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Research Article

Solving Binary Systems of Fractional Integro-Differential Equations by Taylor Wavelets

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Abstract

In this paper, we give a new approach based on Taylor wavelets to solve fractional binary systems of integro-differential equations (FIDEs). To do this, first, we present the function approximation by using Taylor wavelets as well as the operational matrix of fractional integration of these wavelets. Then, by approximating the fractional derivatives of the solutions of the main problem in terms of the Taylor wavelets and using the operational matrix of fractional integration, we approximate the solutions of the main problem. By substituting these approximations in the FIDEs, we obtain a system of nonlinear algebraic equations. Finally, by the help of the proposed method, we solve some numerical examples and show the accuracy and applicability of the method.

Keywords : Taylor wavelets; Fractional integro-differential equations; Binary systems; Nonlinear algebraic equations.

1 Introduction

Fractional calculus which includes integro-differential of any arbitrary order can be considered an old but very important topic in mathematics for its noticeable role in other scientific disciplines.
Many phenomena in fields of physics, chemistry, economics, engineering and other sciences can be explained in form of mathematical models by fractional calculus. In recent years, the frequent appearance of flow mechanism, viscoelastic, biology, electrochemical and other engineering technical issues, have led researchers to do a lot of tasks in this field. Today, many methods have been devised to solve such problems. Scientists have applied these methods to find the exact solution or an approximate one that has the least possible absolute error [1, 2, 3, 4]. One of these procedures is the wavelet method. The wavelets opinion is a relatively new topic which for many mathematicians and researchers is a powerful tool in their researches. The wavelets method is widely used in many engineering and scientific disciplines such as signal processing, time frequency,

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analysis and quick algorithms for simple implementation. Wavelets is a particular type of oscillatory functions that can be used to approximate unknown functions [5, 6, 7]. The most important wavelets ever have been applied to approximate unknown functions are Haar, Legendre, Chebyshev, Bernoulli and CAS wavelets [8, 9, 10, 11, 12]. Each series of wavelets are made based on its polynomials. Orthogonal polynomials are used to solve many fractional order differential equations. It helps us reduce these equations to a system of algebraic equations by using the operational derivative or integral matrix. Althoug Taylor polynomials are not orthogonal, the operational matrix can be calculated for them [13, 14].
Since there has been less task on binary systems of FIDEs, it has encouraged us to solve them numerically. In this way, we have chosen Taylor wavelets method because it has the least absolute error, accuracy and simplicity. Even under certain conditions, approximate solutions of examples will be the exact solution. The general form of these examples is a coupled systems of FIDEs as follows [13]

$$\begin{cases} D^{\alpha}F(x) = f_1(x, F(x), G(x)) \\ \qquad\qquad + \int_0^x f_2(t, F(t), G(t))dt, \\ \\ D^{\beta}G(x) = g_1(x, F(x), G(x)) \\ \qquad\qquad + \int_0^x g_2(t, F(t), G(t))dt, \end{cases} \tag{1.1}$$

where $x, t \in [0, 1], \alpha, \beta \in (0, 1]$, and D^{α}, D^{β} display the Caputo derivative operator.

2 Basic concepts of fractional calculus

We devoted this section to important basic concepts that are needed [15].

Definition 2.1 *The Riemann-Liouville fractional integral for order $m \geq 0$ is a function that is defined as*

$$I^m R(y) = \begin{cases} \frac{1}{\Gamma(m)} \int_0^y (y-t)^{m-1} F(t)dt, \\ \qquad\qquad m > 0, \\ \\ F(y), \qquad\qquad m = 0, \end{cases} \tag{2.2}$$

where

$$\Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t} dt.$$

Definition 2.2 *The Caputo fractional derivative for $m > 0$ has the following definition*

$$D^m f(y) = \begin{cases} \frac{1}{\Gamma(n-m)} \int_0^y (y-t)^{n-m-1} f^{(n)}(t)dt, \\ \qquad\qquad m > 0, n-1 < m < n, \\ \\ \frac{d^{(n)} f(y)}{dy^n}, \qquad m = n, \end{cases} \tag{2.3}$$

where $y \geq 0$, and $n = 1, 2, 3, \ldots$.

The operator Riemann-Liouville integral and Caputo derivative for $y > 0$ has the following relationships

$$D^m I^m f(y) = f(y),$$
$$I^m D^m f(y) = f(y) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0^+)}{j!} y^j, \; k-1 < m < k. \tag{2.4}$$

3 Function approximation and error analysis

3.1 Taylor wavelets

Taylor wavelets are defined on $[0, 1]$ as [16]

$$\Psi_{ij}(y) = \begin{cases} 2^{\alpha-1} \widehat{T}_j(2^{\alpha-1}y - i + 1), \\ \qquad \frac{i-1}{2^{\alpha-1}} \leq y < \frac{i}{2^{\alpha-1}}, \\ \\ 0, \qquad otherwise, \end{cases} \tag{3.5}$$

where $i = 1, 2, \cdots, 2^{\alpha-1}$, $j = 0, 1, 2, \cdots, \beta - 1$, $(\alpha, \beta \in \mathcal{N})$ and $\widehat{T}_j(y) = \sqrt{2j+1} T_j(y)$. The coefficient $\sqrt{2j+1}$ is for normality and $T_j(y)$ are the well-known Taylor polynomials of order j that are written as $T_j(y) = t^j$ and they form a perfect basis on $[0, 1]$ [14].

3.2 Function approximation by using Taylor wavelets

For any function defined on $L^2[0, 1]$ we can have

$$f(y) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} a_{ij} \psi_{ij}(y) = A^T \Psi(y), \tag{3.6}$$

where

$$a_{ij} = \langle f(y), \psi_{ij}(y) \rangle = A^T \Psi(y)$$
$$= \int_0^1 w(y) f(y) \psi_{ij}(y) dy. \tag{3.7}$$

In (3.7), $\langle , \rangle$ displays the linner product with weight function $w(y)$.
The infinite series in Eq. (3.6) can be written as the finite series

$$f(y) = \sum_{i=1}^{2^{\alpha-1}} \sum_{j=0}^{\beta-1} a_{ij}\psi_{ij}(y) = A^T\Psi(y), \qquad (3.8)$$

where A and $\Psi(y)$ are $2^{\alpha-1}\beta \times 1$ matrices given by

$$A = [a_{10}, a_{11}, \cdots, a_{1(\beta-1)}, a_{20}, a_{21}, \cdots, a_{2(\beta-1)}, \cdots, a_{2^{\alpha-1}0}, a_{2^{\alpha-1}1}, \cdots, a_{2^{\alpha-1}(\beta-1)}]^T,$$

$$\begin{aligned}\Psi(y) =&[\psi_{10}(y), \psi_{11}(y), \cdots, \psi_{1(\beta-1)}(y), \psi_{20}(y), \\ &\psi_{21}(y), \cdots, \psi_{2(\beta-1)}(y), \cdots, \psi_{2^{\alpha-1}0}(y), \\ &\psi_{2^{\alpha-1}1}(y), \cdots, \psi_{2^{\alpha-1}(\beta-1)}(y)]^T.\end{aligned}$$

For simplicity we write Eq. (3.8) as follows

$$f_k(y) \approx \sum_{n=1}^{k} a_n\psi_n(y) = A_k^T\Psi_k(y) = \tilde{f}_k(y), \quad (3.9)$$

where $a_n = a_{ij}$, $\psi_n = \psi_{ij}$, $k = 2^{\alpha-1}\beta$, $n = \beta(i-1)+j+1$ and $\tilde{f}_k(y)$ is the best approximation of $f(y)$.
We choose collocation points as $y_n = \dfrac{2n-1}{2k}$ that the Taylor wavelet matrix can be written as

$$\phi_{k\times k} = [\Psi(\frac{1}{2k})\ \Psi(\frac{3}{2k})\ \Psi(\frac{5}{2k})\ \cdots\ \Psi(\frac{2n-1}{2k})]. \qquad (3.10)$$

3.3 Error analysis

Now we find the error bound of the approximate $f(y)$ by using Taylor wavelets.

Theorem 3.1 *Suppose $f(y) \in C^n[c,d]$ and $g_n(y)$ be interpolating polynomial of degree n that agrees with $f(y)$ at the Chebyshev nodes on $[c,d]$. Then we have [17]*

$$\|f(y) - g_n(y)\| \leq \frac{2}{n!}(\frac{d-c}{4})^n \max_{\delta\in[c,d]}|f^{(n)}(\delta)|. \qquad (3.11)$$

Theorem 3.2 *Let $f(y) \in C^n[0,1]$ and*

$$f(y) \approx \sum_{i=1}^{2^{\alpha-1}} \sum_{j=0}^{\beta-1} a_{ij}\psi_{ij}(y) = A^T\Psi(y).$$

Then, the error bound obtain as

$$\|f(y) - A^T\Psi(y)\| \leq \frac{2}{n!\,4^n 2^{n(\alpha-1)}} \max_{\delta\in[0,1]}|f^{(n)}(\delta)|. \qquad (3.12)$$

Proof. We divide the interval $[0,1]$ to $2^{\alpha-1}$ subintervals $I_{\alpha,i} = [\frac{i-1}{2^{\alpha-1}}, \frac{i}{2^{\alpha-1}}]$, $i = 1, 2, \cdots, 2^{\alpha-1}$.

Since $A^T\Psi(y)$ is the best approximation then by using Theorem(3.1) we have

$$\begin{aligned}\|f(y) - A^T\Psi(y)\|_2^2 &= \int_0^1 [f(y) - A^T\Psi(y)]^2 dy \\ &= \sum_{i=1}^{2^{\alpha-1}} \int_{I_{\alpha,i}} [f(y) - A^T\Psi(y)]^2 dy \\ &\leq \sum_{i=1}^{2^{\alpha-1}} \int_{I_{\alpha,i}} [f(y) - g_n(y)]^2 dy \\ &\leq \sum_{i=1}^{2^{\alpha-1}} \int_{I_{\alpha,i}} \Big[\frac{2}{n!}\Big(\frac{1/2^{\alpha-1}}{4}\Big)^n \max_{\delta\in I_{\alpha,i}}|f^{(n)}(\delta)|\Big]^2 dy \\ &\leq \sum_{i=1}^{2^{\alpha-1}} \int_{I_{\alpha,i}} \Big[\frac{2}{n!}\Big(\frac{1/2^{\alpha-1}}{4}\Big)^n \max_{\delta\in[0,1]}|f^{(n)}(\delta)|\Big]^2 dy \\ &= \Big[\frac{2}{n!\,4^n 2^{n(\alpha-1)}} \max_{\delta\in[0,1]}|f^{(n)}(\delta)|\Big]^2 \\ &\leq \Big[\frac{2}{n!\,4^n 2^{n(\alpha-1)}} \max_{\delta\in[0,1]}|f^{(n)}(\delta)|\Big],\end{aligned}$$

where $g_n(y)$ is a polynomial that interpolates $f(y)$ at the Chebyshev nodes on $I_{\alpha,i}$.

4 Operational matrix of fractional order integration

4.1 Block pulse functions

Block pulse functions (BPFs) are defined as [18]

$$b_i(y) = \begin{cases} 1, & \dfrac{(i-1)}{k} \leq y < \dfrac{i}{k}, \\ 0, & otherwise \end{cases} \qquad (4.13)$$

where $i = 1, 2, \cdots, k$ and $k = 2^{\alpha-1}\beta$.
The BPFs have the following two characteristics

$$b_i(y)b_j(y) = \begin{cases} b_i(y), & i = j, \\ 0, & i \neq j, \end{cases} \tag{4.14}$$

$$\int_0^y b_i(y)b_j(y)dy = \begin{cases} \frac{1}{k}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4.15}$$

Definition 4.1 *Let* $P = [p_1, p_2, \cdots, p_k]^T$ *and* $Q = [q_1, q_2, \cdots, q_k]^T$.
We define $P \odot Q = [p_1q_1, p_2q_2, \cdots, p_kq_k]^T$ *and* $P^2 = [p_1^2, p_2^2, \cdots, p_k^2]^T$.

Lemma 4.1 *Suppose that functions* $f(y)$ *and* $g(y)$ *on* $L^2[0,1]$ *are defined, so that* $f(y) \approx F^T B_k(y)$ *and* $g(y) \approx G^T B_k(y)$ *where* $F^T = [f_1, f_2, \cdots, f_k]$, $G^T = [g_1, g_2, \cdots, g_k]$ *and* $B_k^T(y) = [b_1, b_2, \cdots, b_k]^T$.
Then

$$f(y)g(y) = F^T B_k(y) G^T B_k(y) \approx (F^T \odot G^T) B_k(y), \tag{4.16}$$

$$f(y)^2 \approx (F^T B_k(y))^2 = (F^T)^2 B_k(y). \tag{4.17}$$

Proof. By using the properties of BPFs, the proof is distinct.

4.2 Taylor wavelets operational matrix of fractional order integration

The integration of Taylor wavelets $\Psi(y)$ can be obtained as [19]

$$I\Psi_k(y) = \int_0^y \Psi_k(x)dx \approx P_{k\times k}\Psi_k(y), \tag{4.18}$$

where $P_{k\times k}$ is named the integral operational matrix of Taylor wavelets and k displays dimension, also $P^m_{k\times k}$ is named the fractional order integration operational matrix of Taylor wavelets and achieved from

$$I^m\Psi_k(y) \approx P^m_{k\times k}\Psi_k(y). \tag{4.19}$$

The following relation can be deduced by using Taylor wavelets matrix $\phi_{k\times k}$ in (3.10) and the definition of BPFs.

$$\Psi_k(y) \approx \phi_{k\times k} B_k(y). \tag{4.20}$$

The fractional order integration operational matrix of BPFs can be expressed as [20]

$$I^m B_k(y) \approx F^m B_k(y), \tag{4.21}$$

we get from (4.19)-(4.21)

$$\begin{aligned} P^m_{k\times k}\Psi_k(y) &\approx I^m\Psi_k(y) \\ &\approx I^m\phi_{k\times k}B_k(y) = \phi_{k\times k}I^m B_k(y) \\ &\approx \phi_{k\times k}F^m B_k(y) \\ &\approx \phi_{k\times k}F^m\phi^{-1}_{k\times k}\Psi_k(y). \end{aligned} \tag{4.22}$$

Finally, we obtain from (4.22)

$$P^m_{k\times k} \approx \phi_{k\times k}F^m\phi^{-1}_{k\times k}. \tag{4.23}$$

It is necessary to mention that general form matrix F^m and $\phi_{k\times k}$ explained as follow [21]

$$F^m = (\frac{1}{k})^m \frac{1}{\Gamma(m+2)} \begin{bmatrix} 1 & \sigma_1 & \sigma_2 & \cdots & \sigma_{k-1} \\ 0 & 1 & \sigma_1 & \cdots & \sigma_{k-2} \\ 0 & 0 & 1 & \cdots & \sigma_{k-3} \\ 0 & 0 & 0 & \cdots & \sigma_{k-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and $\sigma_i = (i+1)^{m+1} - 2pi^{m+1} + (i-1)^{m+1}$, $i = 1, 2, \ldots, k-m$. Also the matrix $\phi_{k\times k}$ is as

$$\phi_{r\times r} = \begin{bmatrix} T & 0 & 0 & \cdots & 0 \\ 0 & T & 0 & \cdots & 0 \\ 0 & 0 & T & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & T \end{bmatrix},$$

where T is matrix $\beta \times \beta$ as

$$T=\begin{bmatrix} \psi_{10}(\frac{1}{2k}) & \psi_{10}(\frac{3}{2k}) & \psi_{10}(\frac{5}{2k}) & \cdots & \psi_{10}(\frac{2n-1}{2k}) \\ \psi_{11}(\frac{1}{2k}) & \psi_{11}(\frac{3}{2k}) & \psi_{11}(\frac{5}{2k}) & \cdots & \psi_{11}(\frac{2n-1}{2k}) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \psi_{2^{\alpha-1}(\beta-1)}(\frac{1}{2k}) & \psi_{2^{\alpha-1}(\beta-1)}(\frac{3}{2k}) & \psi_{2^{\alpha-1}(\beta-1)}(\frac{5}{2k}) & \cdots & \psi_{2^{\alpha-1}(\beta-1)}(\frac{2n-1}{2k}) \end{bmatrix}.$$

Now we compute marices F^m, $\phi_{k\times k}$ and $P^m_{k\times k}$ for $\alpha=2$ implies $i=1,2$, $\beta=4$ implies $\beta=0,1,2,3$ and $m=0.5$, $(k=2^{\alpha-1}\beta)$,

$$F^{0.5}=\begin{bmatrix} 0.2660 & 0.2203 & 0.1434 & 0.1160 & 0.1001 & 0.0894 & 0.0816 & 0.0755 \\ 0 & 0.2660 & 0.2203 & 0.1434 & 0.1160 & 0.1001 & 0.0894 & 0.0816 \\ 0 & 0 & 0.2660 & 0.2203 & 0.1434 & 0.1160 & 0.1001 & 0.0894 \\ 0 & 0 & 0 & 0.2660 & 0.2203 & 0.1434 & 0.1160 & 0.1001 \\ 0 & 0 & 0 & 0 & 0.2660 & 0.2203 & 0.1434 & 0.1160 \\ 0 & 0 & 0 & 0 & 0 & 0.2660 & 0.2203 & 0.1434 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2660 & 0.2203 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2660 \end{bmatrix},$$

$$\phi_{8\times 8}=\begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 & 0 \\ 0.3062 & 0.9186 & 1.5309 & 2.1433 & 0 & 0 & 0 & 0 \\ 0.0494 & 0.4447 & 1.2353 & 2.4211 & 0 & 0 & 0 & 0 \\ 0.0073 & 0.1973 & 0.9135 & 2.5066 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0 & 0.3062 & 0.9186 & 1.5309 & 2.1433 \\ 0 & 0 & 0 & 0 & 0.0494 & 0.4447 & 1.2353 & 2.4211 \\ 0 & 0 & 0 & 0 & 0.0073 & 0.1973 & 0.9135 & 2.5066 \end{bmatrix},$$

$$P^{0.5}_{8\times 8}=\begin{bmatrix} 0.1115 & 0.7959 & -0.5405 & 0.1994 & 0.6861 & -0.5673 & 0.5029 & -0.1923 \\ 0.0127 & 0.1442 & 0.4030 & -0.0911 & 0.7206 & -0.7244 & 0.6819 & -0.2664 \\ 0.0052 & -0.0257 & 0.2523 & 0.1686 & 0.6814 & -0.7522 & 0.7275 & -0.2871 \\ -0.0121 & 0.0943 & -0.2814 & 0.5558 & 0.6363 & -0.7408 & 0.7273 & -0.2887 \\ 0 & 0 & 0 & 0 & 0.1115 & 0.7959 & -0.5405 & 0.1994 \\ 0 & 0 & 0 & 0 & 0.0127 & 0.1442 & 0.4030 & -0.0911 \\ 0 & 0 & 0 & 0 & 0.0052 & -0.0257 & 0.2523 & 0.1686 \\ 0 & 0 & 0 & 0 & -0.0121 & 0.0943 & -0.2814 & 0.5558 \end{bmatrix}.$$

5 Numerical examples

To demonstrate the accuracy and the efficiency of the proposed method based on the Taylor wavelets we solve the following two examples chosen the [13] and show their graphs for different values of y.

Example 5.1 *[13]*

$$\begin{cases} D^i u(t) = -\frac{1}{2}u^2(t) - v(t) \\ \quad - \int_0^t v(\tau)u(\tau)d\tau + \frac{1}{2},\ 0 < i \leq 1, \\ \\ D^j v(t) = v^2(t) - u^2(t) - \int_0^t v(\tau)d\tau, \\ \quad 0 < j \leq 1, \end{cases} \tag{5.24}$$

where the initial conditions are $u(0) = 1$ and $v(0) = 0$. The exact solutions by $u(t) = \cos t$ and $v(t) = \sin t$ are achieved only for $i = j = 1$ and for $i, j \in (0, 1)$ are unknown.

Let

$$\begin{cases} D^i u(t) \approx A_k^T \Psi_k(t), \\ \\ D^j v(t) \approx C_k^T \Psi_k(t), \end{cases} \tag{5.25}$$

where $A_k^T = [a_1, a_2, a_3, \cdots, a_k]$ and $C_k^T = [c_1, c_2, c_3, \cdots, c_k]$. By using Eqs.(2.4), (4.19), (4.20) and (5.25), we have

$$\begin{cases} u(t) = I^i D^i u(t) + u(0) \approx A_k^T p^i_{k\times k}\Psi_k(t) \\ \quad +1 \approx A_k^T p^i_{k\times k}\phi_{k\times k}B_k(t) + 1, \\ \\ v(t) = I^j D^j v(t) + v(0) \approx C_k^T p^j_{k\times k}\Psi_k(t) \\ \quad \approx C_k^T p^j_{k\times k}\phi_{k\times r}B_k(t), \end{cases} \tag{5.26}$$

from Eqs.(4.16)-(4.18) and (5.26), we have

$$u^2(t) \approx (A_k^T p^i_{k\times k}\phi_{k\times k})^2 B_k(t) + 2A_k^T p^i_{k\times k}\phi_{k\times k}B_k(t) + 1, \tag{5.27}$$

$$v^2(t) \approx C_k^T p^j_{k\times k}\phi_{k\times k})^2 B_k(t),$$

$$\int_0^t v(\tau)d\tau \approx \int_0^t C_k^T p^j_{k\times k}\Psi_k(\tau)d\tau \approx C_k^T p^{1+j}_{k\times k}\phi_{k\times k}B_k(t). \tag{5.28}$$

$$\begin{aligned} v(t)u(t) &\approx (C_k^T p^j_{k\times k}\phi_{k\times k} \\ &\quad B_k(t))(A_k^T p^i_{k\times k}\phi_{k\times k}B_k(t) + 1) \\ &= (C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k}) \\ &\quad B_k(t) + C_k^T p^j_{k\times k}\phi_{k\times k}B_k(t). \end{aligned}$$

$$\begin{aligned} &\int_0^t v(\tau)u(\tau)d\tau \approx \int_0^t (C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k})B_k(\tau)d\tau \\ &+ \int_0^t C_k^T p^j_{k\times k}\phi_{k\times k}B_k(\tau)d\tau \\ &= (C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k}) \int_0^t B_k(\tau)d\tau \\ &+ (C_k^T p^j_{k\times k}\phi_{k\times k}) \int_0^t B_k(\tau)d\tau, \\ &\approx (C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k}) \int_0^t \phi^{-1}_{k\times k}\Psi_k(\tau)d\tau \\ &+ C_k^T p^j_{k\times k}\phi_{k\times k} \int_0^t \phi^{-1}_{k\times k}\Psi_k(\tau)d\tau \\ &\approx (C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k})\phi^{-1}_{k\times k}p_{k\times k}\phi_{k\times k}B_k(t) \\ &+ C_k^T p^{1+j}_{k\times k}\phi_{k\times k}B_k(t). \end{aligned} \tag{5.29}$$

By replacing Eqs. (4.20), and (5.25)-(5.29) into Eq. (5.24), and by the properties of BPFs, we obtain

$$\begin{cases} A_k^T \phi_{k\times k} = \\ -\frac{1}{2}(A_k^T p^i_{k\times k}\phi_{k\times k})^2 - A_k^T p^i_{k\times k}\phi_{k\times k} \\ -C_k^T p^j_{k\times k}\phi_{k\times k} \\ -(C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k} \\ +C_k^T p^j_{k\times k}\phi_{k\times k})\phi^{-1}_{k\times k}p_{k\times k}\phi_{k\times k}, \\ \\ C_k^T \phi_{k\times k} = \\ (C_k^T p^j_{k\times k}\phi_{k\times k})^2 + (A_k^T p^i_{k\times k}\phi_{k\times k})^2 \\ +2A_k^T p^i_{k\times k}\phi_{k\times k} - C_k^T p^{1+j}_{k\times k}\phi_{k\times k} \\ +[1, 1, \ldots, 1]_{1\times k}. \end{cases} \tag{5.30}$$

Now Eq. (5.24) has been converted to Eq.(5.30) and this is a system of nonlinear algebraic equations that has $2k$ unknown coefficients, A_k and C_k, which by calculating nknown coefficients dissolves.

The numerical conclusion of Example (5.1) are displayed in the Tables 1-3. These tables including the approximate and the exact solutions and for distinct values t, k, i and j absolute errors also has been shown.

If you pay attention, will find out that by increasing α and β value of u and v converge to exact solutions, particularly when $i, j \to 1$.

Also convergence functions u and v in figures 1-5 is apparent.

Table 1: Numerical results for Example 5.1 for different valuse of i and j when $\alpha = 4$ and $\beta = 5$, ($k = 2^{\alpha-1}\beta = 40$ and $n = 2, 6, 10, \cdots, 34, 38$).

	i=0.75, j=0.75		i=0.80, j=0.80		i=0.85, j=0.85		i=1, j=1		Exact solution	
$t_n = \frac{2n-1}{2k}$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$
$t_2 = 0.0375$	0.9943	0.0921	0.9962	0.0772	0.9974	0.0646	0.9992	0.0375	0.9993	0.0375
$t_6 = 0.1375$	0.9620	0.2415	0.9709	0.2165	0.9778	0.1941	0.9905	0.1370	0.9906	0.1371
$t_{10} = 0.2375$	0.9151	0.3554	0.9311	0.3297	0.9444	0.3046	0.9719	0.2352	0.9719	0.2353
$t_{14} = 0.3375$	0.8590	0.4480	0.8809	0.4259	0.9002	0.4025	0.9435	0.3311	0.9436	0.3311
$t_{18} = 0.4375$	0.7970	0.5230	0.8231	0.5076	0.8472	0.4891	0.9058	0.4236	0.9058	0.4237
$t_{22} = 0.5375$	0.7316	0.5818	0.7597	0.5755	0.7870	0.5645	0.8590	0.5119	0.8590	0.5120
$t_{26} = 0.6375$	0.6649	0.6248	0.6926	0.6297	0.7213	0.6283	0.8036	0.5950	0.8036	0.5952
$t_{30} = 0.7375$	0.5990	0.6521	0.6238	0.6700	0.6517	0.6802	0.7402	0.6723	0.7402	0.6724
$t_{34} = 0.8375$	0.5357	0.6635	0.5549	0.6959	0.5795	0.7195	0.6694	0.7428	0.6693	0.7430
$t_{38} = 0.9375$	0.4768	0.6590	0.4877	0.7069	0.5065	0.7455	0.5919	0.8059	0.5918	0.8061

Table 2: Numerical results for Example 5.1 for different valuse of i and j when $\alpha = 4$ and $\beta = 6$, ($k = 2^{\alpha-1}\beta = 48$ and $n = 4, 8, 12, \cdots, 36, 40$).

	i=0.85, j=0.85		i=0.88, j=0.88		i=0.95, j=0.95		i=1, j=1		Exact solution	
$t_n = \frac{2n-1}{2k}$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$
$t_4 = 0.0729$	0.9924	0.1138	0.9938	0.1042	0.9962	0.0847	0.9973	0.0728	0.9973	0.0729
$t_8 = 0.1563$	0.9725	0.2159	0.9765	0.2026	0.9839	0.1740	0.9878	0.1556	0.9878	0.1556
$t_{12} = 0.2396$	0.9436	0.3068	0.9506	0.2921	0.9640	0.2592	0.9714	0.2373	0.9714	0.2373
$t_{16} = 0.3229$	0.9073	0.3890	0.9171	0.3745	0.9368	0.3408	0.9483	0.3173	0.9483	0.3173
$t_{20} = 0.4063$	0.8646	0.4633	0.8771	0.4503	0.9029	0.4184	0.9186	0.3951	0.9186	0.3952
$t_{24} = 0.4896$	0.8167	0.5298	0.8314	0.5194	0.8628	0.4918	0.8825	0.4702	0.8825	0.4703
$t_{28} = 0.5729$	0.7643	0.5885	0.7807	0.5816	0.8168	0.5604	0.8403	0.5420	0.8403	0.5421
$t_{32} = 0.6563$	0.7085	0.6390	0.7259	0.6366	0.7655	0.6239	0.7923	0.6100	0.7923	0.6102
$t_{36} = 0.7396$	0.6501	0.6812	0.6677	0.6840	0.7095	0.6818	0.7388	0.6739	0.7387	0.6740
$t_{40} = 0.8229$	0.5901	0.7146	0.6070	0.7235	0.6491	0.7337	0.6801	0.7330	0.6801	0.7331

Table 3: Absolute error for Tables 1 and 2 when $i = j = 1$.

t_n	e_u	e_v	t_n	e_u	e_v
t_2	$7.7905e-05$	$8.7865e-06$	t_4	$5.3709e-05$	$1.1862e-05$
t_6	$7.5336e-05$	$3.2182e-05$	t_8	$5.1753e-05$	$2.5389e-05$
t_{10}	$6.9722e-05$	$5.5448e-05$	t_{12}	$4.8318e-05$	$3.8843e-05$
t_{14}	$6.0970e-05$	$7.8484e-05$	t_{16}	$4.3368e-05$	$5.2186e-05$
t_{18}	$4.9026e-05$	$1.0121e-04$	t_{20}	$3.6880e-05$	$6.5382e-05$
t_{22}	$3.3868e-05$	$1.2358e-04$	t_{24}	$2.8844e-05$	$7.8411e-05$
t_{26}	$1.5508e-05$	$1.4562e-04$	t_{28}	$1.9259e-05$	$9.1270e-05$
t_{30}	$6.0239e-06$	$1.6747e-04$	t_{32}	$8.1339e-06$	$1.0399e-04$
t_{34}	$3.0691e-05$	$1.8939e-04$	t_{36}	$4.5181e-06$	$1.1663e-04$
t_{38}	$5.8465e-05$	$2.1755e-04$	t_{40}	$1.8681e-05$	$1.2930e-04$

Table 4: Numerical for Example 5.2 for different valuse of i and j when $\alpha = 5$ and $\beta = 4$, ($k = 2^{\alpha-1}\beta = 64$ and $n = 1, 8, 15, \cdots, 56, 63$).

	i=0.65, j=0.65		i=0.75, j=0.75		i=0.85, j=0.85		i=1, j=1		Exact solution	
$t_n = \dfrac{2n-1}{2k}$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$
$t_1 = 0.0078$	0.0449	0.0041	0.0274	0.0015	0.0167	0.0006	0.0078	0.0001	0.0078	0.0000
$t_8 = 0.1172$	0.2653	0.1062	0.2144	0.0610	0.1700	0.0341	0.1172	0.0138	0.1172	0.0137
$t_{15} = 0.2266$	0.3913	0.2441	0.3445	0.1623	0.2952	0.1043	0.2266	0.0514	0.2266	0.0513
$t_{22} = 0.3359$	0.4858	0.3936	0.4527	0.2884	0.4085	0.2025	0.3359	0.1129	0.3359	0.1129
$t_{29} = 0.4453$	0.5617	0.5447	0.5466	0.4310	0.5134	0.3242	0.4453	0.1983	0.4453	0.1983
$t_{36} = 0.5547$	0.6251	0.6910	0.6298	0.5842	0.6117	0.4658	0.5547	0.3077	0.5547	0.3077
$t_{43} = 0.6641$	0.6799	0.8283	0.7044	0.7428	0.7042	0.6242	0.6640	0.4410	0.6641	0.4410
$t_{49} = 0.7578$	0.7223	0.9365	0.7629	0.8800	0.7796	0.7711	0.7578	0.5743	0.7578	0.5743
$t_{56} = 0.8672$	0.7687	1.0503	0.8258	1.0378	0.8633	0.9262	0.8671	0.7520	0.8672	0.7520
$t_{63} = 0.9766$	0.8140	1.1500	0.8844	1.1903	0.9428	1.1427	0.9765	0.9536	0.9766	0.9537

Table 5: Numerical results for Example 5.2 for different valuse i and j when $\alpha = 4$ and $\beta = 7$, ($k = 2^{\alpha-1}\beta = 56$ and $n = 5, 10, 15, \cdots, 45, 50$).

	i=0.79, j=0.79		i=0.85, j=0.85		i=0.92, j=0.92		i=1, j=1		Exact solution	
$t_n = \frac{2n-1}{2k}$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$
$t_5 = 0.0804$	0.1459	0.0268	0.1236	0.0180	0.1013	0.0113	0.0804	0.0065	0.0804	0.0065
$t_{10} = 0.1695$	0.2605	0.0867	0.2319	0.0639	0.2011	0.0443	0.1696	0.0289	0.1696	0.0288
$t_{15} = 0.2589$	0.3591	0.1680	0.3297	0.1306	0.2958	0.0963	0.2589	0.0671	0.2589	0.0670
$t_{20} = 0.3482$	0.4476	0.2657	0.4206	0.2150	0.3871	0.1657	0.3482	0.1213	0.3482	0.1213
$t_{25} = 0.4375$	0.5284	0.3764	0.5061	0.3148	0.4756	0.2515	0.4375	0.1915	0.4375	0.1914
$t_{30} = 0.5268$	0.6029	0.4973	0.5872	0.4280	0.5619	0.3527	0.5268	0.2776	0.5268	0.2775
$t_{35} = 0.6696$	0.6721	0.6260	0.6643	0.5528	0.6460	0.4685	0.6160	0.3796	0.6161	0.3795
$t_{40} = 0.7054$	0.7367	0.7603	0.7379	0.6878	0.7282	0.5980	0.7053	0.4976	0.7054	0.4975
$t_{45} = 0.7946$	0.7973	0.8981	0.8082	0.8311	0.8086	0.7403	0.7946	0.6315	0.7946	0.6315
$t_{50} = 0.8839$	0.8544	1.0374	0.8757	0.9813	0.8873	0.8946	0.8839	0.7813	0.8839	0.7813

Table 6: Absolute error for Tables 4 and 5 when $i = j = 1$.

t_n	e_u	e_v	t_n	e_u	e_v
t_1	$3.1972e-07$	$6.0948e-05$	t_5	$4.3955e-06$	$7.8119e-05$
t_8	$4.9669e-06$	$5.8974e-05$	t_{10}	$9.5450e-06$	$7.5057e-05$
t_{15}	$9.9215e-06$	$5.5396e-05$	t_{15}	$1.4941e-05$	$7.0541e-05$
t_{22}	$1.5136e-05$	$5.0071e-05$	t_{20}	$2.0552e-05$	$6.4466e-05$
t_{29}	$2.0567e-05$	$4.2846e-05$	t_{25}	$2.6348e-05$	$5.6721e-05$
t_{36}	$2.6179e-05$	$3.3552e-05$	t_{30}	$3.2303e-05$	$4.7188e-05$
t_{43}	$3.1940e-05$	$2.2010e-05$	t_{35}	$3.8394e-05$	$3.5738e-05$
t_{49}	$3.6972e-05$	$1.0178e-05$	t_{40}	$4.4599e-05$	$2.2238e-05$
t_{56}	$4.2930e-05$	$6.0733e-06$	t_{45}	$5.7271e-05$	$6.5424e-06$
t_{63}	$4.8959e-05$	$2.5177e-05$	t_{50}	$6.3700e-05$	$1.1503e-05$

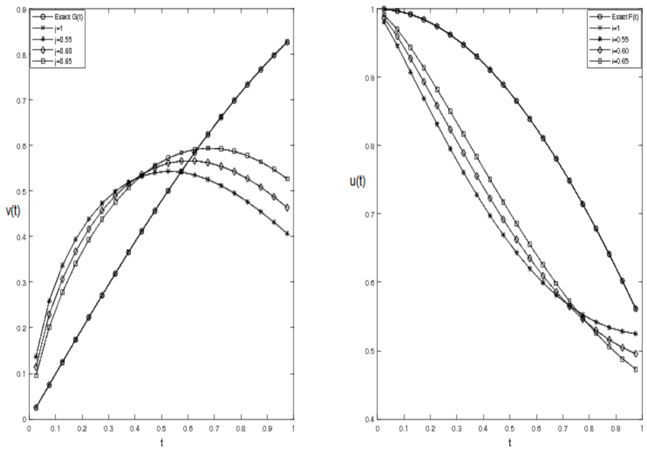

Figure 1: Numerical results for different values of i and j, when $\alpha = 3$, $\beta = 5$ and $k = 20$.

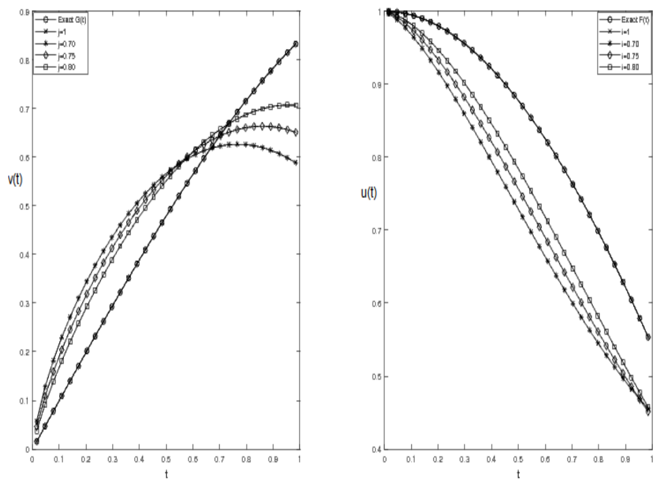

Figure 2: Numerical results for different values of i and j, when $\alpha = 4$, $\beta = 4$ and $k = 32$.

Example 5.2 *[13]*

$$\begin{cases} D^i u(t) = \dfrac{1}{3}v(t)u(t) - v(t) + 1 - \int_0^t [v(\tau) \\ \quad -2u(\tau)]d\tau, \qquad 0 < i \leq 1, \\ \\ D^j v(t) = \dfrac{1}{3}v(t)u(t) + \dfrac{1}{2}u^2(t) + 2u(t)\int_0^t [v(\tau) \\ \quad +u(\tau)]d\tau, \qquad 0 < j \leq 1, \end{cases} \tag{5.31}$$

which the initial conditions are $u(0) = 0$ *and* $v(0) = 0$. *The exact solutions by* $u(t) = t$ *and* $v(t) = t^2$ *are obtained only for* $i = j = 1$ *and for* $i, j \in (0, 1)$ *are unknown.*
Let

$$\begin{cases} D^i u(t) \approx A_k^T \Psi_k(t), \\ \\ D^j v(t) \approx C_k^T \Psi_k(t), \end{cases} \tag{5.32}$$

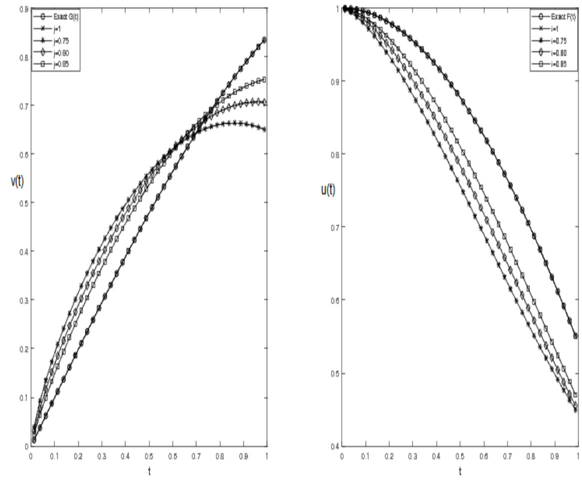

Figure 3: [Related to Table (1)]. Numerical results for different values of i and j, when $\alpha = 4$, $\beta = 5$ and $k = 40$.

where $A_k^T = [a_1, a_2, a_3, \ldots, a_k]$ *and* $C_k^T = [c_1, c_2, c_3, \ldots, c_k]$.
By using Eqs.(2.4), (4.19), (4.20) and (5.32) we have

$$\begin{cases} u(t) = I^i D^i u(t) + u(0) \approx A_k^T p^i_{k\times k}\psi_k(t) \\ \qquad \approx A_k^T p^i_{k\times k}\phi_{k\times k}B_k(t), \\ \\ v(t) = I^j D^j v(t) + v(0) \approx C_k^T p^j_{k\times k}\psi_k(t) \\ \qquad \approx C_k^T q^s_{n'\times n'}\phi_{k\times k}B_k(t). \end{cases} \tag{5.33}$$

From Eqs.(4.16)-(4.18) and (5.33), we obtain

$$v(t)u(t) \approx (C_k^T p^j_{k\times k}\phi_{k\times k}B_k(t))(A_k^T p^i_{k\times k}\phi_{k\times n'}B_k(t)) \\ = (C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k})B_k(t). \tag{5.34}$$

$$u^2(t) \approx (A_k^T p^i_{k\times k}\phi_{k\times k}B_k(t))^2 \\ = (A_k^T p^i_{k\times k}\phi_{k\times k})^2 B_k(t). \tag{5.35}$$

$$\int_0^t u(\tau)d\tau \approx \int_0^t A_k^T p^i_{k\times k}\psi_k(\tau)d\tau \\ = A_k^T p^i_{k\times k}\int_0^t \psi_k(\tau)d\tau \\ \approx A_k^T p^i_{k\times k}p^1_{k\times k}\psi_k(t) \\ \approx A_k^T p^{1+i}_{k\times k}\phi_{k\times k}B_k(t). \tag{5.36}$$

$$\int_0^t v(\tau)d\tau \approx \int_0^t C_k^T p^j_{k\times k}\psi_k(\tau)d\tau \\ \approx C_k^T p^{1+j}_{k\times k}\phi_{k\times k}B_k(t). \tag{5.37}$$

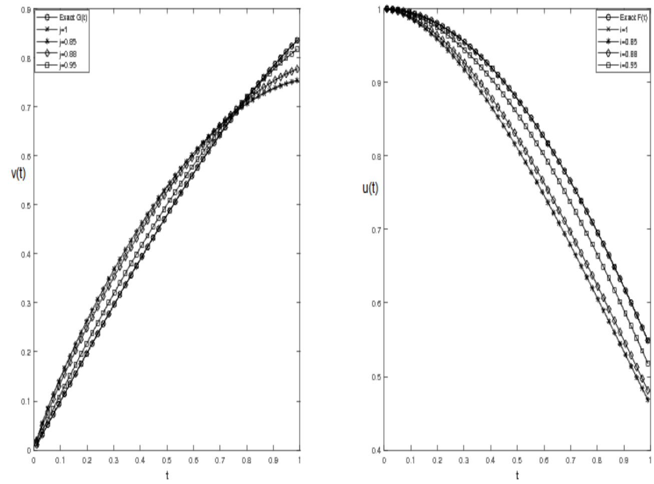

Figure 4: [Related to Table (2)]. Numerical results for different values of i and j, when $\alpha = 4$, $\beta = 6$ and $k = 48$.

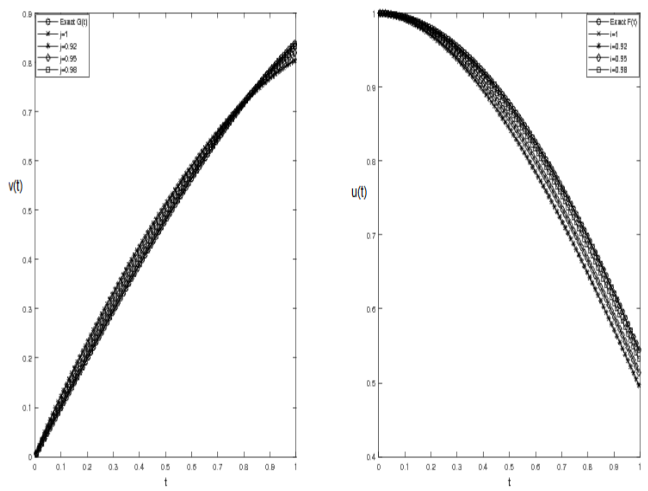

Figure 5: Numerical results for different values of i and j, when $\alpha = 5$, $\beta = 6$ and $k = 96$.

By replacing the Eqs.(4.20), (5.32), and (5.34)-(5.37) into Eq.(5.31), we get

$$\begin{cases} A_k^T \phi_{k\times k} B_k(t) = \\ \dfrac{1}{3}(B_k p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k})B_k(t) \\ -C_k^T p^j_{k\times k}\phi_{k\times k}B_k(t) + [1,1,\ldots,1]_{1\times r}B_k(t) \\ -C_k^T q^{1+j}_{k\times k}\phi_{k\times k}B_k(t) \\ +2A_k^T p^{1+i}_{k\times k}\phi_{k\times k}B_k(t), \\ \\ C_k^T \phi_{k\times k} = \\ \dfrac{1}{3}(C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k})B_k(t) \\ +\dfrac{1}{2}(A_k^T p^i_{k\times k}\phi_{k\times k})^2 B_k(t) \\ +2A_k^T p^i_{k\times k}\phi_{k\times k}B_k(t) \\ -C_k^T p^{1+j}_{k\times k}\phi_{k\times k}B_k(t) - A_k^T p^{1+i}_{k\times k}\phi_{k\times k}B_k(t). \end{cases} \tag{5.38}$$

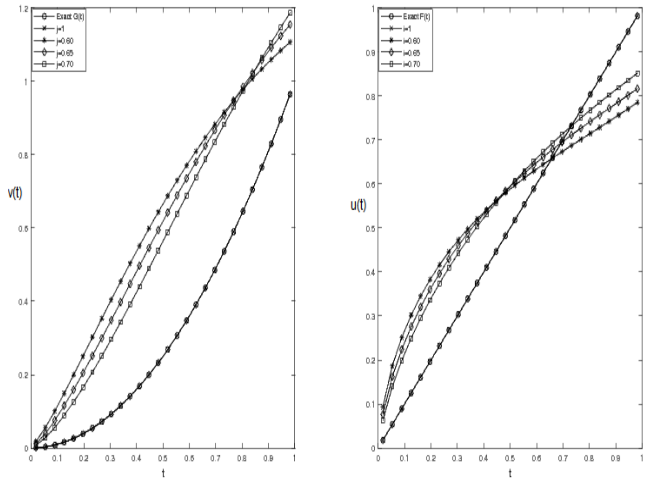

Figure 6: Numerical results for different values of i and j, when $\alpha = 3$, $\beta = 7$ and $k = 28$.

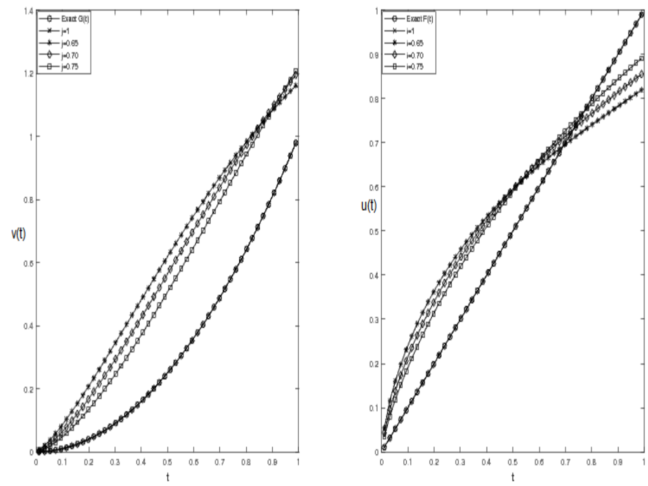

Figure 7: Numerical results for different values of i and j, when $\alpha = 4$, $\beta = 6$ and $k = 48$.

By using the properties of BPFs, we obtain

$$\begin{cases} A_k^T \phi_{k\times k} = \\ \dfrac{1}{3}(C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k}) \\ -C_k^T p^j_{k\times k}\phi_{k\times k} + [1,1,\ldots,1]_{1\times k} \\ -C_k^T p^{1+j}_{k\times k}\phi_{k\times k} + 2A_k^T p^{1+i}_{k\times k}\phi_{k\times k}, \\ \\ A_k^T \phi_{k\times k} = \\ \dfrac{1}{3}(C_k^T p^j_{k\times k}\phi_{k\times k} \odot A_k^T p^i_{k\times k}\phi_{k\times k}) \\ +\dfrac{1}{2}(A_k^T p^i_{k\times k}\phi_{k\times k})^2 + 2A_k^T p^i_{k\times k}\phi_{k\times k} \\ -C_k^T p^{1+j}_{k\times k}\phi_{k\times k} - A_k^T p^{1+i}_{k\times k}\phi_{k\times k}. \end{cases} \tag{5.39}$$

Now Eq.(5.31) has been converted to Eq.(5.39) and this is a system of nonlinear algebraic equations that has $2k$ unknown coefficients, A_k and C_k, which by calculating nknown coefficients dis-

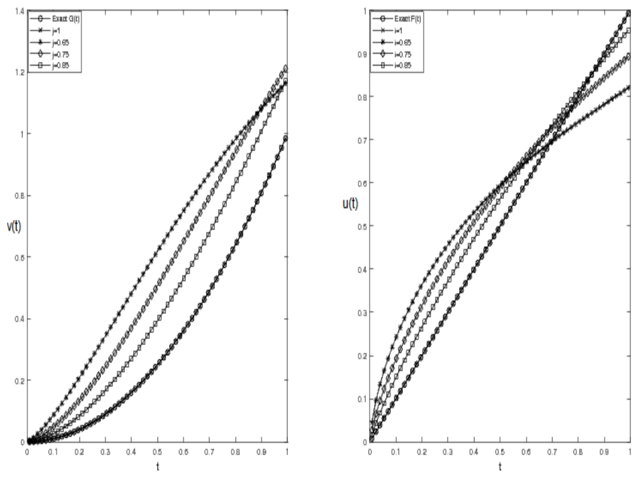

Figure 8: [Related to Table (4)]. Numerical results for different values of i and j, when $\alpha = 5$, $\beta = 4$ and $k = 64$.

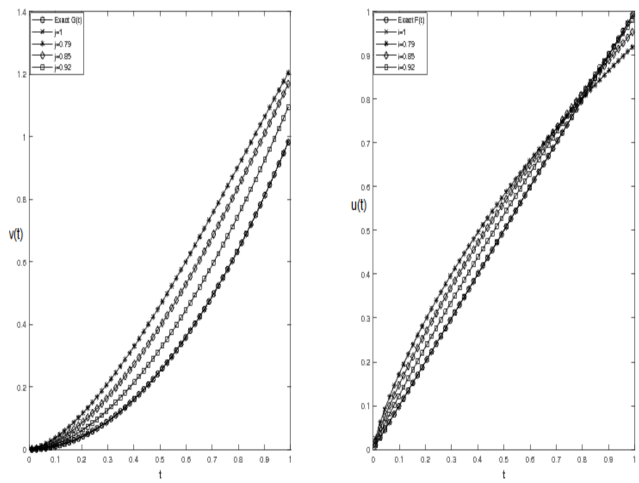

Figure 9: [Related to Table (5)]. Numerical results for different values of i and j, when $\alpha = 4$, $\beta = 7$ and $k = 56$.

solves.
The numerical conclusion of Example (5.2) are displayed in the Tables 4-6. In these tables including the approximate and the exact solutions and also absolute errors for different values of t, k, i and j.
If you look at the tables you will notice that by increasing α and β valuse of u and v converge to exact solutions, particularly when $i, j \to 1$.
Convergence functions u and v in figures 6-10 is apparent.

6 Conclusion

What caused to solve binary systems of FDIEs were two reasons; first, the numerical solution of FDIEs is often impossible or very difficult; second, solving binary systems of this equations are less considered. In this article, we have chosen Taylor wavelets method since this method is less used to solve such systems. Our numerical findings have been compared with the solutions obtained by other numerical methods such as Bernoulli, Legendre, Chebyshev and Haar wavelets. As can be seen, this technique has very high accuracy and efficiency; and its absolute error values are ignorable that this fact is proved by looking at the tables and figures.

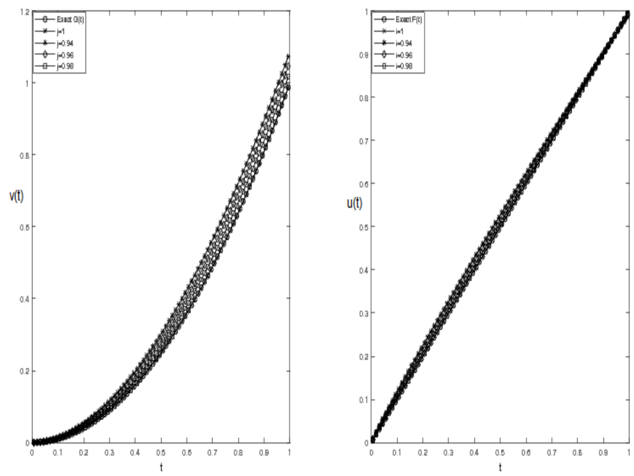

Figure 10: Numerical results for different values of i and j, when $\alpha = 5$, $\beta = 5$ and $k = 80$.

References

[1] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Teory and applications of the fractional differential equations, *Mathematical Studies, Vol, 204, Elsevier. Amsterdam.* 2006.

[2] R. Hilfer, Applications of fractional calculus in physics, *World Scientific Publishing Company, Singapore* 2000.

[3] R. L. Bagley, P. J. Torvic, A theoretical basis for the application of fractional calculus to viscoelasticity, *J. Rheol.* 27 (1983) 201-210.

[4] R. T. Baillie, Long memory process and fractional integration in econometrics, *J. Econom.* 73 (1996) 55-59.

[5] C. K. Chui, Wavelets: A mathematical tool for signal analysis, *SIAM, Philadelphia. PA.* 1997.

[6] G. Beylkin, R. Coifman, V. Rokhlin, Fast wavelet transforms and numerical algorithms, *Commun. Pure. Appl. Math.* 44 (1991) 141-183.

[7] M. R. Razzaghi, S. Yousefi, Legendre wavelets operational matrix of integration, *J. Sys. Sci.* 32 (2001) 495-502.

[8] C. H. Hsiao, S. P. Wu, Numerical solution of time-varying function differential equations via Haar wavelets, *Appl. Math. Comput.* 188 (2007) 1049-1058.

[9] M. H. Heydari, M. R. Hooshmanddasl, Legendre wavelets method for solving fractional partial differential equations with Dirichlet boundary conditions, *J. Appl. Mat. Comput.* 234 (2014) 267-276.

[10] L. Zhu, Q. B. Fan, Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelets, *Commun Nonlinear, Sci Numer. Simulat.* 17 (2012) 2333-2341.

[11] E. Keshavarz, Y. Ordokhani, M. Razzaghi, Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, *J. Appl. Math. Model* 38 (2014) 6038-6051.

[12] E. Fathizade, R. Ezzati, K. Maleknejad, CAS wavelet function method for solving Abel equations with error analysis, *Int. J. Res. Ind. Eng.* 4 (2017) 350-364.

[13] J. Wang, T. Z. Xu, Y. Q. Wei, J. Q. Xie, Numerical solution for coupled systems of nonlinear order integro-differential equations via wavelets method, *J. Appl. Math. Comput.* 324 (2018) 39-50.

[14] M. Razzaghi, M. Razzaghi, Taylor series direct method for variational problems, *J. Franklin. Inst.* 325 (1988) 125-131.

[15] I. Podlubny, Fractional differential equations, *Vol. 198, Academic Press, San Diego, Galif, USA.* 1999.

[16] E. Keshavarz, Y. Ordokhani, M. Razzaghi, The Taylor wavelets method for solving the initial and boundary problems of Bratu-type equations, *J. Appl. Numer Math.* 128 (2018) 205-216.

[17] G. Dahlquist, A. Björck, Numerical methods in scientific computing, *Vol. 1. SIAM, Philadelphia* 2008.

[18] Y. L. Li, N. Sun, Numerical solution of fractional differential equations using the generalized block puls operational matrix, Comput. Math. Appl. 62 (2011) 1046-1054.

[19] M. H. Heydari, M. R. Hooshmanddasl, F. M. M. Ghaini, F. Mohammadi, Wavelets collocation method for solving multi order fractional differential equations, *J. Appl. Math. Vol 2012, Article ID 542401,* 19 pages.

[20] A. Kilicman, A. Zhour, Z. A. Aziz, Kronecker operational matrices for fractional calculus and some applications, *Appl. Math. Comput.* 187 (2007) 250-265.

[21] M. H. Heydari, M. R. Hooshmanddasl, C. Cattani, M. Li, Legendre wavelets method for solving fractional population growth model in a closed system, *Math. Problems in engineering* (2013) article ID 161030, 8 pages.

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