

Fixed points for total asymptotically nonexpansive mappings in a new version of bead space

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Abstract

The notion of a bead metric space is defined as a nice generalization of the uniformly convex normed space such as $CAT(0)$ space, where the curvature is bounded from above by zero. In fact, the bead spaces themselves can be considered in particular as natural extensions of convex sets in uniformly convex spaces and normed bead spaces are identical with uniformly convex spaces. In this paper, we define a new version of bead space and called it CN -bead space. Then the existence of fixed point for asymptotically nonexpansive mapping and total asymptotically nonexpansive mapping in CN -bead space are proved. In other word, Let K be a bounded subset of complete CN -bead space X . Then the fixed point set $F(T)$, where T is a total asymptotically nonexpansive selfmap on K , is nonempty and closed. Moreover, the fixed point set $F(T)$, where T is an asymptotically nonexpansive selfmap on K , is nonempty.

Keywords : Bead space; $CAT(0)$ space; Fixed point; Total asymptotically nonexpansive mapping.

1 Introduction

THE notion of a bead metric space is defined in [4] and [5] as a nice generalization of the uniformly convex normed space. In this paper, we define a new version of bead space and called it CN -bead space. This idea enables us to show the existence of fixed point for total asymptotically nonexpansive mapping in this setting. Due to this, some definitions and a lemma are given as following:

Let K be a subset of a complete metric space X . A mapping $T : K \rightarrow K$ is said to be nonexpansive provided that

$$d(Tx, Ty) \leq d(x, y), \quad x, y \in K.$$

The mapping T is called asymptotically nonexpansive, if there exists a nonnegative sequence

$\{k_n\}_{n \geq 1}$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + k_n)d(x, y),$$

for all $x, y \in K$ and $n \geq 1$. This class of mappings was introduced by Goebel and Kirk [3] as a generalization of the class of nonexpansive mappings. In 2006, Alber et. al. [1] introduced the class of total asymptotically nonexpansive mappings as below:

Definition 1.1 Let K be a subset of complete metric space X . A mapping $T : K \rightarrow K$ is called total asymptotically nonexpansive, if there exist nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$, $n \geq 1$, with $k_n^{(1)}, k_n^{(2)} \rightarrow 0$ as $n \rightarrow \infty$, and strictly increasing and continuous functions $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\Psi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + k_n^{(1)}\psi(d(x, y)) + k_n^{(2)} \quad (1.1)$$

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Remark 1.1 If $\psi(\lambda) = \lambda$, then 1.1 takes the form

$$d(T^n x, T^n y) \leq (1 + k_n^{(1)})d(x, y) + k_n^{(2)} \quad (1.2)$$

In addition, if $k_n^{(2)} = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. If $k_n^{(1)} = 0$ and $k_n^{(2)} = 0$ for all $n \geq 1$, then we obtain from (1.1) the class of nonexpansive mappings.

Definition 1.2 (see [4, Definition 1]) Let (X, d) be a metric space and $\emptyset \neq A \subset X$ a bounded set. An $x \in X$ is a central point for A respect to X if

$$\begin{aligned} r(A) &:= \inf\{t \in (0, \infty) : \exists z \in X, \\ &\quad A \subset B(z, t)\} \\ &= \inf\{t \in (0, \infty) : A \subset B(x, t)\}. \end{aligned}$$

The center $c(A)$ for A is the set of all central points for A , and $r(A)$ is the radius of A .

Definition 1.3 (see [4, Definition 6]) A metric space (X, d) is a bead space if the following is satisfied:

for every $r > 0, \beta > 0$ there exists a $\delta > 0$ such that for each $x, y \in X$ with $d(x, y) \geq \beta$ there exists a $z \in X$ such that

$$B(x, r + \delta) \cap B(y, r + \delta) \subset B(z, r - \delta).$$

Lemma 1.1 (see [4, Lemma 10]) Let (X, d) be a bead space and $\emptyset \neq A \subset X$ be bounded. Then $c(A)$ consists of at most one point. If in addition (X, d) is complete, then $c(A)$ is a singleton.

Finally, we recall the definition and some properties of $CAT(0)$ spaces (see [2] and [6]) as follows: Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(\acute{t})) = |t - \acute{t}|$ for all $t, \acute{t} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y . When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x to y for each $x, y \in X$. A subset $Y \subseteq X$ is said to convex if Y includes every geodesic segment joining any two of its points. A geodesic

triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane E^2 such that $d_{E^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a $CAT(0)$ space [2], if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be a comparison triangle for Δ . Then Δ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$: $d(x, y) \leq d_{E^2}(\bar{x}, \bar{y})$.

Let X be a complete $CAT(0)$ space, (x_n) be a bounded sequence in X and for $x \in X$ set:

$$r(x, (x_n)) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius $r((x_n))$ of (x_n) is given by

$$r((x_n)) = \inf\{r(x, (x_n)) : x \in X\},$$

and the asymptotic center $A((x_n))$ of (x_n) is the set:

$$A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}.$$

In the next section, the notion of CN -bead space is introduced and some fixed point theorems are proved.

2 CN-bead space

The definition of bead space is generalized and called CN -bead space as follows:

Definition 2.1 A metric space (X, d) is a CN -bead space, if it is bead space and satisfy the following inequality:

$$d(w, z)^2 \leq \frac{1}{2}d(x, z)^2 + \frac{1}{2}d(y, z)^2 - \frac{1}{4}d(x, y)^2, \quad (2.3)$$

for specified x, y, z in Definition 1.3 and arbitrary $w \in X$.

Lemma 2.1 Let K be a bounded subset of a complete CN -bead space X . Suppose $\phi(u) = \limsup_{n \rightarrow \infty} d(x_n, u)$, where (x_n) is a sequence in K . Then there exists $x_0 \in K$ such that

$$\phi(x_0) = \inf\{\phi(x) : x \in K\}$$

Set $A := \{x_n : n \in \mathbf{N}\}$. Obviously, A is a bounded set, so by Lemma 1.1, $c(A) = \{x_0\}$. By Definition 1.2, if we set $t = \limsup_n d(x_n, x) = \phi(x)$, then we get

$$r(A) = \inf\{\phi(x) : x \in K\} = \phi(x_0).$$

Lemma 2.2 *Let K be a bounded subset of a complete CN-bead space X , and $T : K \rightarrow K$ be a total asymptotically nonexpansive mapping. Define $\phi(u) = \limsup_n d(T^n x, u)$ for each $u \in K$. Then $\lim_m \phi(T^m(w)) = \phi(w)$, where $w \in K$ is such that $\phi(w) = \inf\{\phi(u) : u \in K\}$.*

Since T is total asymptotically nonexpansive, we have

$$\begin{aligned} d(T^{n+m}(x), T^m(w)) &\leq d(T^n(x), w) \\ &\quad + k_m^{(1)}\psi(d(T^n(x), w)) \\ &\quad + k_m^{(2)} \end{aligned}$$

for any $n, m \geq 1$. Let n go to infinity, then

$$\phi(T^m(w)) \leq \phi(w) + k_m^{(1)}\psi(\phi(w)) + k_m^{(2)}$$

Let m go to infinity, which implies $\lim_{m \rightarrow \infty} \phi(T^m(w)) = \phi(w)$.

Theorem 2.1 *Let K be a bounded subset of complete CN-bead space X . Then any total asymptotically nonexpansive mapping $T : K \rightarrow K$ has a fixed point. Moreover, the fixed point set $F(T)$ is closed.*

For each $u \in K$ define

$$\phi(u) = \limsup_{n \rightarrow \infty} d(T^n(x), u).$$

Let $w \in K$ such that $\phi(w) = \inf\{\phi(u) : u \in K\}$. We have seen that $\phi(T^m(w)) = \phi(w)$ as $m \rightarrow \infty$. Let

$$\begin{aligned} B(T^m(w), r + \delta) \cap B(T^h(w), r + \delta) \\ = B(z, r - \delta), \end{aligned}$$

the inequality (2.3) implies

$$\begin{aligned} d(T^n(x), z)^2 &\leq \frac{1}{2}d(T^n(x), T^m(w))^2 \\ &\quad + \frac{1}{2}d(T^n(x), T^h(w))^2 \\ &\quad - \frac{1}{4}d(T^m(w), T^h(w))^2. \end{aligned}$$

Let n go to infinity, then

$$\begin{aligned} \phi(w)^2 &\leq \phi(z)^2 \\ &\leq \frac{1}{2}\phi(T^m(w))^2 + \frac{1}{2}\phi(T^h(w))^2 \\ &\quad - \frac{1}{4}d(T^m(w), T^h(w))^2 \end{aligned}$$

which implies

$$\limsup_{m, h \rightarrow \infty} d(T^m w, T^h w)^2 \leq 0,$$

therefore $(T^n(w))$ is a Cauchy sequence. Let $v = \lim_{n \rightarrow \infty} T^n(w)$. Since T is continuous, then $T(v) = v$ and this proves that $F(T) \neq \emptyset$. Again, since T is continuous, $F(T)$ is closed. The next two corollaries are straightforward result of Theorem 2.1.

Corollary 2.1 *Let K be a bounded subset of complete CN-bead space X . Then any asymptotically nonexpansive mapping $T : K \rightarrow K$ has a fixed point.*

Corollary 2.2 *Let K be a bounded subset of complete CN-bead space X . Then any nonexpansive mapping $T : K \rightarrow K$ has a fixed point.*

References

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