx} + 2u_{t,x\theta}\right] - B_{11t}\left[\Psi_{xt,x}\sin\alpha + R_t\Psi_{xt,xx}\right] \\
&\quad - B_{12t}\Psi_{\theta t,x\theta} - B_{16t}\left[R_t\Psi_{\theta t,xx} + 2\Psi_{xt,x\theta}\right] + A_{22t}\frac{\sin\alpha}{R_t}\left[v_{t,\theta} + u_t\sin\alpha + w_t\cos\alpha\right] \\
&\quad + A_{26t}\left[v_{t,x}\sin\alpha - \frac{\sin^2\alpha}{R_t}v_t - \frac{1}{R_t}\left(v_{t,\theta\theta} + w_{t,\theta}\cos\alpha\right)\right] \\
&\quad + B_{22t}\frac{\sin\alpha}{R_t}\left[\Psi_{\theta t,x}\sin\alpha - \frac{\sin^2\alpha}{R_t}\Psi_{\theta t} - \frac{1}{R_t}\Psi_{\theta t,\theta\theta}\right] - A_{66t}\left[v_{t,x\theta} + \frac{1}{R_t}\left(u_{t,\theta\theta} - v_{t,\theta}\sin\alpha\right)\right] \\
&\quad - B_{66t}\left[\Psi_{\theta t,x\theta} + \frac{1}{R_t}\left(\Psi_{xt,\theta\theta} - \Psi_{\theta t,\theta}\sin\alpha\right)\right] + \tau_x(\theta,x)\cos\alpha = 0
\end{aligned} \quad (30)$$

$$\begin{aligned}
&-A_{11b}\left[u_{b,x}\sin\alpha + R_b u_{b,xx}\right] - A_{12b}\left[v_{b,x\theta} + w_{b,x}\cos\alpha\right] - A_{16b}\left[R_b v_{b,xx} + 2u_{b,x\theta}\right] - B_{11b}\left[\Psi_{xb,x}\sin\alpha + R_b\Psi_{xb,xx}\right] \\
&\quad - B_{12b}\Psi_{\theta b,x\theta} - B_{16b}\left[R_b\Psi_{\theta b,xx} + 2\Psi_{xb,x\theta}\right] + A_{22b}\frac{\sin\alpha}{R_b}\left[v_{b,\theta} + u_b\sin\alpha + w_b\cos\alpha\right] \\
&\quad + A_{26b}\left[v_{b,x}\sin\alpha - \frac{\sin^2\alpha}{R_b}v_b - \frac{1}{R_b}\left(v_{b,\theta\theta} + w_{b,\theta}\cos\alpha\right)\right] + B_{22b}\frac{\sin\alpha}{R_b}\left[\Psi_{\theta b,\theta} + \Psi_{xb}\sin\alpha\right] \\
&\quad + B_{26b}\left[\Psi_{\theta b,x}\sin\alpha - \frac{\sin^2\alpha}{R_b}\Psi_{\theta b} - \frac{1}{R_b}\Psi_{\theta b,\theta\theta}\right] - A_{66b}\left[v_{b,x\theta} + \frac{1}{R_b}\left(u_{b,\theta\theta} - v_{b,\theta}\sin\alpha\right)\right] \\
&\quad - B_{66b}\left[\Psi_{\theta b,x\theta} + \frac{1}{R_b}\left(\Psi_{xb,\theta\theta} - \Psi_{\theta b,\theta}\sin\alpha\right)\right] - \tau_x(\theta,x)\cos\alpha = 0
\end{aligned} \quad (31)$$

$$-A_{12t}u_{t,x\theta}-A_{22t}\frac{1}{R_t}\left[v_{t,\theta\theta}+u_{t,\theta}\sin\alpha+w_{t,\theta}\cos\alpha\right]-A_{26t}\left[2v_{t,x\theta}+u_{t,x}\sin\alpha+w_{t,x}\cos\alpha+\frac{1}{R_t}u_{t,\theta\theta}+\frac{\sin\alpha}{R_t}(u_t\sin\alpha+w_t\cos\alpha)\right]-B_{12t}\Psi_{xt,x\theta}-B_{22t}\frac{1}{R_t}\left[\Psi_{\theta t,\theta\theta}+\Psi_{xt,\theta}\sin\alpha\right]-B_{26t}\left[2\Psi_{\theta t,x\theta}+\Psi_{xt,x}\sin\alpha+\frac{1}{R_t}\Psi_{xt,\theta\theta}+\frac{\sin^2\alpha}{R_t}\Psi_{xt}\right]-A_{16t}\left[2u_{t,x}\sin\alpha+R_tu_{t,xx}\right]-A_{66t}\left[v_{t,x}\sin\alpha+R_tv_{t,xx}+u_{t,x\theta}+\frac{\sin\alpha}{R_t}\left(u_{t,\theta}-v_t\sin\alpha\right)\right]-B_{16t}\left[2\Psi_{xt,x}\sin\alpha+R_t\Psi_{xt,xx}\right]-B_{66t}\left[\Psi_{\theta t,x}\sin\alpha+\Psi_{xt,x\theta}+R_t\Psi_{\theta t,xx}+\frac{\sin\alpha}{R_t}\left(\Psi_{xt,\theta}-\Psi_{\theta t}\sin\alpha\right)\right]-kG_{\theta zt}d_t\left(\Psi_{\theta t}+\frac{w_{t,\theta}-v_t\cos\alpha}{R_t}\right)\cos\alpha+\frac{\tau_\theta(\theta,x)}{r_{tc}}\cos\alpha=0 \tag{32}$$

$$-A_{12b}u_{b,x\theta}-A_{22b}\frac{1}{R_b}\left[v_{b,\theta\theta}+u_{b,\theta}\sin\alpha+w_{b,\theta}\cos\alpha\right]-A_{26b}\left[2v_{b,x\theta}+u_{b,x}\sin\alpha+w_{b,x}\cos\alpha+\frac{1}{R_b}u_{b,\theta\theta}+\frac{\sin\alpha}{R_b}(u_b\sin\alpha+w_b\cos\alpha)\right]-B_{12b}\Psi_{xb,x\theta}-B_{22b}\frac{1}{R_b}\left[\Psi_{\theta b,\theta\theta}+\Psi_{xb,\theta}\sin\alpha\right]-B_{26b}\left[2\Psi_{\theta b,x\theta}+\Psi_{xb,x}\sin\alpha+\frac{1}{R_b}\Psi_{xb,\theta\theta}+\frac{\sin^2\alpha}{R_b}\Psi_{xb}\right]-A_{16b}\left[2u_{b,x}\sin\alpha+R_bu_{b,xx}\right]-A_{66b}\left[v_{b,x}\sin\alpha+R_bv_{b,xx}+u_{b,x\theta}+\frac{\sin\alpha}{R_b}(u_{b,\theta}-v_b\sin\alpha)\right]-B_{16b}\left[2\Psi_{xb,x}\sin\alpha+R_b\Psi_{xb,xx}\right]-B_{66b}\left[\Psi_{\theta b,x}\sin\alpha+\Psi_{xb,x\theta}+R_t\Psi_{\theta b,xx}+\frac{\sin\alpha}{R_b}\left(\Psi_{xb,\theta}-\Psi_{\theta b}\sin\alpha\right)\right]-kG_{\theta zb}d_b\left(\Psi_{\theta b}+\frac{w_{b,\theta}-v_b\cos\alpha}{R_b}\right)\cos\alpha-\frac{\tau_\theta(\theta,x)}{r_{bc}}\cos\alpha=0 \tag{33}$$

$$A_{12t}u_{t,x}\cos\alpha+A_{22t}\frac{\cos\alpha}{R_t}\left[v_{t,\theta}+u_t\sin\alpha+w_t\cos\alpha\right]+A_{26t}\left[v_{t,x}\cos\alpha+\frac{\cos\alpha}{R_t}\left(u_{t,\theta}-v_t\sin\alpha\right)\right]+B_{12t}\Psi_{xt,x}\cos\alpha+B_{22t}\frac{\cos\alpha}{R_t}\left[\Psi_{\theta t,\theta}+\Psi_{xt}\sin\alpha\right]+B_{26b}\left[\Psi_{\theta t,x}\cos\alpha+\frac{\cos\alpha}{R_t}\left(\Psi_{xt,\theta}-\Psi_{\theta t}\sin\alpha\right)\right]-kG_{xzt}d_t\left(\Psi_{xt}+w_{t,x}\right)\sin\alpha-R_tkG_{xzt}d_t\left(\Psi_{xt,x}+w_{t,xx}\right)-kG_{\theta zt}d_t\left(\Psi_{\theta t,t}+\frac{w_{t,\theta\theta}-v_{t,\theta}\cos\alpha}{R_t}\right)-\left[\left(R_tN_{x0}^tw_{t,x}+N_{x\theta 0}^tw_{t,\theta}\right)_{,x}+\left(\frac{1}{R_t}N_{\theta 0}^tw_{t,\theta}+N_{x\theta 0}^tw_{t,x}\right)_{,\theta}\right]+\frac{\tau_{\theta,\theta}}{r_{tc}\cos\alpha}\left[\frac{r_{tc}-r_{bc}}{r_{bc}ln(\frac{r_{bc}}{r_{tc}})}+1\right]\cos\alpha-\tau_{x,x}\left[\frac{r_{tc}-r_{bc}}{ln(\frac{r_{bc}}{r_{tc}})}+r_{tc}\right]\cos\alpha+\frac{E_c}{ln\left(\frac{r_{bc}}{r_{tc}}\right)}(w_b-w_t)\cos\alpha=0 \tag{34}$$

$$A_{12b}u_{b,x}\cos\alpha+A_{22b}\frac{\cos\alpha}{R_b}\left[v_{b,\theta}+u_b\sin\alpha+w_t\cos\alpha\right]+A_{26t}\left[v_{t,x}\cos\alpha+\frac{\cos\alpha}{R_t}\left(u_{t,\theta}-v_t\sin\alpha\right)\right]+B_{12t}\Psi_{xt,x}\cos\alpha-B_{22t}\frac{\cos\alpha}{R_t}\left[\Psi_{\theta t,\theta}+\Psi_{xt}\sin\alpha\right]+B_{26t}\left[\Psi_{\theta t,x}\cos\alpha+\frac{\cos\alpha}{R_t}\left(\Psi_{xt,\theta}-\Psi_{\theta t}\sin\alpha\right)\right]-kG_{xzt}d_t\left(\Psi_{xt}+w_{t,x}\right)\sin\alpha-R_tkG_{xzt}d_t\left(\Psi_{xt,x}+w_{t,xx}\right)-kG_{\theta zt}d_t\left(\Psi_{\theta t,\theta}+\frac{w_{t,\theta\theta}-v_{t,\theta}\cos\alpha}{R_t}\right)-\left[\left(R_bN_{x0}^bw_{b,x}+N_{x\theta 0}^bw_{b,\theta}\right)_{,x}+\left(\frac{1}{R_b}N_{x\theta 0}^bw_{b,\theta}+N_{x\theta 0}^bw_{b,x}\right)_{,\theta}\right]-\frac{\tau_{\theta,\theta}}{r_{bc}\cos\alpha}\left[\frac{r_{tc}-r_{bc}}{r_{tc}\,ln\left(\frac{r_{bc}}{r_{tc}}\right)}+1\right]\cos\alpha+\tau_{x,x}\left[\frac{r_{tc}-r_{bc}}{ln\left(\frac{r_{bc}}{r_{tc}}\right)}+r_{bc}\right]\cos\alpha-\frac{E_c}{ln\left(\frac{r_{bc}}{r_{tc}}\right)}(w_b-w_t)\cos\alpha=0 \tag{35}$$

$$-B_{11t}\left[u_{t,x}\sin\alpha+R_tu_{t,xx}\right]-B_{12t}\left[v_{t,x\theta}+w_{t,x}\cos\alpha\right]-B_{16t}\left[R_tv_{t,xx}+2u_{t,x\theta}\right]-D_{11t}\left[\Psi_{xt,x}\sin\alpha+R_t\Psi_{xt,xx}\right]$$
$$-D_{12t}\Psi_{\theta t,x\theta}-D_{16t}\left[R_t\Psi_{\theta t,xx}+2\Psi_{xt,x\theta}\right]+B_{22t}\frac{\sin\alpha}{R_t}\left[v_{t,\theta}+u_t\sin\alpha+w_t\cos\alpha\right]$$
$$+B_{26t}\left[v_{t,x}\sin\alpha-\frac{1}{R_t}v_t\sin^2\alpha-\frac{1}{R_t}\left(v_{t,\theta\theta}+w_{t,\theta}\cos\alpha\right)\right]+D_{22t}\frac{\sin\alpha}{R_t}\left[\Psi_{\theta t,t}+\Psi_{xt}\sin\alpha\right]$$
$$+D_{26t}\left[\Psi_{\theta t,x}\sin\alpha-\frac{1}{R_t}\Psi_{\theta t}\sin^2\alpha-\frac{1}{R_t}\Psi_{\theta t,\theta\theta}\right]-B_{66t}\left[v_{t,x\theta}\sin\alpha+\frac{1}{R_t}\left(u_{t,\theta\theta}-v_{t,\theta}\sin\alpha\right)\right]$$
$$-D_{66t}\left[\Psi_{\theta t,x\theta}+\frac{1}{R_t}\left(\Psi_{xt,\theta\theta}-\Psi_{\theta t,\theta}\sin\alpha\right)\right]+R_tkG_{xzt}d_t\left(\Psi_{xt}+w_{t,x}\right)-\tau_x(\theta,x)\frac{d_t}{2}\cos\alpha=0 \quad (36)$$

$$-B_{11b}\left[u_{b,x}\sin\alpha+R_bu_{b,xx}\right]-B_{12b}\left[v_{b,x\theta}+w_{b,x}\cos\alpha\right]-B_{16b}\left[R_bv_{b,xx}+2u_{b,x\theta}\right]-D_{11b}\left[\Psi_{xb,x}\sin\alpha+R_b\Psi_{xb,xx}\right]$$
$$-D_{12b}\Psi_{\theta b,x\theta}-D_{16b}\left[R_b\Psi_{\theta b,xx}+2\Psi_{xb,x\theta}\right]+B_{22b}\frac{\sin\alpha}{R_b}\left[V_{b,\theta}+u_b\sin\alpha+w_b\cos\alpha\right]$$
$$+B_{26b}\left[v_{b,x}\sin\alpha-\frac{1}{R_b}v_b\sin^2\alpha-\frac{1}{R_b}\left(v_{b,\theta\theta}+w_{b,\theta}\cos\alpha\right)\right]+D_{22b}\frac{\sin\alpha}{R_b}\left[\Psi_{\theta b,x}+\Psi_{xb}\sin\alpha\right]$$
$$+D_{26b}\left[\Psi_{\theta b,x}\sin\alpha-\frac{1}{R_b}-\Psi_{\theta b}\sin^2\alpha-\frac{1}{R_b}\Psi_{\theta b,\theta\theta}\right]-B_{66b}\left[v_{b,x\theta}+\frac{1}{R_b}\left(u_{b,\theta\theta}-v_{b,\theta}\sin\alpha\right)\right]$$
$$-D_{66b}\left[\Psi_{\theta b,x\theta}+\frac{1}{R_b}\left(\Psi_{xb,\theta\theta}-\Psi_{\theta b,\theta}\sin\alpha\right)\right]+R_bkG_{xzb}d_b\left(\Psi_{xb}+w_{b,x}\right)-\tau_x(\theta,x)\frac{d_b}{2}\cos\alpha=0 \quad (37)$$

$$-B_{12t}u_{t,x\theta}-B_{22t}\frac{v_{t,\theta\theta}+u_{t,\theta}\sin\alpha+w_{t,\theta}\cos\alpha}{R_t}-B_{26t}\left[2v_{t,x\theta}+u_{t,x}\sin\alpha+w_{t,x}\cos\alpha\right.$$
$$\left.+\frac{u_{t,\theta\theta}+u_t\sin^2\alpha+w_t\sin\alpha\cos\alpha}{R_t}\right]-D_{12t}\Psi_{xt,x\theta}-D_{22t}\frac{\Psi_{\theta t,\theta\theta}+\Psi_{xt,\theta}\sin\alpha}{R_t}-D_{26t}\left[2\Psi_{\theta t,x\theta}\right.$$
$$\left.+\Psi_{xt,x}sin\alpha+\frac{\Psi_{xt,\theta\theta}+\Psi_{xt}\sin^2\alpha}{R_t}\right]-B_{16t}\left[2u_{t,x}\sin\alpha+R_tu_{t,xx}\right]-B_{66t}\left[v_{t,x}\sin\alpha+u_{t,x\theta}+R_tv_{t,xx}\right.$$
$$\left.+\frac{u_{t,\theta}\sin\alpha-v_t\sin^2\alpha}{R_t}\right]-D_{16t}\left[2\Psi_{xt,x}\sin\alpha+R_t\Psi_{xt,xx}\right]-D_{66t}\left[\Psi_{\theta t,x}\sin\alpha+R_t\Psi_{\theta t,xx}+\Psi_{xt,x\theta}\right.$$
$$\left.+\frac{\Psi_{xt,\theta}\sin\alpha-\Psi_{\theta t}\sin^2\alpha}{R_t}\right]+R_tkG_{\theta zt}d_t\left(\Psi_{\theta t}+\frac{w_{t,\theta}-v_t\cos\alpha}{R_t}\right)-\frac{\tau_\theta}{r_{tc}}\frac{d_t}{2}\cos\alpha=0 \quad (38)$$

$$-B_{12b}u_{b,x\theta}-B_{22b}\frac{v_{b,\theta\theta}+u_{b,\theta}\sin\alpha+w_{b,\theta}cos\alpha}{R_b}-B_{26b}\left[2v_{b,x\theta}+u_{b,x}\sin\alpha+w_{b,x}\cos\alpha+\right.$$
$$\left.\frac{u_{b,\theta\theta}+u_b\sin^2\alpha+w_b\sin\alpha\cos\alpha}{R_b}\right]-D_{12b}\Psi_{xb,x\theta}-D_{22b}\frac{\Psi_{\theta b,\theta\theta}+\Psi_{xb,\theta}\sin\alpha}{R_b}-D_{26b}\left[2\Psi_{\theta b,x\theta}+\Psi_{xb,x}\sin\alpha+\right.$$
$$\left.\frac{\Psi_{xb,\theta\theta}+\Psi_{xb}\sin^2\alpha}{R_b}\right]-B_{16b}\left[2u_{b,x}\sin\alpha+R_bu_{b,xx}\right]-B_{66b}\left[v_{b,x}\sin\alpha+u_{b,x\theta}+R_bv_{b,xx}+\right.$$
$$\left.\frac{u_{b,\theta}\sin\alpha-v_b\sin^2\alpha}{R_b}\right]-D_{16b}\left[2\Psi_{xb,x}\sin\alpha+R_b\Psi_{xb,xx}\right]-D_{66b}\left[\Psi_{\theta b,x}\sin\alpha+R_b\Psi_{\theta b,xx}+\Psi_{xb,x\theta}+\frac{\Psi_{xb,\theta}\sin\alpha-\Psi_{\theta b}\sin^2\alpha}{R_b}\right]+$$
$$R_bkG_{\theta zb}d_b\left(\Psi_{\theta b}+\frac{w_{b,\theta}-v_b\cos\alpha}{R_b}\right)-\frac{\tau_\theta}{r_{bc}}\frac{d_b}{2}\cos\alpha=0 \quad (39)$$

Following the solution procedure outlined in [8], let us assume the solutions of Eqs. (30)-(39) as the following:

$$u_i(x,\theta)=\sum_{m=0}^{\infty}a_{mi}x^m\cos n\theta$$
$$v_i(x,\theta)=\sum_{m=0}^{\infty}b_{mi}x^m\sin n\theta$$
$$w_i(x,\theta)=\sum_{m=0}^{\infty}c_{mi}x^m\cos n\theta$$
$$\psi_{xi}(x,\theta)=\sum_{m=0}^{\infty}d_{mi}x^m\cos n\theta \quad (40)$$
$$\psi_{\theta i}(x,\theta)=\sum_{m=0}^{\infty}f_{mi}x^m\sin n\theta$$
$$\tau_x(x,\theta)=\sum_{m=0}^{\infty}g_mx^m\cos n\theta$$

$$\tau_\theta(x,\theta)=\sum_{m=0}^{\infty} h_m x^m \sin n\theta$$

Where n is an integer representing the circumferential wave number of the buckled shell, a_{mi}, b_{mi}, c_{mi}, d_{mi}, f_{mi}, g_m and h_m (i=t,b) are constants to be determined by means of recursive expressions.

Substituting Eqs. (40) into Eqs. (30)-(39) for a finite number of terms in the series, using Eqs. (1), and matching the terms of the same order in x, we develop twelve linear algebraic equations and obtain the recursive relations. As a consequence of the complexity of the equations, it is not possible to obtain a closed form recursive expression. Fortunately, since the governing equations are linear, the superposition principle can be utilized to simplify the solution. A total number of 22 base functions (fundamental solutions) are needed. The final solution is in fact the summation of the fundamental solutions.

Recurrence relations allow that unknown constants $a_{mi}, b_{mi}, c_{mi}, d_{mi}, f_{mi}$, g_m and h_m (i=t,b) and (m ≥ 2) are expressed in terms of $a_{0i}, a_{1i}, b_{0i}, b_{1i}, c_{0i}, c_{1i}, d_{0i}, d_{1i}, f_{0i}, f_{1i}, g_0$ and g_1 . Therefore, the solutions to Eqs. (13)-(25) are:

$$\begin{aligned} u_i(x,\theta)=[&u_1(x)a_{0t}+u_2(x)a_{1t}+u_3(x)a_{0b}+u_4(x)a_{1b}+u_5(x)b_{0t}+u_6(x)b_{1t}+u_7(x)b_{0b}+u_8(x)b_{1b}+u_9(x)c_{0t}\\ &+u_{10}(x)c_{1t}+u_{11}(x)c_{0b}+u_{12}(x)c_{1b}+u_{13}(x)d_{0t}+u_{14}(x)d_{1t}+u_{15}(x)d_{0b}+u_{16}(x)d_{1b}\\ &+u_{17}(x)f_{0t}+u_{18}(x)f_{1t}+u_{19}(x)f_{0b}+u_{20}(x)f_{1b}+u_{21}(x)g_0+u_{22}(x)g_1]\cos n\theta \quad (i=t,b)\end{aligned} \tag{41}$$

$$\begin{aligned} v_i(x,\theta)=[&v_1(x)a_{0t}+v_2(x)a_{1t}+v_3(x)a_{0b}+v_4(x)a_{1b}+v_5(x)b_{0t}+v_6(x)b_{1t}+v_7(x)b_{0b}+v_8(x)b_{1b}+v_9(x)c_{0t}\\ &+v_{10}(x)c_{1t}+v_{11}(x)c_{0b}+v_{12}(x)c_{1b}+v_{13}(x)d_{0t}+v_{14}(x)d_{1t}+v_{15}(x)d_{0b}+v_{16}(x)d_{1b}\\ &+v_{17}(x)f_{0t}+v_{18}(x)f_{1t}+v_{19}(x)f_{0b}+v_{20}(x)f_{1b}+v_{21}(x)g_0+v_{22}(x)g_1]\cos n\theta \quad (i=t,b)\end{aligned} \tag{42}$$

$$\begin{aligned} w_i(x,\theta)=[&w_1(x)a_{0t}+w_2(x)a_{1t}+w_3(x)a_{0b}+w_4(x)a_{1b}+w_5(x)b_{0t}+w_6(x)b_{1t}+w_7(x)b_{0b}+w_8(x)b_{1b}\\ &+w_9(x)c_{0t}+w_{10}(x)c_{1t}+w_{11}(x)c_{0b}+w_{12}(x)c_{1b}+w_{13}(x)d_{0t}+w_{14}(x)d_{1t}+w_{15}(x)d_{0b}\\ &+w_{16}(x)d_{1b}+w_{17}(x)f_{0t}+w_{18}(x)f_{1t}+w_{19}(x)f_{0b}+w_{20}(x)f_{1b}+w_{21}(x)g_0\\ &+w_{22}(x)g_1]\cos n\theta \quad (i=t,b)\end{aligned} \tag{43}$$

$$\begin{aligned} \Psi_i(x,\theta)=[&\Psi_{x1}(x)a_{0t}+\Psi_{x2}(x)a_{1t}+\Psi_{x3}(x)a_{0b}+\Psi_{x4}(x)a_{1b}+\Psi_{x5}(x)b_{0t}+\Psi_{x6}(x)b_{1t}+\Psi_{x7}(x)b_{0b}+\Psi_{x8}(x)b_{1b}\\ &+\Psi_{x9}(x)c_{0t}+\Psi_{x10}(x)c_{1t}+\Psi_{x11}(x)c_{0b}+\Psi_{12}(x)c_{1b}+\Psi_{13}(x)d_{0t}+\Psi_{14}(x)d_{1t}+\Psi_{15}(x)d_{0b}\\ &+\Psi_{x16}(x)d_{1b}+\Psi_{x17}(x)f_{0t}+\Psi_{x18}(x)f_{1t}+\Psi_{x19}(x)f_{0b}+\Psi_{x20}(x)f_{1b}+\Psi_{x21}(x)g_0\\ &+\Psi_{x22}(x)g_1]\cos n\theta \quad (i=t,b)\end{aligned} \tag{44}$$

$$\begin{aligned} \Psi_i(x,\theta)=[&\Psi_{\theta1}(x)a_{0t}+\Psi_{\theta2}(x)a_{1t}+\Psi_{\theta3}(x)a_{0b}+\Psi_{\theta4}(x)a_{1b}+\Psi_{\theta5}(x)b_{0t}+\Psi_{\theta6}(x)b_{1t}+\Psi_{\theta7}(x)b_{0b}+\Psi_{\theta8}(x)b_{1b}\\ &+\Psi_{\theta9}(x)c_{0t}+\Psi_{\theta10}(x)c_{1t}+\Psi_{\theta11}(x)c_{0b}+\Psi_{\theta12}(x)c_{1b}+\Psi_{\theta13}(x)d_{0t}+\Psi_{\theta14}(x)d_{1t}+\Psi_{\theta15}(x)d_{0b}\\ &+\Psi_{\theta16}(x)d_{1b}+\Psi_{\theta17}(x)f_{0t}+\Psi_{\theta18}(x)f_{1t}+\Psi_{\theta19}(x)f_{0b}+\Psi_{\theta20}(x)f_{1b}+\Psi_{\theta21}(x)g_0\\ &+\Psi_{\theta22}(x)g_1]\mathit{sin}\, n\theta \quad (i=t,b)\end{aligned} \tag{45}$$

$$\begin{aligned} \tau_i(x,\theta)=[&T_{x1}(x)a_{0t}+T_{x2}(x)a_{1t}+T_{x3}(x)a_{0b}+T_{x4}(x)a_{1b}+T_{x5}(x)b_{0t}+T_{x6}(x)b_{1t}+T_{x7}(x)b_{0b}+T_{x8}(x)b_{1b}\\ &+T_{x9}(x)c_{0t}+T_{x10}(x)c_{1t}+T_{x11}(x)c_{0b}+T_{12}(x)c_{1b}+T_{13}(x)d_{0t}+T_{14}(x)d_{1t}+T_{15}(x)d_{0b}\\ &+T_{x16}(x)d_{1b}+T_{x17}(x)f_{0t}+T_{x18}(x)f_{1t}+T_{x19}(x)f_{0b}+T_{x20}(x)f_{1b}+T_{x21}(x)g_0\\ &+T_{x22}(x)g_1]\cos n\theta \quad (i=t,b)\end{aligned} \tag{46}$$

$$\begin{aligned} \tau_i(x,\theta)=[&T_{\theta1}(x)a_{0t}+T_{\theta2}(x)a_{1t}+T_{\theta3}(x)a_{0b}+T_{\theta4}(x)a_{1b}+T_{\theta5}(x)b_{0t}+T_{\theta6}(x)b_{1t}+T_{\theta7}(x)b_{0b}+T_{\theta8}(x)b_{1b}\\ &+T_{\theta9}(x)c_{0t}+T_{\theta10}(x)c_{1t}+T_{\theta11}(x)c_{0b}+T_{\theta12}(x)c_{1b}+T_{\theta13}(x)d_{0t}+\Psi_{\theta14}(x)d_{1t}+T_{\theta15}(x)d_{0b}\\ &+T_{\theta16}(x)d_{1b}+T_{\theta17}(x)f_{0t}+T_{\theta18}(x)f_{1t}+T_{\theta19}(x)f_{0b}+T_{\theta20}(x)f_{1b}+T_{\theta21}(x)g_0\\ &+T_{\theta22}(x)g_1]\cos n\theta \qquad (i=t,b)\end{aligned} \tag{47}$$

Where $u_i, v_i, w_i, \Psi_{xi}, \Psi_{\theta i}, \tau_x, \tau_\theta$ $(i=t,b)$ are the fundamental solutions of the equations of motion.

As an example, the first base functions $u_1(x), v_1(x), w_1(x), \Psi_{x1}(x), \Psi_{\theta1}, T_{x1}$ and $T_{\theta1}$ are obtained in terms of the natural frequency by assuming that

$a_{0t} = 1$

$$a_{1t} = a_{0b} = a_{1b} = b_{0t} = b_{1t} = b_{0b} = b_{1b} = c_{0t} = c_{1t} = c_{0b} = c_{1b} = d_{0t} = d_{1t} = d_{0b} = d_{1b} = f_{0t} = f_{1t} = f_{0b} = f_{1b} = g_0 = g_1 = 0 \quad (48)$$

Now, to determine the unknown coefficients, the boundary conditions are applied to the Eqs. (41)-(47). To have a nontrivial solution for the unknown coefficients, the determinant of the coefficient matrix obtained from applying the boundary conditions at both ends of the cone must vanish which gives the critical buckling loads of the conical composite sandwich shell.

4 VALIDATION AND NUMERICAL RESULTS

As the first case study, the convergence of the solution procedure is studied. To this aim, the two boundary conditions of clamped-clamped (C-C) and simply-simply (S-S) edges are considered and the convergence of the critical buckling load is studied. Table1 shows that the number of expansion terms for the C-C boundary conditions is less than those of S-S boundary conditions.

Table 1 Effect of expansion terms on the convergence of the critical buckling load (α = 30, h = 0.02)

	n=0		n=1		n=2	
m	C-C	S-S	C-C	S-S	C-C	S-S
10	0.25e6	0.35e6	0.05e6	0.25e6	0.15e6	0.15e6
20	0.05e6	0.15e6	0.15e6	0.25e6	0.15e6	0.05e6
30	0.05e6	0.25e6	0.05e6	0.05e6	0.35e6	0.05e6
40	0.15e6	0.05e6	**0.15e6**	0.15e6	0.05e6	0.05e6
50	**0.05e6**	0.15e6	0.15e6	**0.25e6**	0.05e6	0.05e6
60	0.05e6	0.25e6	0.15e6	0.25e6	**0.15e6**	0.35e6
70	0.05e6	0.15e6	0.15e6	0.25e6	0.15e6	**0.05e6**
80	0.05e6	**0.05e6**	0.15e6	0.25e6	0.15e6	0.05e6
90	0.05e6	0.05e6	0.15e6	0.25e6	0.15e6	0.05e6
100	0.05e6	0.05e6	0.15e6	0.25e6	0.15e6	0.05e6

Material properties of conical sandwich shell applied for buckling analysis in this research are listed in Table 2.

Table 2 Material properties of conical sandwich shell for buckling analysis [22]

Face sheets	Core
E_1=131Gpa, $E_2=E_3$=10.34Gpa $G_{12}=G_{23}$=6.895Gpa G_{13}=6.205Gpa $\upsilon_{12}=\upsilon_{13}$=0.22 υ_{23}=0.49 ρ=1627kg/m^3	$E_1=E_2=E_3$=0.00689Gpa $G_{12}=G_{23}=G_{13}$=0.00345Gpa υ =0 ρ=97kg/m^3

In order to show the validity and feasibility of the buckling analysis of composite sandwich conical shells using improved higher-order sandwich shell theory, some comparisons are made with the results of a finite element software. After buckling analysis on each boundary condition, the frequency parameters are obtained by using solid elements (6 degrees of freedom). The obtained results from the FEM software and exact solution are tabulated in Table 3.

Therefore, anti-symmetric cross-ply laminated composite sandwich conical shells (L/R_1 = 8) having lamination schemes as (0/90/core/0/90) and (0/90/0/90/core/0/90/0/90) with simply supported and clamped boundary conditions are considered. Some geometric parameters such as thickness have significant effects on the dynamic behavior of the shell. The effect of thickness on the buckling loads is considered in Table4. To this aim conical shell with the semi-vertex angle of α = 30◦ is considered which have h/R_1 = 0.1 to h/R_1 = 0.4. Assuming R_1 and face thickness to core thickness ratio are constant, It is shown that any increase in the thickness ratio results in an increase in the buckling load. Results also show that for any of the C-C and S-S boundary conditions and at any thickness ratio, increasing the thickness to radius ratio increases the corresponding frequency parameter.

Table 3 Comparisons of buckling loads obtained via analytical and FEM methods

	S-S				C-C			
	Analytical		FEM		Analytical		FEM	
Lamination/ critical loads	p_{cr}(N)	q_{cr} (pa)	p_{cr}(N)	q_{cr} (pa)	p_{cr}(N)	q_{cr} (pa)	p_{cr}(N)	q_{cr} (pa)
(0°/90°/0°/90°/core/0°/90°/0°/90°)	2.9e5	1.2e5	2.98e5	1.21e5	2.85e5	1.2e5	2.97e5	1.21e5

(0°/90°/core/0°/90°)	1.3e5	0.9e5	1.31e5	0.9471e5	1.3e5	0.9e5	1.31e5	0.9477e5

Table 4 Effect of thickness on frequency parameter for the conical composite sandwich shell

	S-S		C-C	
h/R1	Analytical	FEM	Analytical	FEM
0.1	0.77	0.77978	0.78	0.78009
0.2	1.5	1.5062	1.501	1.5064
0.4	3	3.013	3.01	3.0131

5 CONCLUSION

Buckling analysis of conical composite sandwich shells has been performed through an Improved Higher-order Sandwich Shell theory. The buckling analysis of composite truncated sandwich conical shells with higher- order theory are presented for the first time. The principle of minimum potential energy has been used to obtain the governing equation and to find the solution for buckling problem. Results show that the implemented method gives the exact results with a finite number of expansion terms. As the thickness ratio increases, the number of terms needed for a reasonable convergence decreases, and the buckling load increases. The results obtained by the analytical method have been compared with the numerical results from FEM analysis and good agreements have been reached.

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