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Research Article

Strong convergence for variational inequalities and equilibrium problems and representations

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Abstract

We introduce an implicit method for finding a common element of the set of solutions of systems of equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings and a representation of nonexpansive mappings. Then we prove the strong convergence of the proposed implicit schemes to the unique solution of a variational inequality, which is the optimality condition for a minimization problem and is also a common fixed point for a sequence of nonexpansive mappings and a representation of nonexpansive mappings.

Keywords : Representation; Equilibrium problem; Fixed point; Nonexpansive mapping; Variational inequality.

1 Introduction

Let H be a Hilbert space and let $G : H\times H \to \mathbb{R}$ be an equilibrium function, that is

$$G(u, u) = 0 \quad \text{for every } u \in H.$$

The Equilibrium Problem is defined as follows: Find $\tilde{x} \in H$ such that

$$G(\tilde{x}, y) \geq 0 \qquad \text{for all } y \in H. \tag{1.1}$$

A solution of (1.1) is said to be an equilibrium point and the set of the equilibrium points is denoted by SEP(G). Let C be a closed convex subset of H. A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. Let f be an α-contraction on H (i.e. $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $x, y \in H$ with $0 \leq \alpha < 1$), and A be a bounded linear operator on H. The following variational inequality problem with viscosity is of great interest [10, 11]. Find x^* in C such that

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in C), \tag{1.2}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \Big(\frac{1}{2}\langle Ax, x\rangle + h(x)\Big),$$

where γ satisfies $\|I - A\| \leq 1 - \alpha\gamma$ and h is a potential function for γf (that is $h'(x) = \gamma f(x)$). S. Takahashi and W. Takahashi [20] introduced the following viscosity approximation method for finding a common element of SEP(G) and Fix(T), where T is a nonexpansive mapping. Starting

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with an arbitrary element $x_1 \in H$, they defined the sequences $\{u_n\}$ and $\{x_n\}$ recursively by

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \\ (y \in H), \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n)Tu_n \\ (n \in \mathbb{N}), \end{cases}$$

and Plubtieng and Punpaeng in [14] proved a strong convergence theorem for an implicit iterative sequence $\{x_n\}$ obtained from the viscosity approximation method for finding a common element in $\mathrm{SEP(G)} \cap \mathrm{Fix(T)}$ which satisfies the variational inequality (1.2):

Theorem 1.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let G be a bifunction from $H \times H$ into $\mathbb{R}$ satisfying*
(A_1) $G(x, x) = 0$ for all $x \in C$;
(A_2) G is monotone, i.e. $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;
(A_3) For all $x, y, z \in C$,

$$\limsup_{t \to 0} G(tz + (1 - t)x, y) \leq G(x, y);$$

(A_4) For all $x \in C$, $y \mapsto G(x, y)$ is convex and lower semicontinuous.
For $x \in H$ and $r > 0$, set $S_r : H \to C$ to be the resolvent of G i.e. $S_r(x)$ is the unique $z \in C$ for which

$$G(z, y) + \frac{1}{r}\Big\langle y - z, z - x\Big\rangle \geq 0, \qquad (y \in C).$$

Let T be a nonexpansive mapping on H such that $\mathrm{SEP(G)} \cap \mathrm{Fix(T)} \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tu_n \\ (n \in \mathbb{N}), \\ G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \\ (y \in H), \end{cases}$$

where $u_n = S_{r_n} x_n$, $\{r_n\} \subset (0, \infty)$ and $\alpha_n \subset [0, 1]$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point z in $\mathrm{Fix(T)} \cap \mathrm{SEP(G)}$ which solves the variational inequality

$$\langle (A - \gamma f)z, z - x\rangle \leq 0 \quad x \in \mathrm{Fix(T)} \cap \mathrm{SEP(G)}.$$

V. Colao, G. L. Acedo and G. Marino proved a strong convergence theorem for the following implicit sequence $\{z_n\}$ for finding a common element in $\bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \bigcap_{k=1}^{K} \mathrm{SEP(G_k)}$ in [4],

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)W_n S_n^K z_n,$$

where

$$S_n^K = S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}}$$

and $n \in \mathbb{N}$. In this paper, motivated by Lau, Miyake and Takahashi [9], Atsushiba and Takahashi [2], Shimizu and Takahashi [16] and Takahashi [21], in Theorem 3.1, we use the harmonic concepts for improving the results proved in [4], in other word we use the amenability concepts and the theory of representations in our results but V. Colao, G. L. Acedo and G. Marino have not used these concepts in [4]. We introduce the following general implicit algorithm for finding a common element of the set of solutions of a system of equilibrium problems $\mathrm{SEP}(\wp)$ for a family $\wp = \{G_k; k = 1, 2 \cdots, K\}$ of bifunctions and the set of fixed points of a family $\{T_i\}_{i\in\mathbb{N}}$ of nonexpansive mappings from C into itself and a representation $\varrho = \{T_t : t \in S\}$ of a semigroup S as nonexpansive mappings from C into itself, with respect to W-mappings and a left regular sequence $\{\mu_n\}$ of means defined on an appropriate subspace of bounded real-valued functions of the semigroup:

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)T_{\mu_n} W_n S_n^K z_n,$$

where

$$S_n^K = S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}}$$

and $\mathrm{n} \in \mathbb{N}$.
Our goal is to prove some results of strong convergence for implicit schemes to approach a solution x^* of the problem (1.2) such that

$$x^* \in \bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \mathrm{Fix}(\mathcal{S}) \cap \mathrm{SEP}(\wp).$$

2 Preliminaries

Throughout this paper H denotes a Hilbert space. Moreover we assume that A is a bounded strongly positive operator on H with constant $\overline{\gamma}$; that is there exists $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \quad (x \in H).$$

For a map $T : H \to H$ we denote by $\mathrm{Fix(T)} := \{\mathrm{x \in H : x = Tx}\}$ the fixed point set of T. Note that if T is a nonexpansive mapping, Fix(T) is closed and convex (see [6]).

Let S be a semigroup. We denote by $B(S)$ the Banach space of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$ and $f \in B(S)$ we define l_s and r_s in $B(S)$ by
$(l_sf)(t) = f(st)$, $\quad (r_sf)(t) = f(ts), \quad (t \in S)$.
Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\|= \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e. $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_sf) = \mu(f)$ (resp. $\mu(r_sf) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $B(S)$ is amenable when S is a commutative semigroup (see page 29 of [19]). A net $\{\mu_\alpha\}$ of means on X is said to be left regular if

$$\lim_\alpha\|l_s^*\mu_\alpha - \mu_\alpha\|= 0,$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let f be a function of semigroup S into a reflexive Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let X be a subspace of $B(S)$ containing all the functions $t \to \left\langle f(t), x^*\right\rangle$ with $x^* \in E^*$. We know from [7] that for any $\mu \in X^*$, there exists a unique element f_μ in E such that $\left\langle f_\mu, x^*\right\rangle = \mu_t\left\langle f(t), x^*\right\rangle$ for all $x^* \in E^*$. We denote such f_μ by $\int f(t)\mu(t)$. Moreover, if μ is a mean on X then from [8], $\int f(t)\mu(t) \in \overline{\mathrm{co}}\,\{f(t) : t \in S\}$.

Let C be a nonempty closed and convex subset of H. Then, a family $\varrho = \{T_s : s \in S\}$ of mappings from C into itself is said to be a representation of S as nonexpansive mapping on C into itself if satisfies the following :
(1) $T_{st}x = T_sT_tx$ for all $s, t \in S$ and $x \in C$;
(2) for every $s \in S$ the mapping $T_s : C \to C$ is nonexpansive.
We denote by Fix(ϱ) the set of common fixed points of , that is Fix(ϱ)=$\{x \in C : T_sx = x, \ (s \in S)\}$.

For an equilibrium function $G : H \times H \to \mathbb{R}$, $\mathrm{SEP(G)} := \{\mathrm{x \in H : G(x, y) \geq 0 \ , (y \in H)}\}$ is the set of solutions of the related equilibrium problem.

Let C be a closed convex subset of a Hilbert space H. Recall that the (nearest) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_Cx \in C$ satisfying the property

$$\|x - P_Cx\|= \min_{y\in C}\|x - y\|.$$

The following Lemma characterizes the projection P_C.

Lemma 2.1 ([19]). *Let C be a closed convex subset of a real Hilbert space H, $x \in H$ and $y \in C$. Then $P_Cx = y$ if and only if it satisfies the inequality*

$$\left\langle x - y, y - z\right\rangle \geq 0 \quad (z \in C).$$

Lemma 2.2 ([10]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$ Then $\|I - \rho A\|\leq 1 - \rho\overline{\gamma}$.*

Theorem 2.1 ([18]). *Let S be a semigroup, C be a closed convex subset of a Hilbert space H, $\varrho = \{T_s : s \in S\}$ be a representation of S as nonexpansive mapping from C into itself such that $Fix() \neq \emptyset$ and X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \left\langle T(t)x, y\right\rangle$ be an element of X for each $x \in C$ and $y \in H$, and μ be a mean on X. If we write $T_\mu x$ instead of $\int T_tx\, d\mu(t)$, then the following hold.*
(i) *T_μ is a nonexpansive mapping from C into C.*
(ii) *$T_\mu x = x$ for each $x \in Fix(\varrho)$.*
(iii) *$T_\mu x \in \overline{co}\,\{T_tx : t \in S\}$ for each $x \in C$.*
(iv) *If μ is left invariant, then T_μ is a nonexpansive retraction from C onto $Fix(S)$.*

Theorem 2.2 ([5]). *Let C be a nonempty closed convex subset of a Hilbert space H and $G : H \times H \to \mathbb{R}$ satisfy,*
(A_1) *$G(x,x) = 0$ for all $x \in C$;*
(A_2) *G is monotone, i.e. $G(x,y) + G(y,x) \leq 0$ for all $x, y \in C$;*
(A_3) *For all $x, y, z \in C$,*

$$\limsup_{t\to 0} G(tz+(1-t)x, y) \leq G(x,y);$$

(A_4) *For all $x \in C, y \mapsto G(x,y)$ is convex and lower semicontinuous.*
For $x \in H$ and $r > 0$, set $S_r : H \to C$ to be

$$S_r(x) := \{z \in C : G(z,y) + \frac{1}{r}\Big\langle y - z, z - x\Big\rangle \geq 0, \quad (y \in C)\},$$

then S_r is well defined and the followings are valid:
(i) *S_r is single-valued;*
(ii) *S_r is firmly nonexpansive, i.e.*

$$\|S_r x - S_r y\|^2 \leq \Big\langle S_r x - S_r y, x - y\Big\rangle,$$

for all $x, y \in H$;
(iii) $\mathrm{Fix} S_r = \mathrm{SEP}(G)$;
(iv) $\mathrm{SEP}(G)$; is closed and convex.

Theorem 2.3 ([4]). *Let $\{r_n\} \subset (0,\infty)$ be a sequence converging to $r > 0$. For a bifunction $G : H \times H \to \mathbb{R}$, satisfying conditions (A_1)- (A_4), define S_r and S_{r_n} for $n \in \mathbb{N}$ as in* Theorem 2.5, then for every $x \in H$, we have $\lim_n \|S_{r_n} - S_r\| = 0$.

Let C be a nonempty subset of a Hilbert space H and $T : C \to H$ be a mapping. Then T is said to be demiclosed at $v \in H$ if for any sequence $\{x_n\}$ in C, the following implication holds: $x_n \rightharpoonup u \in C, \quad Tx_n \to v$ imply $Tu = v$, where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence.

Lemma 2.3 ([1]). *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T : C \to H$ is nonexpansive. Then, the mapping $I - T$ is demiclosed at zero.*

Remark 2.1 *Every Hilbert space is a uniformly convex Banach space, and therefore is a strictly convex Banach space (see pages 95, 98 of [19]).*

Definition 2.1 *A vector space X is said to satisfy Opial's condition, if for each sequence $\{x_n\}$ in X which converges weakly to point $x \in X$,*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad (y \in X, \ y \neq x).$$

Note that every Hilbert space satisfies the Opial's condition (see [12] and [15]).

Definition 2.2 *Let K be a nonempty subset of a Banach space X and $\{x_n\}$ be a sequence in K. The set of the asymptotic center of $\{x_n\}$ with respect to K, defined by*

$$A(\{x_n\}) = \Big\{x \in K : \limsup_{n\to\infty} \|x_n - x\| = \inf_{y\in K} \limsup_{n\to\infty} \|x_n - y\|\Big\}.$$

Lemma 2.4 ([1]). *Let X be a uniformly convex Banach space satisfying the Opial's condition and K be a nonempty closed convex subset of X. If a sequence $\{z_n\} \subset K$ converges weakly to a point z_0, then $\{z_0\}$ is the asymptotic center of $\{z_n\}$ with respect to K.*

Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings of C into itself and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq 1$ for every $i \in \mathbb{N}$. Following [17], for any $n \geq 1$, we define a mapping W_n of C into itself as follows,

$$\begin{aligned}
U_{n,n+1} &:= I,\\
U_{n,n} &:= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I,\\
&\vdots\\
U_{n,k} &:= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \qquad (2.3)\\
&\vdots\\
U_{n,2} &:= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I,\\
W_n := U_{n,1} &:= \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I.
\end{aligned}$$

The following results hold for the mappings W_n.

Theorem 2.4 ([17]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty}\mathrm{Fix(T_i)} \neq \emptyset$ and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq b < 1$ for every $i \in \mathbb{N}$. Then*
(1) *W_n is nonexpansive and* $\mathrm{Fix(W_n)} = \bigcap_{i=1}^{n}\mathrm{Fix(T_i)}$ for each $n \geq 1$,

(2) *for each $x \in C$ and for each positive integer j, the limit $\lim_{n\to\infty} U_{n,j}x$ exists.*

(3) *The mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} \quad (\mathrm{x} \in \mathrm{C}),$$

is a nonexpansive mapping satisfying $\mathrm{Fix(W)} = \bigcap_{i=1}^{\infty}\mathrm{Fix(T_i)}$ *and it is called the W-mapping generated by $\{T_i\}_{i\in\mathbb{N}}$, and $\{\lambda_i\}_{i\in\mathbb{N}}$.*

Theorem 2.5 ([13]). *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty}\mathrm{Fix(T_i)} \neq \emptyset$, $\{\breve{}_{\mathrm{i}}\}$ be a real sequence such that $0 < \lambda_i \leq b < 1$, $(i \geq 1)$. If D is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

Throughout the rest of this paper, the open ball of radius r centered at 0 is denoted by B_r. For $\epsilon > 0$ and a mapping $T : D \to H$, we let $F_\epsilon(T; D)$ be the set of ϵ-approximate fixed points of T, i.e.

$$F_\epsilon(T; D) = \{x \in D : \|x - Tx\| \leq \epsilon\}.$$

3 Main results

In this Section, we deal with the strong convergence approximation scheme for finding a common element of the set of solutions of a system of an equilibrium problem and the set of common fixed points of a sequence of nonexpansive mappings and left amenable nonexpansive semigroup in a Hilbert space. These results extend the main result of [4] and many others.

Theorem 3.1 *Let S be a semigroup and let C be a closed convex subset of a Hilbert space H. Suppose that $\varrho = \{T_s : s \in S\}$ be a representation of S as nonexpansive mapping from C into itself and suppose $\mathrm{Fix}(\varrho) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, y\rangle$ is an element of X for each $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix}(\varrho)) \subseteq \mathrm{Fix}(\varrho)$forevery $i \in \mathbb{N}$, and $\wp = \{G_k : k = 1, 2, \cdots K\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ and f is an α-contraction on H. Moreover, let $\{r_{k,n}\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $0 < \epsilon_n < 1$ and $0 < \lambda_n \leq b < 1$, and γ is a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that,*

(i) *for every $k \in \{1, 2, \cdots, K\}$, the function G_k satisfies* $(A_1) - (A_4)$ *of* Theorem 2.5,

(ii) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}}\mathrm{Fix(T_n)} \cap \mathrm{Fix}(\mathcal{S}) \cap \mathrm{SEP}(\wp) \neq \emptyset$,

(iii) $\lim_n \epsilon_n = 0$ and,

(iv) for every $k \in \{1, 2, \cdots, K\}$, $\lim_n r_{k,n}$ exists and is a positive real number.

For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.3), for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$. Let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in Theorem 2.5. If $\{z_n\}$ is the sequence generated by

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)T_{\mu_n} W_n S^K_n z_n, \tag{3.4}$$

where $S^K_n = S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}}$ for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$. Then $\{z_n\}$ strongly converges to $x^* \in \mathfrak{F}$, where x^* is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}), \tag{3.5}$$

or, equivalently,

$$x^* = \mathrm{P}_{\mathfrak{F}}(I - (A - \gamma f))x^*,$$

or, equivalently, x^* is the unique solution of the minimization problem

$$\min_{x\in\mathfrak{F}}\Big(\frac{1}{2}\Big\langle Ax,x\Big\rangle+h(x)\Big),$$

where h is a potential function for γf.

Since $\epsilon_n \to 0$, we may assume that $\epsilon_n \leq \min\Big\{\|A\|^{-1},\frac{1}{\bar{\gamma}}\Big\}$. We observe that if $\|p\|=1$, then

$$\begin{aligned}\langle(I-\epsilon_n A)p\ ,\ p\rangle &= 1-\epsilon_n\langle Ap\ ,\ p\rangle\\ &\geq 1-\epsilon_n\|A\|\geq 0.\end{aligned}$$

Hence, if $\|p\|\neq 1$ and $p\neq 0$, then we have

$$\begin{aligned}&\langle(I-\epsilon_n A)p\ ,\ p\rangle\\ &\quad=\|p\|^2\langle(I-\epsilon_n A)\frac{p}{\|p\|}\ ,\ \frac{p}{\|p\|}\rangle\geq 0.\end{aligned}$$

We also have $\langle(I-\epsilon_n A)p\ ,\ p\rangle=0$, if $p=0$. Hence $\langle(I-\epsilon_n A)p\ ,\ p\rangle\geq 0$, for all $p\in C$.
By Lemma 2.2, we have

$$\|I-\epsilon_n A\|\leq 1-\epsilon_n\bar{\gamma}$$

. We shall divide the proof into eight steps.

Step 1. The existence of z_n which satisfies (3.4).
Proof. This follows immediately from the fact that for every $n\in\mathbb{N}$, the mapping N_n given by

$$N_n x:=\epsilon_n\gamma f(x)+(I-\epsilon_n A)T_{\mu_n}W_nS_n^Kx \qquad (x\in H),$$

is a contraction. To see this, put $\beta_n=1+\epsilon_n\gamma\alpha-\epsilon_n\bar{\gamma}$, then $0\leq\beta_n<1$ $(n\in\mathbb{N})$. We have,

$$\begin{aligned}&\|N_nx-N_ny\|\\ &\leq\epsilon_n\gamma\|f(x)-f(y)\|\\ &\quad+\Big(1-\epsilon_n\bar{\gamma}\Big)\|T_{\mu_n}W_nS_n^Kx-T_{\mu_n}W_nS_n^Ky\|\\ &\leq\epsilon_n\gamma\alpha\|x-y\|+(1-\epsilon_n\bar{\gamma})\|x-y\|\\ &=(1+\epsilon_n\gamma\alpha-\epsilon_n\bar{\gamma})\|x-y\|=\beta_n\|x-y\|.\end{aligned}$$

Therefore, by Banach Contraction Principle ([19], p. 4), there exist a unique point z_n such that $N_nz_n=z_n$.

Step 2. $\{z_n\}$ is bounded.
Proof. Let $p\in\mathfrak{F}$. We have

$$\begin{aligned}\|z_n-p\|^2&=\Big\langle\epsilon_n\gamma f(z_n)\\ &\quad+(I-\epsilon_n A)T_{\mu_n}W_nS_n^Kz_n-p\ ,\ z_n-p\Big\rangle\\ &=\epsilon_n\gamma\Big\langle f(z_n)-f(p),z_n-p\Big\rangle\\ &\quad+\epsilon_n\Big\langle\gamma f(p)-Ap\ ,\ z_n-p\Big\rangle\\ &\quad+\Big\langle(I-\epsilon_n A)(T_{\mu_n}W_nS_n^Kz_n\\ &\quad-T_{\mu_n}W_nS_n^Kp)\ ,\ z_n-p\Big\rangle\\ &\leq\epsilon_n\gamma\alpha\|z_n-p\|^2+(1-\epsilon_n\bar{\gamma})\|z_n-p\|^2\\ &\quad+\epsilon_n\Big\langle\gamma f(p)-Ap\ ,\ z_n-p\Big\rangle.\end{aligned}$$

Thus,

$$\|z_n-p\|^2\leq\frac{1}{\bar{\gamma}-\alpha\gamma}\Big\langle\gamma f(p)-Ap\ ,\ z_n-p\Big\rangle. \quad (3.6)$$

Hence,

$$\|z_n-p\|\leq\frac{1}{\bar{\gamma}-\alpha\gamma}\|\gamma f(p)-Ap\|.$$

That is, the sequence $\{z_n\}$ is bounded.

Step 3. For every fixed $k\in\{1,2,\cdots,K\}$, we have

$$\lim_n\|z_n-S_{r_{k,n}}^kz_n\|=0. \quad (3.7)$$

Proof. Let $k\in\{1,2,\cdots,K\}$, since by (ii) of Theorem 2.5, $S_{r_{k,n}}^k$ is firmly nonexpansive, we conclude that

$$\begin{aligned}&\|S_{r_{k,n}}^kz_n-p\|^2\\ &\quad=\|S_{r_{k,n}}^kz_n-S_{r_{k,n}}^kp\|^2\\ &\quad\leq\Big\langle S_{r_{k,n}}^kz_n-S_{r_{k,n}}^kp\ ,\ z_n-p\Big\rangle\\ &\quad=\frac{1}{2}\Big(\|S_{r_{k,n}}^kz_n-p\|^2\\ &\qquad+\|z_n-p\|^2-\|z_n-S_{r_{k,n}}^kz_n\|^2\Big).\end{aligned}$$

Therefore,

$$\|z_n-S_{r_{k,n}}^kz_n\|^2\leq\|z_n-p\|^2-\|S_{r_{k,n}}^kz_n-p\|^2. \quad (3.8)$$

If we put
$L_n := 2\left\langle \gamma f(z_n) - AT_{\mu_n} W_n S_n^K z_n \,,\, z_n - p \right\rangle$, then by using the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\left\langle y, x+y \right\rangle, \qquad (3.9)$$

we obtain

$$\begin{aligned}
&\|z_n - p\|^2 \\
&\quad = \|\epsilon_n \gamma f(z_n) \\
&\qquad + (I - \epsilon_n A) T_{\mu_n} W_n S^1_{r_{1,n}} S^2_{r_{2,n}} \\
&\qquad \cdots S^K_{r_{K,n}} z_n - p\|^2 \\
&\quad \leq \|T_{\mu_n} W_n S^1_{r_{1,n}} S^2_{r_{2,n}} \\
&\qquad \cdots S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n \\
&\quad \leq \|S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n.
\end{aligned}$$

So by (3.8), we have

$$\|z_n - S^k_{r_{k,n}} z_n\|^2 \leq \epsilon_n L_n.$$

That $\{L_n\}_{n\in\mathbb{N}}$ is a bounded sequence, implies

$$\lim_n \|z_n - S^k_{r_{k,n}} z_n\| = 0.$$

By induction we assume that (3.7) holds for every $k > \overline{k}$, and we prove it for $\overline{k}$.
Indeed, we have

$$\begin{aligned}
&\|z_n - p\|^2 \\
&\quad \leq \|T_{\mu_n} W_n S^1_{r_{1,n}} S^2_{r_{2,n}} \\
&\qquad \cdots S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n \\
&\quad \leq \|S^{\overline{k}}_{r_{\overline{k},n}} \cdots S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n. \qquad (3.10)
\end{aligned}$$

Observe that

$$\begin{aligned}
&\|S_{r_{\overline{k},n}}{}^{\overline{k}} \cdots S^K_{r_{K,n}} z_n - p\| \\
&\quad = \|S^{\overline{k}}_{r_{\overline{k},n}} \cdots S^K_{r_{K,n}} z_n - S^{\overline{k}}_{r_{\overline{k},n}} z_n \\
&\qquad + S^{\overline{k}}_{r_{\overline{k},n}} z_n - p\| \\
&\quad \leq \|S^{\overline{k}+1}_{r_{\overline{k}+1,n}} \cdots S^K_{r_{K,n}} z_n - z_n\| \\
&\qquad + \|S^{\overline{k}}_{r_{\overline{k},n}} z_n - p\| \\
&\quad \leq \|S^{\overline{k}+1}_{r_{\overline{k}+1,n}} \cdots S^K_{r_{K,n}} z_n - S^{\overline{k}+1}_{r_{\overline{k}+1,n}} z_n\| \\
&\qquad + \|S^{\overline{k}+1}_{r_{\overline{k}+1,n}} z_n - z_n\| + \|S^{\overline{k}}_{r_{\overline{k},n}} z_n - p\| \\
&\quad \leq \|S^{\overline{k}+2}_{r_{\overline{k}+2,n}} \cdots S^K_{r_{K,n}} z_n - z_n\| \\
&\qquad + \|S^{\overline{k}+1}_{r_{\overline{k}+1,n}} z_n - z_n\| + \|S^{\overline{k}}_{r_{\overline{k},n}} z_n - p\| \\
&\quad \vdots \\
&\quad \leq \|S^{\overline{k}}_{r_{\overline{k},n}} z_n - p\| + \sum_{k=\overline{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\|.
\end{aligned}$$

Inequality (3.10) gives,

$$\begin{aligned}
&\|z_n - p\|^2 \\
&\leq \Bigg(\sum_{k=\overline{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| \\
&\quad + 2\|S^{\overline{k}}_{r_{\overline{k},n}} z_n - p\| \Bigg) \Bigg(\sum_{k=\overline{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| \Bigg) \\
&\quad + \|S^{\overline{k}}_{r_{\overline{k},n}} z_n - p\|^2 + \epsilon_n L_n.
\end{aligned}$$

From this inequality and (3.8), we obtain

$$\begin{aligned}
&\|z_n - S^{\overline{k}}_{r_{\overline{k},n}} z_n\|^2 \\
&\leq \Bigg(\sum_{k=\overline{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| \\
&\quad + 2\|S^{\overline{k}}_{r_{\overline{k},n}} z_n - p\| \Bigg) \Bigg(\sum_{k=\overline{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| \Bigg) \\
&\quad + \epsilon_n L_n.
\end{aligned}$$

Since by assumption,

$$\lim_n \sum_{k=\overline{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| = 0,$$

hence

$$\lim_n \|z_n - S^{\overline{k}}_{r_{\overline{k},n}} z_n\| = 0,$$

as required.

Step 4. $\lim_n \|z_n - T_{\mu_n} W_n z_n\| = 0$.
Proof. To see this, put

$$M_n := 2\Big\langle \gamma f(z_n) - A T_{\mu_n} W_n S^K_n z_n , z_n - T_{\mu_n} W_n z_n \Big\rangle.$$

It is obvious that $\{M_n\}_{n\in\mathbb{N}}$ is a bounded sequence. By using (3.9), we have

$$\begin{aligned}
&\|z_n - T_{\mu_n} W_n z_n\|^2 \\
&= \|\epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_{\mu_n} W_n S^K_n z_n - T_{\mu_n} W_n z_n\|^2 \\
&\leq \|S^K_n z_n - z_n\|^2 + \epsilon_n M_n,
\end{aligned}$$

and

$$\begin{aligned}
&\|S^K_n z_n - z_n\| \\
&\leq \|S^1_{r_{1,n}} \cdots S^K_{r_{K,n}} z_n - S^1_{r_{1,n}} z_n\| + \|S^1_{r_{1,n}} z_n - z_n\| \\
&\leq \|S^2_{r_{2,n}} \cdots S^K_{r_{K,n}} z_n - z_n\| + \|S^1_{r_{1,n}} z_n - z_n\| \\
&\quad \vdots \\
&\leq \sum_{k=1}^{K} \|S^k_{r_{k,n}} z_n - z_n\|.
\end{aligned}$$

Using (3.7) and the fact that $\{M_n\}_{n\in\mathbb{N}}$ is a bounded sequence, we can conclude that,

$$\begin{aligned}
&\lim_n \|z_n - T_{\mu_n} W_n z_n\|^2 \\
&\leq \Big(\lim_n \sum_{k=1}^{K} \|S^k_{r_{k,n}} z_n - z_n\|\Big)^2 + \lim_n \epsilon_n M_n = 0 .
\end{aligned}$$

Step 5. $\lim_{n\to\infty} \|z_n - T_t z_n\| = 0$, for all $t \in S$.
Proof. Let $p \in \mathfrak{F}$ and put

$$M_0 = \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \alpha\gamma}.$$

Let $D = \{y \in H : \|y - p\| \leq M_0\}$. It is clear that D is a bounded closed convex set, and $\{z_n : n \in \mathbb{N}\} \subseteq D$. It is also obvious that D is invariant under $\{S^k_{r_{k,n}} : k = 1, 2, \ldots K, n \in \mathbb{N}\}$, W_n for every $n \in \mathbb{N}$, and . We will show that

$$\limsup_{n\to\infty} \sup_{y\in D} \|T_{\mu_n} y - T_t T_{\mu_n} y\| = 0 \quad (\mathrm{t} \in \mathrm{S}). \tag{3.11}$$

Let $\epsilon > 0$. By Theorem 2.1 of [3], there exists $\delta > 0$ such that

$$\overline{\mathrm{co}} F_\delta(T_t; D) + B_\delta \subseteq F_\epsilon(T_t; D) \quad (\mathrm{t} \in \mathrm{S}). \tag{3.12}$$

Also by Corollary 1.1 of [3], there exists a natural number N such that

$$\Big\| \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y - T_t \Big(\frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \Big) \Big\| \leq \delta, \tag{3.13}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$, since $\{\mu_n\}$ is strongly left regular, there exists $N_0 \in \mathbb{N}$ such that $\|\mu_n - l^*_{t^i}\mu_n\| \leq \frac{\delta}{(M_0 + \|p\|)}$ for $n \geq N_0$ and $i = 1, 2, \cdots, N$. Then, we have

$$\begin{aligned}
&\sup_{y\in D} \Big\| T_{\mu_n} y - \int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \mu_n(s) \Big\| \\
&= \sup_{y\in D} \sup_{\|z\|=1} \Big| \langle T_{\mu_n} y, z\rangle - \Big\langle \int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \mu_n(s), z \Big\rangle \Big| \\
&= \sup_{y\in D} \sup_{\|z\|=1} \Big| \frac{1}{N+1} \sum_{i=0}^{N} (\mu_n)_s \langle T_s y, z\rangle - \frac{1}{N+1} \sum_{i=0}^{N} (\mu_n)_s \langle T_{t^i s} y, z\rangle \Big| \\
&\leq \frac{1}{N+1} \sum_{i=0}^{N} \sup_{y\in D} \sup_{\|z\|=1} |(\mu_n)_s \langle T_s y, z\rangle - (l^*_{t^i}\mu_n)_s \langle T_s y, z\rangle| \\
&\leq \max_{i=1,2,\cdots,N} \|\mu_n - l^*_{t^i}\mu_n\| (M_0 + \|p\|) \\
&\leq \delta \quad (\mathrm{n} \geq \mathrm{N}_0).
\end{aligned} \tag{3.14}$$

By Theorem 2.3 we have

$$\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s} y \mu_n(s) \in \overline{\text{co}}\Big\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i}(T_s y) : s \in S\Big\}. \quad (3.15)$$

It follows from (3.12)-(3.15) that

$$T_{\mu_n} y \in \overline{\text{co}}\Big\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s} y : s \in S\Big\} + B_\delta \subset \overline{\text{co}} F_\delta(T_t; D) + B_\delta \subset F_\epsilon(T_t; D),$$

for all $y \in D$ and $n \geq N_0$. Therefore,

$$\limsup_{n\to\infty} \sup_{y\in D} \|T_t(T_{\mu_n} y) - T_{\mu_n} y\| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get (3.11).
Let $t \in S$ and $\epsilon > 0$, then there exists $\delta > 0$, which satisfies (3.12). Take $L_0 = (\gamma\alpha + \|A\|)M_0 + \|\gamma f(p) - Ap\|$. Now from (3.11) and condition (iii) there exists $N_1 \in \mathbb{N}$ such that $T_{\mu_n} y \in F_\delta(T_t; D)$ for all $y \in D$ and $\epsilon_n < \frac{\delta}{2L_0}$ for all $n \geq N_1$. We note that

$$\begin{aligned}
&\epsilon_n \|\gamma f(z_n) - A T_{\mu_n} W_n S_n^K z_n\| \\
\leq &\epsilon_n \Big(\|\gamma f(z_n) - \gamma f(p)\| + \|\gamma f(p) - Ap\| \\
&+ \|Ap - A T_{\mu_n} W_n S_n^K z_n\| \Big) \\
\leq &\epsilon_n \Big(\gamma\alpha \|z_n - p\| \\
&+ \|\gamma f(p) - Ap\| + \|A\| \|z_n - p\| \Big) \\
\leq &\epsilon_n \Big((\gamma\alpha + \|A\|) M_0 + \|\gamma f(p) - Ap\| \Big) \\
= &\epsilon_n L_0 \leq \frac{\delta}{2},
\end{aligned}$$

for all $n \geq N_1$. Observe that

$$\begin{aligned}
z_n = &\epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_{\mu_n} W_n S_n^K z_n \\
= &T_{\mu_n} W_n S_n^K z_n + \epsilon_n \Big(\gamma f(z_n) \\
&- A T_{\mu_n} W_n S_n^K z_n \Big) \\
\in &F_\delta(T_t; D) + B_{\frac{\delta}{2}} \\
\subseteq &F_\delta(T_t; D) + B_\delta \\
\subseteq &F_\epsilon(T_t; D),
\end{aligned}$$

for all $n \geq N_1$. This show that

$$\|z_n - T_t z_n\| \leq \epsilon \quad (\text{n} \geq \text{N}_1).$$

Since $\epsilon > 0$ is arbitrary, we get $\lim_{n\to\infty} \|z_n - T_t z_n\| = 0$.

Step 6. The weak ω-limit set of $\{z_n\}$ which is denoted by $\omega_\omega\{z_n\}$ is a subset of $\mathfrak{F}$.
Proof. Let $\hat{z} \in \omega_\omega\{z_n\}$ and let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_j} \rightharpoonup \hat{z}$. We need to show that $\hat{z} \in \mathfrak{F}$. In terms of Lemma 2.4 and Step 5, we conclude that $\hat{z} \in \text{Fix}(\mathcal{S})$. By Theorems 2.2, 2.3, the mapping $W : C \to C$, given by $Wx := \lim_n W_n x$ satisfies

$$\limsup_{n\to\infty} \|W_n \hat{z} - W \hat{z}\| = 0. \quad (3.16)$$

Putting $\lim_n r_{k,n} = \hat{r}_k$ for every $k \in \{1, 2, \cdots, K\}$, by Theorem 2.5, we have

$$S_{\hat{r}_k}^k x = \lim_n S_{r_{k,n}}^k x \qquad (x \in H). \quad (3.17)$$

Since $\hat{z} \in \text{Fix}(\mathcal{S})$, by our assumption, we have $T_i \hat{z} \in \text{Fix}(\mathcal{S})$ for all $i \in \mathbb{N}$ and then $W_n \hat{z} \in \text{Fix}(\mathcal{S})$. Hence, by (ii) of Theorem 2.3, $T_{\mu_n} W_n \hat{z} = W_n \hat{z}$.
Consider the set of the asymptotic center $A(z_{n_j})$ of $\{z_{n_j}\}$ with respect to H. Since $z_{n_j} \rightharpoonup \hat{z}$, Lemma 2.4 implies that $A(z_{n_j}) = \{\hat{z}\}$. By the definition of $A(z_{n_j})$, we have

$$\limsup_{j\to\infty} \|z_{n_j} - z\| \leq \limsup_{j\to\infty} \|z_{n_j} - T_t z_{n_j}\| \quad (t \in S),$$

for all $z \in A(z_{n_j})$. Since $A(z_{n_j}) = \{\hat{z}\}$, by Step 5, we get $z_{n_j} \to \hat{z}$. Using (3.16) and Step 4, we have

$$\begin{aligned}
&\limsup_{j\to\infty}\|z_{n_j}-W\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j}-T_{\mu_{n_j}}W_{n_j}z_{n_j}\| \\
&\qquad+\limsup_{j\to\infty}\|T_{\mu_{n_j}}W_{n_j}z_{n_j}-T_{\mu_{n_j}}W_{n_j}\hat{z}\| \\
&\qquad+\limsup_{j\to\infty}\|T_{\mu_{n_j}}W_{n_j}\hat{z}-W\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j}-T_{\mu_{n_j}}W_{n_j}z_{n_j}\| \\
&\qquad+\limsup_{j\to\infty}\|z_{n_j}-\hat{z}\| \\
&\qquad+\limsup_{j\to\infty}\|W_{n_j}\hat{z}-W\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j}-\hat{z}\|=0.
\end{aligned}$$

This implies that $W(\hat{z})=\hat{z}$.
Using Theorem 2.4 and (3.17) and Step 3, we have

$$\begin{aligned}
&\limsup_{j\to\infty}\|z_{n_j}-S^k_{\hat{r}_k}\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j}-S^k_{r_{k,n_j}}z_{n_j}\| \\
&\qquad+\limsup_{j\to\infty}\|S^k_{r_{k,n_j}}z_{n_j}-S^k_{r_{k,n_j}}\hat{z}\| \\
&\qquad+\limsup_{j\to\infty}\|S^k_{r_{k,n_j}}\hat{z}-S^k_{\hat{r}_k}\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j}-\hat{z}\|=0. \qquad (3.18)
\end{aligned}$$

This implies that $S^k_{\hat{r}_k}(\hat{z})=\hat{z}$ for every $k\in\{1,2,\cdots,K\}$.
Therefore, $\hat{z}\in \mathrm{Fix(W)}\cap(\bigcap_{\mathrm{k=1}}^{\mathrm{K}}\mathrm{Fix}(\mathrm{S}^{\mathrm{k}}_{\hat{\mathrm{r}}_{\mathrm{k}}}))$. In terms of Theorems 2.4 and 2.5, we conclude that $\hat{z}\in(\bigcap_{i=1}^{\infty}\mathrm{Fix}(\mathrm{T_i}))\cap\mathrm{SEP}()$. Since $\hat{z}\in\mathrm{Fix}(\mathcal{S})$, therefore, $\hat{z}\in\mathfrak{F}$.

Step 7. There exists a unique solution $x^*\in\mathfrak{F}$ of the variational inequality (3.5), such that

$$\Gamma:=\limsup_n\left\langle(\gamma f-A)x^*,z_n-x^*\right\rangle\le 0. \qquad (3.19)$$

Proof. Banach Contraction Mapping Principle guarantees that $P_{\mathfrak{F}}(I-(A-\gamma f))$ has a unique fixed point x^* which is, by Lemma 2.1, the unique solution of the variational inequality :

$$\left\langle(\gamma f-A)x^*,x-x^*\right\rangle\le 0 \qquad (x\in\mathfrak{F}).$$

Note that, from the definition of Γ and the fact that z_n is a bounded sequence, we can select a subsequence z_{n_j} of z_n with the following properties:
(i) $\lim_j\left\langle(\gamma f-A)x^*,z_{n_j}-x^*\right\rangle=\Gamma$;
(ii) z_{n_j} is weakly converge to a point $\hat{z}$;
by Step 6, we have $\hat{z}\in\mathfrak{F}$ and then

$$\begin{aligned}
\Gamma=&\lim_j\left\langle(\gamma f-A)x^*,z_{n_j}-x^*\right\rangle\\
=&\left\langle(\gamma f-A)x^*,\hat{z}-x^*\right\rangle\le 0,
\end{aligned}$$

as $x^*\in\mathfrak{F}$ is the unique solution of (3.5).

Step 8. $\{z_n\}$ strongly converges to x^*.

Proof. Indeed, from (3.6), (3.19) and that $x^*\in\mathfrak{F}$, we conclude

$$\begin{aligned}
&\limsup_n\|z_n-x^*\|^2\\
\le&\frac{1}{\overline{\gamma}-\alpha\gamma}\limsup_n\left\langle(\gamma f-A)x^*,z_n-x^*\right\rangle\le 0.
\end{aligned}$$

That is $z_n\to x^*$.

Theorem 3.2 *Let H be a real Hilbert space, T be a nonexpansive mapping of C into itself such that $\mathrm{Fix(T)}\neq\emptyset$, $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix(T)})\subseteq\mathrm{Fix(T)}$ for every $i\in\mathbb{N}$, and $\wp=\{G_k:k=1,2,\cdots K\}$ be a finite family of bifunctions from $H\times H$ into $\mathbb{R}$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, and f be an α-contraction on H. Moreover, let $\{r_{k,n}\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n}>0$, $0<\epsilon_n<1$ and $0<\lambda_n\le b<1$, and γ is a real number such that $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Assume that,*

(i) *for every $k\in\{1,2,\cdots,K\}$, the function G_k satisfies $(A_1)-(A_4)$ of Theorem 2.5 ,*

(ii) $\mathfrak{F}:=\bigcap_{n\in\mathbb{N}}\mathrm{Fix}(\mathrm{T_n})\cap\mathrm{Fix(T)}\cap\mathrm{SEP}(\wp)\neq\emptyset$,

(iii) $\lim_n\epsilon_n=0$ *and,*

(iv) *for every $k\in\{1,2,\cdots,K\}$, $\lim_n r_{k,n}$ exists and is a positive real number.*

For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.3), for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$, let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in Theorem 2.5. If $\{z_n\}$ is the sequence generated by

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)\frac{1}{n}\sum_{k=1}^{n} T^k W_n S^K_n z_n$$
$$(n \in \mathbb{N}).$$

Then $\{z_n\}$ strongly converges to $x^* \in \mathfrak{F}$, where x^* is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Proof. Let $S = \{1, 2, ...\}$, $= \{T^i : i \in S\}$. For $f = (z_1, z_2, \cdots) \in B(S)$, define

$$\mu_n(f) = \frac{1}{n}\sum_{k=1}^{n} z_k \qquad (n \in \mathbb{N}).$$

Then $\{\mu_n\}$ is a regular sequence of means on $B(S)$; for more details, see [19]. Next for each $x \in H$ and $n \in \mathbb{N}$, we have

$$T_{\mu_n} x = \frac{1}{n}\sum_{k=1}^{n} T^k x.$$

Therefore, it follows from Theorem 3.1 that the sequence $\{z_n\}$ converges strongly to $x^* \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Theorem 3.3 *Let H be a real Hilbert space, T be a nonexpansive mapping of C into itself such that* $\mathrm{Fix(T)} \neq \emptyset$*, $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix(T)}) \subseteq \mathrm{Fix(T)}$ for every $i \in \mathbb{N}$, $\wp = \{G_k : k = 1, 2, \cdots K\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, and f be an α-contraction on H. Moreover, let $\{r_{k,n}\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $0 < \epsilon_n < 1$ and $0 < \lambda_n \leq b < 1$, and γ is a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that,*

(i) *for every $k \in \{1, 2, \cdots, K\}$, the function G_k satisfies $(A_1) - (A_4)$ of Theorem t2.5,*

(ii) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \mathrm{Fix(T)} \cap \mathrm{SEP}(\wp) \neq \emptyset$,

(iii) $\lim_n \epsilon_n = 0$ *and,*

(iv) *for every $k \in \{1, 2, \cdots, K\}$, $\lim_n r_{k,n}$ exists and is a positive real number.*

For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.3), for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$, let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in Theorem 2.5. If $\{z_n\}$ is the sequence generated by

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)\frac{1 - a_n}{a_n}\sum_{k=1}^{\infty} (a_n)^k T^k W_n S^K_n z_n$$
$$(n \in \mathbb{N}),$$

where $\{a_n\}$ is an increasing sequence in $(0, 1)$ such that $\lim_n a_n = 1$. Then $\{z_n\}$ strongly converges to $x^* \in \mathfrak{F}$, where x^* is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Proof. Let $S = \{1, 2, ...\}$, $\varrho = \{T^i : i \in S\}$. For $f = (z_1, z_2, \cdots) \in B(S)$, define

$$\mu_n(f) = \frac{1 - a_n}{a_n}\sum_{k=1}^{\infty} (a_n)^k z_k \qquad (n \in \mathbb{N}).$$

Then $\{\mu_n\}$ is a regular sequence of means on $B(S)$; for more details, see ([19], p. 79). Next for each $x \in H$ and $n \in \mathbb{N}$, we have

$$T_{\mu_n} x = \frac{1 - a_n}{a_n}\sum_{k=1}^{\infty} (a_n)^k T^k x.$$

Therefore, it follows from Theorem 3.1 that the sequence $\{z_n\}$ converges strongly to $x^* \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\Big\langle (A-\gamma f)x^*, x-x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Theorem 3.4 *Let H be a real Hilbert space, and C be a nonempty closed convex subset of a Hilbert space H, and $S=\mathbb{R}^+=\{t\in\mathbb{R}: 0\leq t< +\infty\}$, ϱ=$\{T\dot{}t$:$t\in \mathbb{R}^+\}$, and $\varrho=\{T_t : t\in\mathbb{R}^+\}$ be a representation of S as nonexpansive mappings of C into itself and suppose $\mathrm{Fix}(\varrho)\neq\emptyset$. Let X be a left invariant subspace of $B(\mathbb{R}^+)$ such that $1\in X$ and the function $t\mapsto \Big\langle T_t x, y\Big\rangle$ is an element of X for each $x\in C$, $y\in H$, $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix}(\varrho))\subseteq \mathrm{Fix}(\varrho)$for $i\in\mathbb{N}$, $\wp=\{G_k : k=1,2,\cdots K\}$ be a finite family of bifunctions from $H\times H$ into $\mathbb{R}$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, and f is an α-contraction on H. Moreover, let $\{r_{k,n}\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n}>0$, $0<\epsilon_n<1$ and $0<\lambda_n\leq b<1$, and γ is a real number such that $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Assume that,*

(i) *for every $k\in\{1,2,\cdots,K\}$, the function G_k satisfies $(\mathrm{A}_1)-(\mathrm{A}_4)$ of Theorem 2.5,*

(ii) $\mathfrak{F}:=\bigcap_{n\in\mathbb{N}}\mathrm{Fix}(\mathrm{T_n})\cap\mathrm{Fix}(\varrho)\cap\mathrm{SEP}(\wp)\neq\emptyset$,

(iii) $\lim_n \epsilon_n=0$ *and,*

(iv) *for every $k\in\{1,2,\cdots,K\}$, $\lim_n r_{k,n}$ exists and is a positive real number.*

For every $n\in\mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.3), for every $k\in\{1,2,\cdots,K\}$ and $n\in\mathbb{N}$, let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in Theorem 2.5. If $\{z_n\}$ is the sequence generated by

$$z_n = \epsilon_n\gamma f(z_n) + (I-\epsilon_n A)\frac{1}{a_n}\int_0^{a_n} T_t W_n S_n^K z_n t \quad (n \in\mathbb{N}),$$

where $\{a_n\}$ is an increasing sequence in $(0,\infty)$ such that $\lim_n a_n=\infty$. Then $\{z_n\}$ strongly converges to $x^*\in\mathfrak{F}$, where x^* is the unique solution of the variational inequality

$$\Big\langle (A-\gamma f)x^*, x-x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Proof. For $f\in B(\mathbb{R}^+)$, define

$$\mu_n(f)=\frac{1}{a_n}\int_0^{a_n} f(t)t \qquad (n\in\mathbb{N}).$$

Then $\{\mu_n\}$ is a regular sequence of means on $B(\mathbb{R}^+)$; for more details, see ([19], p. 80). Next for each $x\in H$ and $n\in\mathbb{N}$, we have

$$T_{\mu_n}x=\frac{1}{a_n}\int_0^{a_n} T_t x t \qquad (n\in\mathbb{N}).$$

Therefore, it follows from Theorem 3.1 that the sequence $\{z_n\}$ converges strongly to $x^*\in\mathfrak{F}$, which is the unique solution of the variational inequality:

$$\Big\langle (A-\gamma f)x^*, x-x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Theorem 3.5 *Let $\varrho=\{T_t : t\in S\}$ be a representation of S as nonexpansive mappings of H into itself such that $\mathrm{Fix}(\varrho)\neq\emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1\in X$, and the function $t\mapsto\Big\langle T_t x, y\Big\rangle$ is an element of X for each $x,y\in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ and f is an α-contraction on H. Moreover, let $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $0<\epsilon_n<1$, $\lim_n\epsilon_n=0$, $0<\lambda_n\leq b<1$, and γ is a real number such that $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. If $\{z_n\}$ is the sequence generated by*

$$z_n=\epsilon_n\gamma f(z_n)+(I-\epsilon_n A)T_{\mu_n}z_n \quad (n\in\mathbb{N}).$$

Then $\{z_n\}$ *strongly converges to* $x^* \in \mathrm{Fix}(\varrho)$.
Proof. Take $G_k = 0$ for every $k \in \{1, 2, \cdots K\}$, $T_i = I$ for every $i \in \mathbb{N}$ and $C = H$ in Theorem 3.1. Then we have $S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}} z_n = z_n$ and $W_n = I$ for all $n \in \mathbb{N}$. So from Theorem 3.1 the sequences $\{z_n\}$ converges strongly to $x^* \in \mathrm{Fix}(\varrho)$.

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