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Research Article

Extensions of Regular Rings

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Abstract

Let R be an associative ring with identity. An element $x \in R$ is called $\mathbb{Z}G$-regular (resp. strongly $\mathbb{Z}G$-regular) if there exist $g \in G$, $n \in \mathbb{Z}$ and $r \in R$ such that $x^{ng} = x^{ng}rx^{ng}$ (resp. $x^{ng} = x^{(n+1)g}$). A ring R is called $\mathbb{Z}G$-regular (resp. strongly $\mathbb{Z}G$-regular) if every element of R is $\mathbb{Z}G$-regular (resp. strongly $\mathbb{Z}G$-regular). In this paper, we characterize $\mathbb{Z}G$-regular (resp. strongly $\mathbb{Z}G$-regular) rings. Furthermore, this paper includes a brief discussion of $\mathbb{Z}G$-regularity in group rings.

Keywords : Group ring; π-Regular; $\mathbb{Z}G$-Regular; Strongly $\mathbb{Z}G$-regular.

1 Introduction

Recall that an element x in R is said to be regular if $xyx = x$, for some $y \in R$, the ring R is regular if every element of R is regular and an element $x \in R$ is said to be strongly (Von Neumann) regular if there exists $y \in R$ such that $x = x^2y$, the ring R is strongly regular if each of elements R is strongly regular. More properties of regular and strongly regular rings can be found for example in [2, 7, 10]. An element $a \in R$ is said to be π-regular if there exist $b \in R$ and a positive integer n such that $a^n = a^nba^n$. An element $a \in R$ is said to be strongly π-regular if $a^n = a^{n+1}b$. The ring R is π-regular if every element of R is π-regular and is strongly π-regular if every element of R strongly π-regular. By a result of Azumaya [3] and Dischinger [9], the element a can be chosen to commute with b. In particular this definition is left-right symmetric. π-regular and strongly π-regular rings, are studied in particular in [3, 2, 4, 5, 6, 8]. Denote by $\mathbb{Z}G$ the integral group ring of a finite group G. An element $x \in R$ is said to be G-regular if there exist $y \in R$ and $g \in G$ such that $x^g = x^gyx^g$. The ring R is G-regular if each elements of R is G-regular. An element $x \in R$ is said to be strongly G-regular if there exist an element $y \in R$ and $g \in G$ such that $x^g = x^{2g}y$, with this property that $(x^2)^g = (x^g)^2$. A ring R is strongly G-regular if every element of is strongly G-regular. A ring R is abelian if every idempotent element of R is central. A ring R is called locally finite if every finite subset in it generates a finite semigroup multiplicatively. A group is locally finite if every finitely generated subgroup in it, is finite. The $n \times n$ full triangular matrix ring , the $n \times n$ upper triangular matrix ring, the $n \times n$ lower triangular matrix ring over denote by $M_n(R)$, $U_n(R)$, $L_n(R)$ respectively. In Section 2 we define $\mathbb{Z}G$-regular and strongly $\mathbb{Z}G$-regular rings and investigate some characterization of them. Let G be a group and X a set. Then a group action (or just action) of G on X is a binary operation:

$$\mu : X \times G \longrightarrow X$$

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(If there is no fear of confusion, we write $\mu(x,g)$ simply as by x^g) such that

(i) $(x^g)^h = x^{gh}$ for all $x \in X$ and $g, h \in G$,

(ii) $x^1 = x$ for all $x \in X$.

If S is a subset of R and $\prod_{i\in I} R_i$ is a finite direct product of $\{R_i\}_{i\in I}$, then we define: $a^{ng} = (a^g)^n$, $(x_i)^g_{i\in I} = (x^g_i)_{i\in I}$, $S^g = \{x^g | x \in S\}$. For each $(x_i)_{i\in I} \in \prod_{i\in I} R_i$, $g \in G$, $n \in \mathbb{Z}$. The main purpose of Section 3 is to characterize $\mathbb{Z}G$-regular and strongly $\mathbb{Z}G$-regular group rings.

2 Preliminarie

Definition 2.1 *An element $x \in R$ is called $\mathbb{Z}G$-regular (resp. Strongly $\mathbb{Z}G$-regular) if there exist $g \in G$, $n \in \mathbb{Z}$ and $r \in R$ such that $x^{ng} = x^{ng} r x^{ng}$ (resp. $x^{ng} = x^{(n+1)g} r$). A ring R is called $\mathbb{Z}G$-regular (resp. Strongly $\mathbb{Z}G$-regular) if every element of R is $\mathbb{Z}G$-regular (resp. strongly $\mathbb{Z}G$-regular). So an element $x \in R$ is $\mathbb{Z}G$-regular (resp. Strongly $\mathbb{Z}G$-regular) if there exists $g \in G$ such that x^g is π-regular (resp. Strongly π-regular).*

Now we define a $\mathbb{Z}G$-regular ideal as follows: Let $\mu : R \times G \longrightarrow R$ be a group action and I be a two-sided ideal of R. Then group G acts on R/I by the rule $\mu(r + I, g) = \mu(r, g) + I$.

Definition 2.2 *Let J be a two-sided ideal of a ring R. J is a $\mathbb{Z}G$-regular ideal of R in case for any $x \in J$, there exist $n \in \mathbb{Z}$ and $y \in R$ such that $x^{ng} = x^{ng} y x^{ng}$.*

Theorem 2.1 *Any factor ring of a $\mathbb{Z}G$-regular (resp. Strongly $\mathbb{Z}G$-regular) ring is $\mathbb{Z}G$-regular (resp. Strongly $\mathbb{Z}G$-regular). In particular a homomorphic image of a $\mathbb{Z}G$-regular (resp. Strongly $\mathbb{Z}G$-regular) ring is $\mathbb{Z}G$-regular (resp. Strongly $\mathbb{Z}G$-regular).*

Proof. Let R be $\mathbb{Z}G$-regular (resp. Strongly $\mathbb{Z}G$-regular) and I be a two-sided ideal of R. Let $\overline{x} = x + I \in R/I$. Since R is $\mathbb{Z}G$-regular, then there exist $g \in G$, $n \in \mathbb{Z}$ and $r \in R$ such that $x^{ng} = x^{ng} r x^{ng}$ (resp. $x^{ng} = x^{(n+1)g} r$). This implies $\overline{x^{ng}} = \overline{x^{ng}}\,\overline{r}\,\overline{x^{ng}}$ (resp. $\overline{x^{ng}} = \overline{x^{ng}}\,\overline{r}$). Thus by definition we have $\overline{x}^{ng} = \overline{x}^{ng}\overline{r}\overline{x}^{ng}$ (resp. $\overline{x}^{ng} = \overline{x}^{(n+1)g}\overline{r}$).

Theorem 2.2 *Let R be a ring. Then the following conditions are equivalent:*

(i) *R is strongly $\mathbb{Z}G$-regular.*

(ii) *R/N is strongly $\mathbb{Z}G$-regular that is the prime radical N of R.*

(iii) *Every prime factor ring of R is strongly $\mathbb{Z}G$-regular.*

Proof. It suffices to show that (iii) implies (i). Suppose R contains an element a that is not strongly $\mathbb{Z}G$-regular. Then by Zorn s lemma, there exists an ideal I of R that is maximal with respect to the property that $\overline{a}$ is not strongly $\mathbb{Z}G$-regular in $\overline{R} = R/I$. Since I can not be prime, there exist ideals K, L properly containing I such that $KL \subseteq I$ Then we can find a $n \in \mathbb{Z}$ such that $a^{ng_1} - a^{(n+1)g_1} x \in K$ and $a^{ng_2} - a^{(n+1)g_2} y \in L$ for some $x, y \in R$, $g_1, g_2 \in G$. But

$$a^{n(g_1+g_2)} - a^{(n+1)(g_1+g_2)}(a^{-g_1} y + a^{-g_2} x + xy) = (a^{ng_1} - a^{(n+1)g_1} x)(a^{ng_2} - a^{(n+1)g_2} y) \in KL \subseteq I$$

Which is a contradiction.

Lemma 2.1 *Let G be a group acts on the ring R by this property that $(xy)^g = x^g y^g$ for each $x, y \in R$. If $x, y \in R$, $g \in G$ and $x' = x^{ng} - x^{ng} y x^{ng}$, and if $x'^{n'h} = x'^{n'h} a^{n'} x'^{n'h}$ for some $a \in R$ and some $h \in G$. Then $x^{ng} = x^{ng} b x^{ng}$ for some $b \in R$.*

Proof. We have

$$\begin{aligned}
x^{ng} &= x' + x^{ng} y x^{ng} \\
&= (x'^{n'h} a^{n'} x'^{n'h})^{n'^{-1}h^{-1}} + x^{ng} y x^{ng} \\
&= x' a^{h^{-1}} x' + x^{ng} y x^{ng} \\
&= (x^{ng} - x^{ng} y x^{ng}) a^{h^{-1}} (x^{ng} - x^{ng} y x^{ng}) \\
&\quad + x^{ng} y x^{ng} \\
&= (x^{ng} a^{h^{-1}} - x^{ng} y x^{ng} a^{h^{-1}})(x^{ng} - x^{ng} y x^{ng}) \\
&\quad + x^{ng} y x^{ng} \\
&= x^{ng} a^{h^{-1}} x^{ng} - x^{ng} y x^{ng} a^{h^{-1}} x^{ng} \\
&\quad - x^{ng} a^{h^{-1}} x^{ng} y x^{ng} + x^{ng} y x^{ng} a^{h^{-1}} x^{ng} y x^{ng} \\
&\quad + x^{ng} y x^{ng} \\
&= x^{ng} (a^{h^{-1}} - y x^{ng} a^{h^{-1}} - a^{h^{-1}} x^{ng} y \\
&\quad + y x^{ng} a^{h^{-1}} x^{ng} y + y) x^{ng}
\end{aligned} \tag{2.1}$$

Now by taking $b = a^{h^{-1}} - y x^{ng} a^{h^{-1}} - a^{h^{-1}} x^{ng} y + y x^{ng} a^{h^{-1}} x^{ng} y + y$ we have $x^{ng} = x^{ng} b x^{ng}$.

Theorem 2.3 *Let $J \subseteq K$ be two sided ideals in a ring R. So J and K/J are both $\mathbb{Z}G$-regular if and only if K is $\mathbb{Z}G$-regular.*

Proof. Assume that J and K/J are both $\mathbb{Z}G$-regular. Given $x \in J$, it follows from the regularity of K/J that $x^{ng} - x^{ng}yx^{ng} \in J$ for some $y \in K$ and $n \in \mathbb{Z}$. Consequently, $(x^{ng} - x^{ng}yx^{ng})^{n'h} = (x^{ng} - x^{ng}yx^{ng})^{n'h}z(x^{ng} - x^{ng}yx^{ng})^{n'h}$ for some $z \in J$, from which by lemma 2.1 we conclude that $x^{ng} = x^{ng}wx^{ng}$ for some $w \in K$. Thus, K is $\mathbb{Z}G$-regular. Conversely, assume that K is a $\mathbb{Z}G$-regular ring. Clearly K/J is $\mathbb{Z}G$-regular. It suffices to show that J is $\mathbb{Z}G$-regular. Since K is a $\mathbb{Z}G$-regular ring then for any $\forall x \in J$, there exist $g \in G$ and $y \in K$ and $n \in \mathbb{Z}$ such that

$$x^{ng} \in J,\, x^{ng} = x^{ng}yx^{ng}$$

Now by taking $z = yx^{ng}y \in J$ we have:

$$x^{ng} = x^{ng}zx^{ng}$$

Therefore, J is a $\mathbb{Z}G$-regular ideal.

Lemma 2.2 *A finite direct product $\prod_{i \in I} R_i$ (I is a finite set) of $\mathbb{Z}G$-regular rings $\{R_i\}_{i \in I}$ is $\mathbb{Z}G$-regular.*

Proof. At first we prove that direct product of two $\mathbb{Z}G$-regular rings is $\mathbb{Z}G$-regular. Let R_1 and R_2 be two $\mathbb{Z}G$-regular rings. Then for every $(a_1, a_2) \in R_1 \times R_2$ there exist $g_1, g_2 \in G$, $(r_1, r_2) \in R_1 \times R_2$ and $n_1, n_2 \in \mathbb{Z}$ such that $a_1^{n_1g_1} = a_1^{n_1g_1}r_1a_1^{n_1g_1}$ and $a_2^{n_2g_2} = a_2^{n_2g_2}r_2a_2^{n_2g_2}$. Now by setting $ng = n_1n_2g_1g_2$ we have

$$\begin{aligned}
&(a_1, a_2)^{ng} = (a_1^{ng}, a_2^{ng}) \\
&= ((a_1^{n_1g_1})^{n_2g_2}, (a_2^{n_2g_2})^{n_1g_1}) \\
&= ((a_1^{n_1g_1}r_1a_1^{n_1g_1})^{n_2g_2}, (a_2^{n_2g_2}r_2a_2^{n_2g_2})^{n_1g_1}) \\
&= (a_1, a_2)^{ng}(r_1^{n_2g_2}, r_2^{n_1g_1})(a_1, a_2)^{ng}
\end{aligned} \tag{2.2}$$

Thus by induction any finite direct product of $\mathbb{Z}G$-regular rings is $\mathbb{Z}G$-regular.

Theorem 2.4 **(i)** *Let $x \in R$ be $\mathbb{Z}G$-regular, then there exist $g \in G$, $n \in \mathbb{Z}$ and $r \in R$ such that $x^{ng}r$ is idempotent.*

(ii) *If an element $x \in R$ is π-regular, then it is $\mathbb{Z}G$-regular by taking G to be trivial group.*

(iii) *An element $x \in R$ is $\mathbb{Z}G$-regular if there exist $g \in G$, $n \in \mathbb{Z}$ such that x^{ng} is Von Neumann.*

Proof. (i) Since $x \in R$ is $\mathbb{Z}G$-regular thus there exist $g \in G$, $n \in \mathbb{Z}$ and $r \in R$ such that $x^{ng} = x^{ng}rx^{ng}$ therefore $x^{ng}r = x^{ng}rx^{ng}r = (x^{ng}r)^2$. (ii), (iii) are trivial.

Theorem 2.5 *Let S be the center of $\mathbb{Z}G$-regular ring R with the property that $S^{ng} \subseteq S$, for any $g \in G$, $n \in \mathbb{Z}$. Then S is $\mathbb{Z}G$-regular.*

Proof. Let R be a ring with center S, and let $x \in S$. There exist $y \in R$, $n \in \mathbb{Z}$ and $g \in G$ such that $x^{ng}yx^{ng} = x^{ng}$, and we set $z = yx^{ng}y$. Note that

$$x^{ng}zx^{ng} = x^{ng}yx^{ng}yx^{ng} = x^{ng}$$

For any $r \in R$, we have

$$\begin{aligned}
zr = \;& yx^{ng}yr \\
&= y^2rx^{ng} \\
&= y^2rx^{ng}yx^{ng} \\
&= y^2rx^{ng}x^{ng}y \\
&= yx^{ng}yrx^{ng}y \\
&= yx^{ng}yx^{ng}ry \\
&= yx^{ng}ry
\end{aligned} \tag{2.3}$$

Similarly we have $rz = yrx^{ng}y$, so $rz = yrx^{ng}y = yx^{ng}ry = zr$, therefore $z \in S$. Thus S is also $\mathbb{Z}G$-regular.

Proposition 2.1 *A ring R is strongly $\mathbb{Z}G$-regular, if and only if R satisfies the descending chain condition on principal right ideals of the form $a^gR \supseteq a^{2g}R \supseteq ...$, for every $a \in R$ and an element $g \in G$.*

Proof. One direction is clear. Assume R is not strongly $\mathbb{Z}G$-regular. Then there exists an element $a \in R$ such that $x^{ng} \neq x^{(n+1)g}r$ for any $r \in R$ and $g \in G$ and $n \in \mathbb{Z}$. We have a descending chain $a^gR \supset a^{2g}R \supset ...$ of ideals of R which does not terminate, which is a contradiction.

Lemma 2.3 *Let R be a ring. If R is locally finite and $a \in R$, then a^t is an idempotent for some positive t.*

Proof. see [11].

Theorem 2.6 *Let R be a ring. If R is a locally finite ring, then R is strongly $\mathbb{Z}G$-regular.*

By lemma 2.3, a locally finite ring R satisfies the descending chain condition on principal right ideals of form $aR \supseteq a^2R \supseteq ...$, for every a in R; then R satisfies the descending chain condition on principal right ideals of form $a^gR \supseteq a^{2g}R \supseteq ...$, for every $a \in R$ and $g \in G$. Therefore R is strongly $\mathbb{Z}G$-regular by proposition 2.1.

Proposition 2.2 *Let the $n \times n$ full triangular matrix ring over R be $\mathbb{Z}G$-regular. Then R is $\mathbb{Z}G$-regular.*

Proof. It is obvious. We introduced π-regular (resp. Strongly π-regular) rings as an example of $\mathbb{Z}G$-regular (resp. Strongly $\mathbb{Z}G$-regular) rings by taking G to be trivial group. Lee and Kim showed in [12], that the n by n full matrix rings over strongly π-regular ring R, need not be strongly π-regular (see: Example 2.1), so we conclude that the $n \times n$ full triangular matrix ring over R need not be strongly $\mathbb{Z}G$-regular rings.

Theorem 2.7 *For a ring R and a positive integer m, the following conditions are equivalent.*

(a) R is locally finite.

(b) $M_n(R)$ is locally finite.

(c) $U_n(R)$ is locally finite.

(d) $L_n(R)$ is locally finite.

Proof. see [11].

Example 2.1 *Let R be a locally finite ring. $M_n(R)$, $U_n(R)$, $L_n(R)$ R are examples of strongly $\mathbb{Z}G$-regular rings by Theorems 2.6 and 2.7.*

Lemma 2.4 *Let R be an abelian $\mathbb{Z}G$-regular ring. Then for each $x \in R$, there exist $r \in R, g \in G$ and $n \in \mathbb{Z}$ such that $x^{ng}r = rx^{ng}$.*

Since R is $\mathbb{Z}G$-regular, then by theorem 2.4 (i), for each $x \in R$, there exist $g \in G$, $n \in \mathbb{Z}$, $r \in R$ such that $x^{ng}r, rx^{ng} \in Id(R)$ and since R is abelian then $x^{ng}r, rx^{ng} \in Z(R)$, therefore we have:

$$\begin{aligned} x^{ng}r &= (x^{ng}rx^{ng})r \\ &= x^{ng}(rx^{ng})r \\ &= x^{ng}r(rx^{ng}) = r(x^{ng}rx^{ng}) \\ &= rx^{ng} \end{aligned} \tag{2.4}$$

Definition 2.3 *An element $x \in R$ is said unit $\mathbb{Z}G$-regular if there exist $g \in G$ and $u \in U(R)$ and $n \in \mathbb{Z}$ depending on x such that $x^{ng} = x^{ng}ux^{ng}$. R is unit $\mathbb{Z}G$-regular if every element of R is unit $\mathbb{Z}G$-regular.*

Theorem 2.8 *Let R be an abelian $\mathbb{Z}G$-regular ring. Then R is unit $\mathbb{Z}G$-regular.*

Since R is abelian $\mathbb{Z}G$-regular by lemma 2.4, for each $x \in R$, there exist $g \in G$, $y \in R$ and $n \in \mathbb{Z}$, such that $x^{ng}y = yx^{ng}$.

Let $u = x^{ng} + x^{ng}y - 1$ and $v = x^{ng}y + x^{ng}y^2 - 1$. Since $x^{ng}y = yx^{ng}$, then we have:

$$\begin{aligned} uv &= (x^{ng} + x^{ng}y - 1)(x^{ng}y + x^{ng}y^2 - 1) \\ &= x^{ng}(x^{ng}y) + x^{ng}x^{ng}y^2 - x^{ng} \\ &+(x^{ng}y)(x^{ng}y) + (x^{ng}y)(x^{ng}y^2) \\ &-x^{ng}y - x^{ng}y - x^{ng}y^2 + 1 \\ &= x^{ng}(yx^{ng}) + (x^{ng}yx^{ng})y - x^{ng} \\ &+(x^{ng}yx^{ng})y + (x^{ng}yx^{ng})y^2 \\ &-x^{ng}y - x^{ng}y - x^{ng}y^2 + 1 \\ &= x^{ng} + x^{ng}y - x^{ng} + x^{ng}y \\ &+x^{ng}y^2 - x^{ng}y - x^{ng}y - x^{ng}y^2 + 1 = 1 \end{aligned} \tag{2.5}$$

And

$$\begin{aligned} vu &= (x^{ng}y + x^{ng}y^2 - 1)(x^{ng} + x^{ng}y - 1) \\ &= x^{ng}yx^{ng} + x^{ng}yx^{ng}y - x^{ng}y \\ &+x^{ng}y^2x^{ng} + x^{ng}y^2x^{ng}y \\ &-x^{ng}y^2 - x^{ng} - x^{ng}y + 1 \\ &= x^{ng}yx^{ng} + (x^{ng}yx^{ng})y - x^{ng}y \\ &+x^{ng}y(yx^{ng}) + x^{ng}y(yx^{ng})y- \\ &x^{ng}y^2 - x^{ng} - x^{ng}y + 1 \\ &= x^{ng} + x^{ng}y - x^{ng}y + (x^{ng}yx^{ng})y \\ &+x^{ng}yx^{ng}y^2 - x^{ng}y^2 - x^{ng} - x^{ng}y + 1 \\ &= x^{ng} + x^{ng}y - x^{ng}y + x^{ng}y \\ &+x^{ng}y^2 - x^{ng}y^2 - x^{ng} - x^{ng}y + 1 = 1 \end{aligned} \tag{2.6}$$

Therefore, $uv = vu = 1$. Moreover,

$$\begin{aligned} x^{ng}vx^{ng} &= x^{ng}(x^{ng}y \\ &+x^{ng}y^2 - 1)x^{ng} \\ &= x^{ng}x^{ng}yx^{ng} \\ &+x^{ng}x^{ng}y^2x^{ng} - x^{ng}x^{ng} \\ &= x^{ng}x^{ng} + x^{ng}x^{ng}yyx^{ng} \\ &-x^{ng}x^{ng} \\ &= (x^{ng}yx^{ng})yx^{ng} \\ &= x^{ng}yx^{ng} \\ &= x^{ng} \end{aligned} \tag{2.7}$$

Theorem 2.9 *Let R be an abelian $\mathbb{Z}G$-regular ring, and $x \in R$. Then there exist $g \in G$, $n \in \mathbb{Z}$ such that $x^{ng} = eu$, for some $e \in Id(R)$ and $u \in U(R)$.*

Proof. By theorem 2.8, R is unit $\mathbb{Z}G$-regular. Thus there exists $v \in U(R)$ such that $x^{ng} = x^{ng}vx^{ng}$. Let u be the multiplicative inverse of v in R, then $x^{ng} = x^{ng}uv = x^{ng}vu = eu$. Since $e = x^{ng}v \in Id(R)$. Thus $x^{ng} = eu$ for some $e \in Id(R)$ and $u \in U(R)$.

Theorem 2.10 *Let R be an abelian ring. Then the following statements are equivalent:*

(i) *R is a unit $\mathbb{Z}G$-regular ring.*

(ii) *For every $a \in R$, there exist $g \in G$, $n \in \mathbb{Z}$ such that a^{ng} can be written as a product of a unit, and an idempotent of R.*

(iii) *For every $a \in R$, there exist $g \in G$, $n \in \mathbb{Z}$ such that a^{ng} can be written as a product of an idempotent and a unit of R.*

Proof. (i $\Rightarrow$ ii) By theorem 2.9, is clear.
(ii $\Rightarrow$ i) Suppose there exists $g \in G$, $n \in \mathbb{Z}$ such that $a^{ng} = ve$ where $v \in U(R)$ and $e^2 = e$. The latter implies $v^{-1}a^{ng}=v^{-1}a^{ng}v^{-1}a^{ng}$, so $a^{ng} = a^{ng}v^{-1}a^{ng}$, as desired.

3 $\mathbb{Z}G$-regular group ring

Let R be a ring and G a group. We shall denote the group ring of G over R as RG. The augmentation ideal of RG is generated by $\{1, g\}$. We shall use Δ to denote the augmentation ideal of RG. It is known that R is a homomorphic image of RG. Since $RG/\Delta \cong R$. For any element $x = \sum_{g\in G} x_g g \in RG$, the support of x, written as $Supp(x)$, is the subset of G consisting of all those $g \in G$ such that $x_g \neq 0$. Since $x_g \neq 0$ for only finitely many $g \in G$, so Supp(x) is a finite subset of G.

Corollary 3.1 *Let R be a ring and G a group. If RG is a $\mathbb{Z}G$-regular ring, then R is a $\mathbb{Z}G$-regular ring.*

Proof. Since R is homomorphic image of RG, then R is $\mathbb{Z}G$-regular by theorem 2.1. For any idempotent e in a ring, we have the following peirce decomposition:

$$R = eRe \oplus eRf \oplus fRe \oplus fRf$$

Where $f = 1 - e$ is the complementary idempotent to e. Two ring eRe and fRf be characterized by the equation:
$eRe = \{rR : er = r = re\}$,
$fRf = \{rR : fr = r = rf\}$

Lemma 3.1 *e is a central idempotent iff $eRf = fRe = 0$.*

Proof. For $r \in R$, $erf = 0$ and $fre = 0$ amount to $er = ere = re$.

Proposition 3.1 *Let $e \neq 0$ be any central idempotent in R. If eRe and fRf are $\mathbb{Z}G$-regular, then R is a $\mathbb{Z}G$-regular ring.*

Since e is a central idempotent, then we have the peirce decomposition:

$$R = eRe \oplus fRf$$

Thus by lemma 2.2, since eRe and fRf are $\mathbb{Z}G$-regular then R is $\mathbb{Z}G$-regular.

Theorem 3.1 *Let $e_1 + \ldots + e_n = 1$ be a decomposition of 1 into sums of orthogonal idempotents. If e_iRe_i is $\mathbb{Z}G$-regular for each i, then R is $\mathbb{Z}G$-regular.*

Proof. It is obvious from Lemma 2.2 and Proposition 3.1.

Theorem 3.2 *Let R be a commutative semiperfect ring and G a group, and let $(eRe)G$ be $\mathbb{Z}G$-regular for each local idempotent e in R. Then RG is $\mathbb{Z}G$-regular.*

Proof. Since R is semiperfect, so by theorem 6.27 of [1], R has a complete orthogonal set $e_1, e_2, ..., e_n$ of idempotent R. So e_i is a local idempotent for each $i \in \{0, ..., n\}$. Now by hypothesis, $(e_iRe_i)G$ is $\mathbb{Z}G$-regular. Since, $(e_iRe_i)G \cong e_i(RG)e_i$ for each i, it follows that $e_i(RG)e_i$ is $\mathbb{Z}G$-regular. Hence RG is $\mathbb{Z}G$-regular by proposition 3.1.

Theorem 3.3 *Let R be a ring in which 2 is invertible and $G = \{1, g\}$ be a group. Then RG is $\mathbb{Z}G$-regular if and only if R is $\mathbb{Z}G$-regular.*

Proof. If RG is $\mathbb{Z}G$-regular, then by corollary 3.1, R is $\mathbb{Z}G$-regular. Conversely, since R is $\mathbb{Z}G$-regular and 2 is invertible in R, then $RG \cong R \times R$ via the map $a + bg \Longleftrightarrow (a + b, a - b)$. Hence RG is $\mathbb{Z}G$-regular by lemma 2.2.

Theorem 3.4 *Let R be a ring and G a group. Then RG is strongly $\mathbb{Z}G$-regular if and only if $(R/P)G$ is strongly $\mathbb{Z}G$-regular for every prime ideal P of R.*

Proof. If RG is strongly $\mathbb{Z}G$-regular, and I is an ideal of R, then since

$$(R/I)G \cong RG/IG$$

and homomorphic images of strongly $\mathbb{Z}G$-regular rings strongly $\mathbb{Z}G$-regular, it follows that $(R/I)G$ is strongly $\mathbb{Z}G$-regular.
Conversly, suppos to the contrary that RG is not strongly $\mathbb{Z}G$-regular. Then there exists an element $x \in RG$ such that for any $n \in \mathbb{Z}$ and $g \in G$,

$x^{ng} \neq x^{(n+1)g}y$ for any $y \in RG$. Therefore the sequence
$x^g RG \supseteq x^{2g} RG$
$\supseteq ... \supseteq x^{ng}RG \supseteq x^{(n+1)g}RG \supseteq ...$
ideals of does not terminate. Let $\Im$ be the set of all ideals I of R such that the sequence
$(x + IG)^g(RG/IG)$
$\supseteq (x + IG)^{2g}(RG/IG) \supseteq ...$
does not terminate. Note that $\Im \neq \emptyset$, since $(0) \in \Im$. Furthermore, $\Im$ is partially ordered by inclusion. Let $(I_\alpha)_{\alpha\in\Omega}$ be a chain of elements of $\Im$ and let $J = \cup_{\alpha\in\Omega} I_\alpha$. Clearly, J is an ideal of R and $I_\alpha \subset J$ for all $\alpha \in \Omega$. We show that $J \in \Im$. Suppose that $J \notin \Im$. Then $z = x^{ng} - x^{(n+1)g}r \in JG$ for some $r \in RG$, $g \in G$ and $n \in \mathbb{Z}$. Since $Supp(z)$ is finite, there exists some $\alpha \in \Omega$ such that $z \in I_\alpha G$. It follows that the sequence $(x + I_\alpha G)^g(RG/I_\alpha G)$
$\supseteq (x + I_\alpha G)^{2g}(RG/I_\alpha G) \supseteq ...$
terminates, which is a contradiction. Therefore $J \in \Im$ and thus by Zorn 's Lemma, $\Im$ contains a maximal element M. Since $(R/M)G \cong RG/MG$ is not strongly $\mathbb{Z}G$-regular, it follows by hypothesis that M is not a prime ideal. Therefore there exist ideals A, B of R such that $AB \subseteq M$ but $A, B \nsubseteq M$. Let $A' = M + A$ and $B' = B + M$. Then M is strictly contained in A' and B', and we also have that:

$$A'B' = (M + A)(M + B) \subseteq M$$

By the maximality of M in $\Im$, the sequences
$(x + A'G)^{2g}(RG/A'G)$
$\supseteq (x + A'G)^{2g}(RG/A'G) \supseteq ...$
And
$(x + B'G)^{2g}(RG/B'G)$
$\supseteq (x + B'G)^{2g}(RG/B'G) \supseteq ...$
both terminate. Hence there exists $m \in \mathbb{Z}$ such that $(x^{mg} + A'G)(RG/A'G) = (x^{(2m+1)g} + A'G)(RG/A'G)$ and $(x^{mg} + B'G)(RG/B'G) = (x^{(2m+1)g} + B'G)(RG/B'G)$.
It follows that $x^{mg} - x^{(2m+1)g}s \in A'G$ and $x^{mg} - x^{(2m+1)g}t \in B'G$ for some $s, t \in RG$. Therefore:
$(x^{mg} - x^{(2m+1)g}s)(x^{mg} - x^{(2m+1)g}t)$
$\in (A'B')G \subseteq MG$
Form which it follows that $x^{mg} - x^{(2m+1)g}w \in MG$ for some $w \in RG$. Hence the sequence:
$(x + MG)^g(RG/MG)$
$\supseteq (x + MG)^{2g}(RG/MG) \supseteq ...$
terminates; contradicting the fact that $M \in \Im$. We thus have that RG must be a strongly $\mathbb{Z}G$ ring.

Theorem 3.5 *Let R be a ring with artinian prime factors and G be a locally finite group. Then RG is strongly $\mathbb{Z}G$-regular.*

Proof. Let P be a prime ideal of R and $x = \sum_{g\in G} r_g g \in (R/P)G$. Let H_x be the subgroup of G generated by the support of x. Since Supp(x) is finite and G is locally finite, it follows that H_x is finite. It is clear that $x \in (R/P)H_x$ is strongly $\mathbb{Z}G$-regular. Indeed, since R/P is artinian and H_x is finite, so $(R/P)H_x$ is artinian; hence strongly $\mathbb{Z}G$-regular. Since x is arbitrary in $(R/P)G$, so $(R/P)G$ is also strongly $\mathbb{Z}G$-regular. By theorem 3.3, it follows that RG is strongly $\mathbb{Z}G$-regular.

Theorem 3.6 *Let R be a ring, G be a group and $U_n(RG)$ be strongly $\mathbb{Z}G$-regular for $n \geq 2$. Then R is strongly $\mathbb{Z}G$-regular.*

Proof. As $U_n(RG)$ is strongly $\mathbb{Z}G$-regular, so by example 2.1, RG is strongly $\mathbb{Z}G$-regular. Hence by Corallary 3.1, R is strongly $\mathbb{Z}G$-regular.

4 Examples

Here we give some examples of $\mathbb{Z}G$-regular rings.

Example 4.1 *It is clear that if G is a trivial group (group with only one element) then R is $\mathbb{Z}G$-regular for $n \geq 1$ iff R is π-regular.*

Example 4.2 *One easily checks that $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ are $\mathbb{Z}G$-regular rings because they are π-regular rings.*

Example 4.3 *Let $G = U(R)$ (where $U(R)$ is the group of units in R) and X a set, then a regular action μ of G on X is a function:*

$$\mu : X \times G \longrightarrow X : (x, g) = gx \tag{4.8}$$

And conjugate action is a function:

$$\mu : X \times G \longrightarrow X : (x, g) = gxg^{-1} \tag{4.9}$$

Example 4.4 *Let $G = U(R)$. An element $x \in R$ is said unitary π-regular (resp. strongly unitary R-regular) if there exist $g \in G$ and $r \in R$ and $n \in \mathbb{Z}$ such that $(gx)^n = (gx)^n r(gx)^n$ (resp. $(gx)^n = (gx)^{(n+1)}r$). R is unitary π-regular (resp. strongly unitary π-regular) if every element of R is unitary π-regular (resp. strongly unitary π-regular).*

Example 4.5 *Let $G = U(R)$. An element $x \in R$ is said conjugate π-regular (resp. strongly conjugate π-regular) if there exist $g \in G$ and $r \in R$ and $n \in \mathbb{Z}$ such that $(x)^n = (x)^n g^{-1} r g (x)^n$ (resp. $(x)^n = (x)^{n+1} g x g$). R is conjugate π-regular (resp. strongly conjugate π-regular) if every element of R is conjugate π-regular (resp. strongly conjugate π-regular).*

Example 4.6 *Let $Aut(R)$ be automorphism group of R. An element $x \in R$ is called Automorphic π-regular ((Aut) π-regular) if there exist $\alpha \in Aut(R)$, $r \in R$ and $n \in \mathbb{Z}$ such that $(x^\alpha)^n = (x^\alpha)^n r (x^\alpha)^n$. R is Automorphic π-regular every element of R is automorphic π-regular.*

5 Conclusion

Ring theory is a subject of central importance in algebra. Historically, some of major discoveries in ring theory have helped shape the course of development of abstract algebra. In the moment, ring theory is a fertile meeting ground for group rings. In this paper, we characterized $\mathbb{Z}G$-regular and strongly $\mathbb{Z}G$-regular group rings.

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