Available online at http://ijim.srbiau.ac.ir/

Int. J. Industrial Mathematics (ISSN 2008-5621)

Vol. 5, No. 4, 2013 Article ID IJIM-00296, 15 pages

Research Article

Some traveling wave solutions of soliton family

S. Dhawan * †‡ , D. Grover § , S. Kumar ¶

Abstract

Solitons are ubiquitous and exist in almost every area from sky to bottom. For solitons to appear, the relevant equation of motion must be nonlinear. In the present study, we deal with the Korteweg-de Vries (KdV), Modified Korteweg-de Vries (mKdV) and Regularised Long Wave (RLW) equations using Homotopy Perturbation method (HPM). The algorithm makes use of the HPM to determine the initial expansion coefficients using the initial value and boundary conditions. The physical structures of the nonlinear dispersive equation have been investigated for different parameters involved. It is shown how the nature of the waves look like in a simple way by considering the value of a certain single combination of constant parameters. The proposed scheme is standard, direct and computerized, which allow us to do complicated and tedious algebraic calculations. The ease of using this method to determine shock or solitary type of solutions, shows its power.

Keywords : Nonlinear partial differential equations; solitary waves; Homotopy perturbation method (HPM).

1 Introduction

It is significant to seek solutions of nonlinear model problems as they interpret a variety of physical phenomenon. Apart from some particular cases, most of them do not have a precise analytical solution. Therefore, some various approximate methods have recently been developed to tackle them [1]. It has also been observed from the literature that as an important nonlinear topic, nowadays, the solitary waves are being studied extensively both theoretically and experimentally. It is so because in various fields of science and engineering, nonlinear evaluation equations, as well as their analytic and numerical solutions, are of fundamental importance. One of the most attractive and surprising wave phenomenon is the creation of solitary waves or solitons. Solitons are self-localized wave packets arising from a robust balance between dispersion and nonlinearity. In small amplitude approximation, one ends up deriving some forms of nonlinear differential equations like Korteweg-de Vries (KdV) or modified Korteweg-de Vries (mKdV) or nonlinear Schrodinger equation, etc. which have solitary or solitonic solutions. It was ap-

*Corresponding author. dhawan311@gmail.com

†Department Of Mathematics, Dr. B. R. Ambedkar National Institute Of Technology Jalandhar, India.

‡Department Of Mathematics, D. A. V. University Jalandhar, India.

§Graphic Era University Dehradun, India.

¶Department Of Mathematics, Dr. B. R. Ambedkar National Institute Of Technology Jalandhar, India.

proximately two centuries ago that an adequate theory for solitary waves was developed for the well-known Korteweg-de Vries (KdV) equation. Historically, these types of equations first arose in the study of 2D shallow wave propagation, but have since appeared as limiting cases of many dispersive models. In 1895 Korteweg and deVries [2] showed that long waves, in water of relatively shallow depth, could be described approximately by a nonlinear equation and can be used as a model describing the lossless propagation of shallow water waves, magneto hydrodynamics waves in warm plasma, ion-acoustic waves in plasma, acoustic waves in an inharmonic crystal and ion-acoustic waves [3, 4]. It is well known that the Korteweg-de Vries equation is the generic outcome of a weakly nonlinear long-wave asymptotic analysis of many physical systems. It is categorized by its family of solitary wave solutions, with the familiar $sech^2$ profile.

Due to its properties, the KdV equation was the source of many applications and results in a large area of nonlinear physics [5]. Certain theoretical physics phenomena in the quantum mechanics domain are explained by means of a KdV model. It is used in fluid dynamics, aerodynamics, and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behavior, and mass transport. The alternative equation of the non-linear dispersive waves to the more usual KdV-equation, modelled to govern a large number of physical phenomena such as shallow waters and plasma waves, is the Regularised Long Wave (RLW) equation u˙t+ u˙x+u u˙x- u˙xxt=0, The regularized long wave (RLW) equation belongs to a class of the nonlinear evolution equations which provide good models for predicting a variety of physical phenomena. Solitary waves are wave packets or pulses which propagate in nonlinear media. Due to dynamical balance between the non-linear and dispersive effects these waves retain a stable waveform. The regularized long wave (RLW) equation was originally introduced to describe the behavior of the undular bore [6]. It has also been derived from the study of water waves and ion acoustic plasma waves. It was first proposed by Peregrine [7] for modelling the propagation of unidirectional weakly nonlinear and weakly dispersive water waves. A rather interesting property of the RLW equation is that the collision of two solitary waves may result in the creation of secondary solitary waves or sinusoidal solutions. This phenomenon is somewhat analogous to what happens in subatomic physics where the collisions of particles create another particles and/or radiation. Therefore, a study of RLW equation provide the opportunity of investigating the creation of secondary solitary waves and/or radiation to get insight into the corresponding processes of particle physics [8].

The homotopy perturbation method of He [11]-[15] is well addressed and needs less computations in addition to high accuracy. Recently, Grover et.al. [16] have used HPM to obtain approximate analytic solutions of parabolic (heat) PDE and non-linear equations. Fascinated by the efficiency of HPM, in this paper we design a numerical technique i.e. homotopy perturbation method for new solitary-wave solutions of the KdV, mKdV and RLW equations. In HPM the solution is considered as the summation of an infinite series which converges rapidly to exact solution. The methods presented here can also be used to obtain the numerical solution of some other wider class of PDEs describing wave propagation. At present, we consider following two nonlinear models:

Model 1 : The given KdV equation be u˙t-6uu˙x+u˙xxx=0, $0 \leq x \leq l$
$u(0,t) = 0 = u(l,t), \quad u(x,0) = a\sin\frac{\pi x}{l}$

Model 2 : The given mKdV equation be u˙t+24uˆ2 u˙x+u˙xxx=0

Model 3 :The given RLW equation be × equation u˙t+u˙x+uu˙x-u˙xxt=0,
u(0,t)=0=u(l,t), u(x,0)= a sin $\frac{\pi x}{l}$

where the subscripts t and s denote differentiation with respect to time t and space variable x, respectively.

2 Homotopy Perturbation Method (HPM)

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM), [17]-[20]. Recently, it has been successfully applied to solve many types of nonlinear problems [21]-[26]. Moreover, in [26], the basic idea of the HAM is introduced and then its application in some heat transfer equations is studied by making different comparisons. More theoretical details about the HAM can be found in [26].

Next, we outline the general procedure of homotopy perturbation method developed and advanced by He [11]. We consider the differential equation

$$A(u) - f(r) = 0, \;\; r \in \Omega \tag{2.1}$$

$$B\left(u, \frac{\partial u}{\partial x}\right), r \in \Gamma \tag{2.2}$$

where A is a general differential operator, linear or non-linear, $f(r)$ is a known analytic function, B is boundary operator and Γ is the boundary of the domain Ω. The operator A can be generally divided into two operators, L and N , where L is linear and N is a non-linear operator. Equation (2.1) can be written as

$$L(u) + N(u) - f(r) = 0 \tag{2.3}$$

Using homotopy technique, we can construct a homotopy

$$v(r,p) : \Omega \times [0,1] \to R \tag{2.4}$$

which satisfies the relation

$$H(v,p) = (1-p)[L(v)-L(u_0)] + p[A(v)-f(r)] = 0.$$

Here $p \in [0,1]$ is called homotopy parameter and u_0 is an initial approximation for the solution of Equation (2.1) which satisfies the boundary conditions. Clearly, from equation (2.5), we have

$$H(v,0) = L(v) - L(u_0) \tag{2.5}$$

$$H(v,1) = A(v) - f(r) \tag{2.6}$$

We assume that the solution of Equation (2.5) can be expressed as a series in p as follows:

$$\mathbf{v} = \mathbf{v}_0 + p\mathbf{v}_1 + p^2\mathbf{v}_2 + p^3\mathbf{v}_3 + ... \tag{2.7}$$

On setting $p = 1$, we obtain the approximate solution of equation (2.1) as

$$u = \lim_{p\to 1} \mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + ... \tag{2.8}$$

Above series (2.8) is convergent for most of the cases. The convergent rate depends upon the nonlinear operator used. Some of the suggested opinions are [11].

- The second derivative of $N(\mathbf{v})$ w.r.t. V must be s mall because the parameter may be relatively large.
- The norm of $L^{-1}\frac{\partial N}{\partial \mathbf{v}}$ must be smaller than one so that the series converges.

In the next Section we illustrate the application of HPM for the model problems considered above. According to the HPM, we can initially use the embedding parameter p as a small parameter and assume that the solutions of can be represented as a power series in p. To demonstrate the convergence of the scheme, the results of the numerical example are presented in the next Sections to obtain accurate solutions.

3 HPM for KdV equation

The given RLW equation be

$$u_t - 6uu_x + u_{xxx} = 0, \quad 0 \le x \le l \tag{3.9}$$

The homotopy equation for the model problem under consideration is

$$\frac{\partial \mathbf{v}}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(\frac{\partial u_0}{\partial t} - 6\mathbf{v}\frac{\partial \mathbf{v}}{\partial x} + \frac{\partial^3 \mathbf{v}}{\partial x^3}\right) = 0. \tag{3.10}$$

with initial approximation $u_0 = a \sin \frac{\pi x}{l}$ which satisfies the given boundary conditions. Let the solution of (3.9) be of the form

$$\mathbf{v} = \mathbf{v}_0 + p\mathbf{v}_1 + p^2\mathbf{v}_2 + p^3\mathbf{v}_3 + ... \tag{3.11}$$

Using (3.11) in (3.10) and comparing the like powers of p, we have

$$\frac{\partial \mathbf{v}_0}{\partial t} = \frac{\partial u_0}{\partial t}, \frac{\partial \mathbf{v}_1}{\partial t} = 6\mathbf{v}_0\frac{\partial \mathbf{v}_0}{\partial x} - \frac{\partial^3 \mathbf{v}_0}{\partial x^3} - \frac{\partial u_0}{\partial t},$$

$$\mathbf{v}_1 = a\sin\frac{\pi x}{l} \text{ at } t=0\ \forall x,$$

$$\frac{\partial \mathbf{v}_2}{\partial t} = 6\mathbf{v}_1\frac{\partial \mathbf{v}_0}{\partial x} + 6\mathbf{v}_0\frac{\partial \mathbf{v}_1}{\partial x} - \frac{\partial^3 \mathbf{v}_1}{\partial x^3},$$

$$\mathbf{v}_2 = a\sin\frac{\pi x}{l} \text{ at } t=0\ \forall x,$$

$$\frac{\partial \mathbf{v}_3}{\partial t} = 6\mathbf{v}_2\frac{\partial \mathbf{v}_0}{\partial x} + 6\mathbf{v}_1\frac{\partial \mathbf{v}_1}{\partial x} + 6\mathbf{v}_0\frac{\partial^3 \mathbf{v}_2}{\partial x^3} - \frac{\partial^3 \mathbf{v}_2}{\partial x^3},$$

$$\mathbf{v}_3 = a\sin\frac{\pi x}{l} \text{ at } t=0\ \forall x,$$

$$\frac{\partial \mathbf{v}_4}{\partial t} = 6\mathbf{v}_3\frac{\partial \mathbf{v}_0}{\partial x} + 6\mathbf{v}_2\frac{\partial \mathbf{v}_1}{\partial x} + 6\mathbf{v}_1\frac{\partial \mathbf{v}_2}{\partial x} + 6\mathbf{v}_0\frac{\partial \mathbf{v}_3}{\partial x} - \frac{\partial^3 \mathbf{v}_3}{\partial x^3},$$

$$\mathbf{v}_4 = a\sin\frac{\pi x}{l} \text{ at } t=0\ \forall x,$$

$$\frac{\partial \mathbf{v}_5}{\partial t} = 6\mathbf{v}_4\frac{\partial \mathbf{v}_0}{\partial x} + 6\mathbf{v}_3\frac{\partial \mathbf{v}_1}{\partial x} + 6\mathbf{v}_2\frac{\partial \mathbf{v}_2}{\partial x} + 6\mathbf{v}_1\frac{\partial \mathbf{v}_3}{\partial x} - \frac{\partial^3 \mathbf{v}_4}{\partial x^3},$$

$$\mathbf{v}_5 = a\sin\frac{\pi x}{l} \text{ at } t=0\ \forall x.$$

and so on proceeding in the same way for other terms. Solving this system of equations (4.13), we get

$$\mathbf{v}_0 = a\ \sin[\frac{\pi x}{l}]$$

$$\mathbf{v}_1 = t\left(\frac{a\pi^3\cos[\frac{\pi x}{l}]}{l^3} + \frac{6a^2\pi\cos[\frac{\pi x}{l}]\sin[\frac{\pi x}{l}]}{l}\right)$$

$$\mathbf{v}_2 = \frac{15a^2\pi^4t^2\cos[\frac{2\pi x}{l}]}{l^4} + \frac{9a^3\pi^2t^2\sin[\frac{\pi x}{l}]}{l^2} - \frac{a\pi^6t^2\sin[\frac{\pi x}{l}]}{2l^6} + \frac{27a^3\pi^2t^2\cos[\frac{2\pi x}{l}]\sin[\frac{\pi x}{l}]}{l^2}$$

$$\mathbf{v}_3 = -\frac{189a^3\pi^5t^3\cos[\frac{\pi x}{l}]}{l^5} - \frac{a\pi^9t^3\cos[\frac{\pi x}{l}]}{6l^9} + \frac{351a^3\pi^5t^3\cos[\frac{\pi x}{l}]\cos[\frac{2\pi x}{l}]}{l^5} - \frac{216a^4\pi^3t^3\cos[\frac{\pi x}{l}]\sin[\frac{\pi x}{l}]}{l^3} - \frac{84a^2\pi^7t^3\cos[\frac{\pi x}{l}]\sin[\frac{\pi x}{l}]}{l^7} + \frac{144a^4\pi^3t^3\cos[\frac{\pi x}{l}]\sin[\frac{3\pi x}{l}]}{l^3}$$

$$\mathbf{v}_4 = -\frac{342a^4\pi^6t^4\cos[\frac{2\pi x}{l}]}{l^6} - \frac{85a^2\pi^{10}t^4\cos[\frac{2\pi x}{l}]}{l^{10}} + \frac{1854a^4\pi^6t^4\cos[\frac{4\pi x}{l}]}{l^6} + \frac{621a^5\pi^4t^4\sin[\frac{\pi x}{l}]}{4l^4} - \frac{2583a^3\pi^8t^4\sin[\frac{\pi x}{l}]}{2l^8} + \frac{a\pi^{12}t^4\sin[\frac{\pi x}{l}]}{24l^{12}} + \frac{297a^5\pi^4t^4\cos[\frac{2\pi x}{l}]\sin[\frac{\pi x}{l}]}{l^4} - \frac{5265a^3\pi^8t^4\cos[\frac{2\pi x}{l}]\sin[\frac{\pi x}{l}]}{2l^8} + \frac{3375a^5\pi^4t^4\cos[\frac{4\pi x}{l}]\sin[\frac{\pi x}{l}]}{4l^4}$$

$$\mathbf{v}_5 = \frac{96309a^5\pi^7t^5\cos[\frac{\pi x}{l}]}{4l^7} + \frac{73857a^3\pi^{11}t^5\cos[\frac{\pi x}{l}]}{10l^{11}} + \frac{a\pi^{15}t^5\cos[\frac{\pi x}{l}]}{120l^{15}} - \frac{48141a^5\pi^7t^5\cos[\frac{\pi x}{l}]\cos[\frac{2\pi x}{l}]}{l^7} - \frac{29403a^3\pi^{11}t^5\cos[\frac{\pi x}{l}]\cos[\frac{2\pi x}{l}]}{2l^{11}} + \frac{148095a^5\pi^7t^5\cos[\frac{\pi x}{l}]\cos[\frac{4\pi x}{l}]}{4l^7} + \frac{9720a^6\pi^5t^5\cos[\frac{\pi x}{l}]\sin[\frac{\pi x}{l}]}{l^5} + \frac{298728a^4\pi^9t^5\cos[\frac{\pi x}{l}]\sin[\frac{\pi x}{l}]}{5l^9} + \frac{1364a^2\pi^{13}t^5\cos[\frac{\pi x}{l}]\sin[\frac{\pi x}{l}]}{5l^{13}} - \frac{9396a^6\pi^5t^5\cos[\frac{\pi x}{l}]\sin[\frac{3\pi x}{l}]}{l^5} - \frac{55800a^4\pi^9t^5\cos[\frac{\pi x}{l}]\sin[\frac{3\pi x}{l}]}{l^9} + \frac{26244a^6\pi^5t^5\cos[\frac{\pi x}{l}]\sin[\frac{5\pi x}{l}]}{5l^5}$$

$$\mathbf{v}_6 = \frac{1701a^6\pi^8t^6\cos[\frac{2\pi x}{l}]}{l^8} + \frac{44088a^4\pi^{12}t^6\cos[\frac{2\pi x}{l}]}{5l^{12}} + \frac{182a^2\pi^{16}t^6\cos[\frac{2\pi x}{l}]}{l^{16}} - \frac{375192a^6\pi^8t^6\cos[\frac{4\pi x}{l}]}{5l^8} - \frac{316884a^4\pi^{12}t^6\cos[\frac{4\pi x}{l}]}{l^{12}} + \frac{891567a^6\pi^8t^6\cos[\frac{6\pi x}{l}]}{5l^8} - \frac{81a^7\pi^6t^6\sin[\frac{\pi x}{l}]}{16l^6} + \frac{3429a^5\pi^{10}t^6\sin[\frac{\pi x}{l}]}{40l^{10}} - \frac{3267a^3\pi^{14}t^6\sin[\frac{\pi x}{l}]}{80l^{14}} - \frac{a\pi^{18}t^6\sin[\frac{\pi x}{l}]}{720l^{18}} + \frac{177147a^7\pi^6t^6\sin[\frac{3\pi x}{l}]}{80l^6} + \frac{5466771a^5\pi^{10}t^6\sin[\frac{3\pi x}{l}]}{80l^{10}} + \frac{2675673a^3\pi^{14}t^6\sin[\frac{3\pi x}{l}]}{80l^{14}} - \frac{253125a^7\pi^6t^6\sin[\frac{5\pi x}{l}]}{16l^6} - \frac{7648155a^5\pi^{10}t^6\sin[\frac{5\pi x}{l}]}{16l^{10}} + \frac{1361367a^7\pi^6t^6\sin[\frac{7\pi x}{l}]}{80l^6}$$

and proceeding in the same way for other terms, we obtain $\mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9....$ By taking $p = 1$, in (4.13), we can obtain the solution of given model problem (3.9), so that $u = \lim_{p\to 1} \mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + ...$

4 HPM for mKdV equation

We have mKdV equation in hand u˙t+24uˆ2 u˙x+u˙xxx=0 Following the procedure mentioned above, the homotopy equation for the model problem under consideration is given by

$$\frac{\partial \vartheta}{\partial t} - \frac{\partial u_0}{\partial t} + \chi\left(\frac{\partial u_0}{\partial t} + 24\,\vartheta^2\,\frac{\partial \vartheta}{\partial x} + \frac{\partial^3 \vartheta}{\partial x^3}\right) = 0. \tag{4.12}$$

with initial approximation $u_0 = a\sin\frac{\pi x}{l}$ which satisfies the given boundary conditions. Let the solution of (4) be of the form $\vartheta = \vartheta_0 + \chi\vartheta_1 + \chi^2\vartheta_2 + \chi^3\vartheta_3 + ...$ Using this in (4.12) and comparing the like powers of χ, we have relations

$$\frac{\partial \vartheta_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0,$$

$$\frac{\partial \vartheta_1}{\partial t} + 24\vartheta_0^2\frac{\partial \vartheta_0}{\partial x} + \frac{\partial^3 \vartheta_0}{\partial x^3} + \frac{\partial u_0}{\partial t} = 0,$$

$$\frac{\partial \vartheta_2}{\partial t} + 24\vartheta_0\vartheta_1\frac{\partial v_0}{\partial x} + 24\vartheta_0^2\frac{\partial \vartheta_1}{\partial x} + \frac{\partial^3 \vartheta_1}{\partial x^3} = 0,$$

$$\frac{\partial \vartheta_3}{\partial t} + 24(2\vartheta_0\vartheta_2 + \vartheta_1^2)\frac{\partial v_0}{\partial x} + 48\vartheta_0\vartheta_1\frac{\partial \vartheta_1}{\partial x} + 24\vartheta_0^2\frac{\partial^3 \vartheta_2}{\partial x^3} + \frac{\partial^3 \vartheta_2}{\partial x^3} = 0,$$

$$\frac{\partial \vartheta_4}{\partial t} + 24(\vartheta_0\vartheta_3 + 2\vartheta_1\vartheta_2)\frac{\partial v_0}{\partial x} + 24(2\vartheta_0\vartheta_2 + \vartheta_1^2)\frac{\partial v_1}{\partial x} + 48\vartheta_0\vartheta_1\frac{\partial \vartheta_2}{\partial x} + \frac{\partial^3 v_2}{\partial x^3} = 0.$$

and so on. Solving this system of equations (4.13), we get

$$\vartheta_0 = 0.8\sqrt{sech[kx]}$$

$$\vartheta_1 = t(-1.4k^3 sech[kx]^{3/2}\sinh[kx] + 6.144k sech[kx]^{5/2}\sinh[kx] + 1.5k^3 sech[kx]^{7/2}\sinh[kx]^3)$$

$$\vartheta_2 = \frac{1}{2}k^2t^2\sqrt{sech[kx]}(334.08k^2 sech[kx]$$
$$(-0.9166 + \tanh[kx]^2)(-0.200642 + \tanh[kx]^2)$$
$$+ 129.938k^4(-0.999486 + \tanh[kx]^2)(-0.783017$$

$$+\tanh[kx]^2)(-0.136689+\tanh[kx]^2)$$
$$+ sech[kx]^2(-94.3718$$
$$+283.116)\tanh[kx]^2)$$

Similarly we obtain $\vartheta_3, \vartheta_4, \vartheta_5 \ldots$. By taking $\chi = 1$, in (4.13), we can obtain the solution of given model problem (4), so that $u = \lim_{\chi\to 1}\vartheta = \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 + \ldots$.

5 HPM for RLW equation

The given RLW equation be u˙t+u˙x+uu˙x-u˙xxt=0, subject to the conditions

$$u(0,t) = 0 = u(l,t), \quad u(x,0) = a\sin\frac{\pi x}{l}$$

The homotopy equation for the model problem under consideration be

$$\frac{\partial \mathbf{v}}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(\frac{\partial u_0}{\partial t} + \frac{\partial \mathbf{v}}{\partial x} + v\frac{\partial \mathbf{v}}{\partial x} + \frac{\partial^3 \mathbf{v}}{\partial x^3}\right) = 0.$$

with initial approximation $u_0 = a\sin\frac{\pi x}{l}$ which satisfies the given boundary conditions. Let the solution of (5) be of the form **v=v˙0+pv˙1+ p ˆ2 v˙2+p ˆ3 v˙3+...** . Using (5) in (5.13) and comparing the like powers of p, we have

$$\frac{\partial \mathbf{v}_0}{\partial t} = \frac{\partial u_0}{\partial t},$$

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\frac{\partial u_0}{\partial t} - \frac{\partial \mathbf{v}_0}{\partial x} - \mathbf{v}_0\frac{\partial \mathbf{v}_0}{\partial x} - \frac{\partial^3 \mathbf{v}_0}{\partial x^3},$$
$$\mathbf{v}_1 = a\sin\frac{\pi x}{l} \text{ at } t = 0\ \forall x,$$

$$\frac{\partial \mathbf{v}_2}{\partial t} = -\frac{\partial \mathbf{v}_1}{\partial x} - \mathbf{v}_1\frac{\partial \mathbf{v}_0}{\partial x} - v_0\frac{\partial \mathbf{v}_1}{\partial x} - \frac{\partial^3 \mathbf{v}_1}{\partial x^3},$$
$$\mathbf{v}_2 = a\sin\frac{\pi x}{l} \text{ at } t = 0\ \forall x,$$

$$\frac{\partial \mathbf{v}_3}{\partial t} = -\frac{\partial \mathbf{v}_2}{\partial x} - \mathbf{v}_2\frac{\partial \mathbf{v}_0}{\partial x} - \mathbf{v}_1\frac{\partial \mathbf{v}_1}{\partial x} - \mathbf{v}_0\frac{\partial \mathbf{v}_2}{\partial x} - \frac{\partial^3 \mathbf{v}_2}{\partial x^3},$$
$$\mathbf{v}_3 = a\sin\frac{\pi x}{l} \text{ at } t = 0\ \forall x,$$

$$\frac{\partial \mathbf{v}_4}{\partial t} = -\frac{\partial \mathbf{v}_3}{\partial x} - \mathbf{v}_3\frac{\partial \mathbf{v}_0}{\partial x} - \mathbf{v}_2\frac{\partial \mathbf{v}_1}{\partial x} - \mathbf{v}_1\frac{\partial \mathbf{v}_2}{\partial x} - \mathbf{v}_0\frac{\partial \mathbf{v}_3}{\partial x} - \frac{\partial^3 \mathbf{v}_3}{\partial x^3},$$
$$\mathbf{v}_4 = a\sin\frac{\pi x}{l} \text{ at } t = 0\ \forall x,$$

and so on proceeding in the same way for other terms. Solving this system of equations (5.13), we get

$$\mathbf{v}_0 = a\ \sin[\frac{\pi x}{l}]$$

$$\mathbf{v}_1 = t(-\frac{a\pi\cos[\frac{\pi x}{l}]}{l} + \frac{a\pi^3\cos[\frac{\pi x}{l}]}{l^3} - \frac{a^2\pi\cos[\frac{\pi x}{l}]\sin[\frac{\pi x}{l}]}{l})$$

$$\mathbf{v}_2 = \frac{a\pi^3 t\cos[\frac{\pi x}{l}]}{l^3} + \frac{a^2\pi^2 t^2\cos[\frac{2\pi x}{l}]}{l^2} - \frac{a^2\pi^4 t^2\cos[\frac{2\pi x}{l}]}{2l^4} - \frac{a\pi^2 t^2\sin[\frac{\pi x}{l}]}{2l^2} + \frac{a^3\pi^2 t^2\sin[\frac{\pi x}{l}]}{4l^2} + \frac{a\pi^4 t^2\sin[\frac{\pi x}{l}]}{2l^4} + \frac{3a^3\pi^2 t^2\cos[\frac{2\pi x}{l}]\sin[\frac{\pi x}{l}]}{4l^2}$$

$$\mathbf{v}_3 = \frac{a\pi^3 t^3\cos[\frac{\pi x}{l}]}{6l^3} + \frac{a^3\pi^3 t^3\cos[\frac{\pi x}{l}]}{8l^3} - \frac{a\pi^5 t^3\cos[\frac{\pi x}{l}]}{3l^5} - \frac{a^3\pi^5 t^3\cos[\frac{\pi x}{l}]}{24l^5} + \frac{a\pi^7 t^3\cos[\frac{\pi x}{l}]}{6l^7} - \frac{a^2\pi^4 t^2\cos[\frac{2\pi x}{l}]}{2l^4} - \frac{9a^3\pi^3 t^3\cos[\frac{3\pi x}{l}]}{8l^3} + \frac{31a^3\pi^5 t^3\cos[\frac{3\pi x}{l}]}{8l^5} + \frac{a\pi^4 t^2\sin[\frac{\pi x}{l}]}{2l^4} - \frac{a\pi^6 t^2\sin[\frac{\pi x}{l}]}{2l^6} + \frac{a^2\pi^3 t^3\sin[\frac{2\pi x}{l}]}{l^3} + \frac{a^4\pi^3 t^3\sin[\frac{2\pi x}{l}]}{6l^3} - \frac{7a^2\pi^5 t^3\sin[\frac{2\pi x}{l}]}{2l^5} + \frac{3a^2\pi^7 t^3\sin[\frac{2\pi x}{l}]}{2l^7} - \frac{a^4\pi^3 t^3\sin[\frac{4\pi x}{l}]}{3l^3}$$

$$\mathbf{v}_4 = -\frac{a\pi^5 t^3\cos[\frac{\pi x}{l}]}{6l^5} + \frac{a\pi^7 t^3\cos[\frac{\pi x}{l}]}{3l^7} - \frac{a\pi^9 t^3\cos[\frac{\pi x}{l}]}{6l^9} - \frac{2a^2\pi^4 t^4\cos[\frac{2\pi x}{l}]}{3l^4}$$

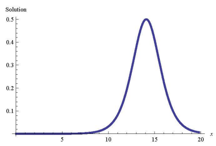
$$-\frac{a^4\pi^4t^4\cos[\frac{2\pi x}{l}]}{3l^4}+\frac{49a^2\pi^6t^4\cos[\frac{2\pi x}{l}]}{12l^6}$$
$$+\frac{5a^4\pi^6t^4\cos[\frac{2\pi x}{l}]}{4l^6}-\frac{95a^2\pi^8t^4\cos[\frac{2\pi x}{l}]}{12l^8}$$
$$+\frac{3a^2\pi^{10}t^4\cos[\frac{2\pi x}{l}]}{l^{10}}+\frac{a^3\pi^5t^3\cos[\frac{3\pi x}{l}]}{2l^5}$$
$$+\frac{4a^4\pi^4t^4\cos[\frac{4\pi x}{l}]}{3l^4}-\frac{91a^4\pi^6t^4\cos[\frac{4\pi x}{l}]}{12l^6}$$
$$+\frac{a\pi^4t^4\sin[\frac{\pi x}{l}]}{24l^4}+\frac{a^3\pi^4t^4\sin[\frac{\pi x}{l}]}{16l^4}$$
$$+\frac{a^5\pi^4t^4\sin[\frac{\pi x}{l}]}{192l^4}-\frac{a\pi^6t^4\sin[\frac{\pi x}{l}]}{8l^6}$$
$$-\frac{31a^3\pi^6t^4\sin[\frac{\pi x}{l}]}{96l^6}+\frac{a\pi^8t^4\sin[\frac{\pi x}{l}]}{8l^8}$$
$$+\frac{13a^3\pi^8t^4\sin[\frac{\pi x}{l}]}{96l^8}-\frac{a\pi^{10}t^4\sin[\frac{\pi x}{l}]}{24l^{10}}$$
$$-\frac{5a^2\pi^5t^3\sin[\frac{2\pi x}{l}]}{6l^5}+\frac{11a^2\pi^7t^3\sin[\frac{2\pi x}{l}]}{6l^7}$$
$$-\frac{27a^3\pi^4t^4\sin[\frac{3\pi x}{l}]}{16l^4}-\frac{27a^5\pi^4t^4\sin[\frac{3\pi x}{l}]}{128l^4}$$
$$+\frac{399a^3\pi^6t^4\sin[\frac{3\pi x}{l}]}{32l^6}-\frac{861a^3\pi^8t^4\sin[\frac{3\pi x}{l}]}{32l^8}$$
$$+\frac{125a^5\pi^4t^4\sin[\frac{5\pi x}{l}]}{384l^4}$$

and proceeding in the same way for other terms, we obtain $\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7....$ By taking $p = 1$, in (5.13), we can obtain the solution of given model problem (5), so that $u = \lim_{p\to 1}\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + ...$

6 Numerical Experiments

In this Section, we reveal physical behaviors of solution profiles for the equations discussed above by solving different cases with different ranges of all the parameters under consideration. In all the cases, we observe that for $t > 0$ the solution evolves so that all the conservation laws are satisfied.

Example 6.1 *In case I, the KdV equation* $u_t+u_x+6\,u\,u_x+u_{xxx}=0, \quad 0 \le x \le l.$



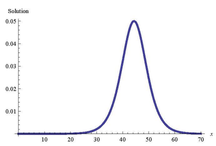
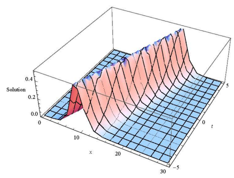


Figure 1: Wave profiles at time 0.1, 1 for $\rho = 1, 0.1$ respectively.

with the initial condition at $t = 0$ is given by $u_0(x) = \frac{\rho}{2}sech^2\left(\frac{\sqrt{\rho}x}{2} - 7\right)$. Solution profiles achieved in this case are shown in Figs 1. These results are obtained within the range $0 \le x \le 70$



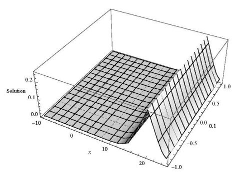


Figure 2: Wave profiles at different times $t = 0.6, 0.8, 1$ and $t = 0.01$ respectively.

at time $0.1, 1$ for $\rho = 1, 0.1$ respectively. Here we have monitored sharp peaks achieving maximum height $0.005089, 0.0002188$ at $x = 14, 43$. Surface plots for this case study can be seen in Fig. 2 for $\rho = 1, 0.5$ in the time interval $-1 \leq t \leq 1$. If we look at these plots, it is observed that the continuous solution profile that comes into picture is associated with dispersive wave component of the solution. next wave solution is depicted in Fig. 3 within the range $-10 \leq x \leq 10$ at times $t = 0.5, 1$ and corresponding surface plot is plotted in Fig. 4 within the spatial range $-10 \leq x \leq 10$ and time range $-0.5 \leq t \leq 0.5$, $-1 \leq t \leq 1$ respectively. In case II, we have KdV equation (6.1), for the initial state $u_0(x) = -n(n+1)sec^2x$, which results in n solitons that propagate with different velocities. Particularly for $n = 2$, we have $u_0(x) = -6sec^2x$ for which results are shown through figs. 5 achieving different depth levels. At negative times, the deeper soliton, which moves faster, approaches the shallower one. At $t = 0$ they combine to give $u_0(x) = -6sec^2x$, which is a single trough of depth 6 and, after the encounter, the deeper soliton has overtaken the shallower one and both resume their original shape and speed. However, as a result of the interaction, the shallower soliton experiences a delay and the deeper soliton is speeded up. The wave profile plotted at different times $t = 0.6, 0.8, 1$ and $t = 0.01$ is shown in Fig. 5. In Fig. 6, the solution profile depicts two waves where the taller one catches the shorter coalesces to form a single wave. The interaction seems visible at the very first sight. A careful examination of the wave profiles shows that the taller one has moved forward and the shorter goes backward.

Example 6.2 *In the second example, we consider numerical experiments made for the problem (4).*

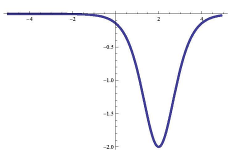
In the first case we have the given initial condition $u_0 = 0.8\sqrt{sech[kx]}$. In Fig. 7, simulation is carried out at different times $t = 0, 0.5, 1, 1.5, 2$ by taking $a = 2, b = 1, k = 1, c = 4$ in the range $-15 \leq x \leq 20$ which produces each soliton of amplitude 1.215 moving to the left.

In the next case, for the initial condition

$$u(x,0) = \left(\frac{12c}{A' + B'}\right)^{1/2},$$

where

$$A' = a(1 + \sqrt{1+\alpha^2}\cosh[2\sqrt{c/b}x],$$



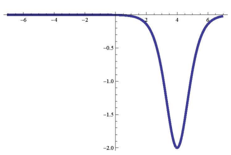
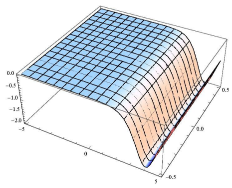


Figure 3: Wave profiles plotted at different times $t = 0.6, 0.8, 1$ and $t = 0.01$.

$$B' = \alpha \sinh[2\sqrt{c/b}x]$$

corresponding analytical solution is

$$u(x,t) = \left(\frac{12c}{A+B}\right)^{1/2}$$



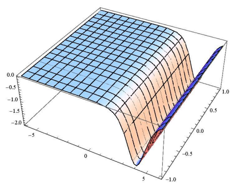


Figure 4: Surface plots within the time range $-0.5 \leq t \leq 0.5$, $-1 \leq t \leq 1$ respectively.

where

$$A = a(1 + \sqrt{1+\alpha^2}\cosh[2\sqrt{c/b}(x-ct)],$$

$$B = \alpha \sinh[2\sqrt{c/b}(x-ct)]$$

Figure 5: Wave profiles plotted at different times $t = 0.6, 0.8, 1$ and $t = 0.01$.

Figure 6: Wave profiles plotted at $t = 0.05$ and $t = 0.1$.

In 2D case, results are produced by taking $a = 24, b = 1, \alpha = 0.1$ within the range $-15 \leq x \leq 20$ with $c = 2$, at times $0.1, 0.5, 1$ and $0, 0.8, 1.5$ respectively. Here, we observe soliton amplitude to be 0.7137 and 0.4986 for 2D plots (x ranging from -5 to 6). Surface plots for this case can be seen with the assumption $a = 24, b = 1, \alpha = 0.1, c = 1$ with space and time ranges $([-6, 6], [0, 1])$ and $([-6, 6], [-2, 2])$ respectively. For $bc < 0$, periodic

Figure 7: Solution $u(x,t)$ at different times $t = 0, 0.5, 1, 1.5, 2$ with $c = 4, 3$ respectively.

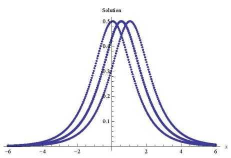
Figure 8: Solution $u(x,t)$ at different times $t = 0.1, 0.5, 1$ and $t = 0, 0.8, 1.5$ with $c = 2, 1$ respectively for 2D plots and surface plot within the time range $t \in [0, 1]$.

solution is obtained for this case is

$$u(x,t) = \left(\frac{12c}{A1 + B1}\right)^{1/2}.$$

where

$$A1 = a(1 + \sqrt{1 + \alpha^2} \cosh[2\sqrt{-c/b}(x - ct)],$$



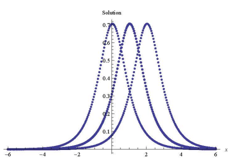
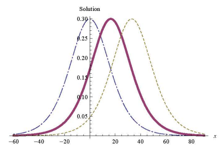


Figure 9: Solution $u(x,t)$ at different times $t = 0, 0.5, 1$ with $c = 2, 1$ respectively for 2D plots and surface plot within the time range $t \in [0,1]$ and $t \in [-7,7]$ respectively.

B1=$\imath\alpha \sinh[2\sqrt{-c/b}(x-ct)]$A perspective view of the solution for this case is given in Fig. 9 we observe the periodic solution profiles and emerge on



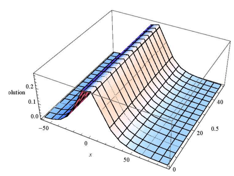


Figure 10: Wave profiles at $t = 0, 15, 30$ and surface plot for the time range $0 \leq t \leq 50$.

the left-hand side inverted but with unchanged amplitude and velocity and having undergone a phase shift. In these simulations, we have $a = 24, b = 1, \alpha = 0.1, c = 1, 2$ with $x \in [-6,6]$ and time range $t \in [0,1]$ and $t \in [-7,7]$ respec-

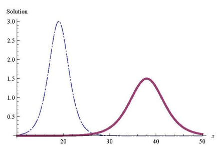


Figure 11: Wave profiles at $t = 2$ and surface plot for the time range $0 \le t \le 2$.

tively. It leads to wave amplitudes 0.5035 and 0.7062. Snapshots of the function profile in the form of surface can also be seen with $c = 2$, with $x \in [-6, 6], t \in [0, 1]$ and $x \in [-4, 4], t \in [-7, 7]$ respectively.

Example 6.3 *This time we deal with RLW equation. With the boundary conditions $u \to 0$ as $x \to \pm\infty$, and initial condition $u(x, 0) = 3c sech^2(k(x - x_0))$, this solution corresponds to the motion of a single solitary wave with amplitude 3c and width k, initially centered at x0 is presented trough figs.*

In the first simulation, a wave profile is observed at different times $t = 0, 15, 30$. At the very beginning, near $x = 0$, sharp edge at $t = 0$ is observed and later as the wave front moves on, its position is changed with the passage of time from $x = 20, 40$ and corresponding surface plot is shown in Fig. 10. In the next case, we study the interaction of two solitary waves. As initial condition, we have $u(x, 0) = 3c_1 sech^2(k_1(x - x_1)) + 3c_2 sech^2(k_2(x - x_2))$ where $k_1 = 1/2\sqrt{\varepsilon c_1/\mu(1 + \varepsilon c_1)}$, and $k_2 = 1/2\sqrt{\varepsilon c_2/\mu(1 + \varepsilon c_2)}$. Here, the one of the amplitude is $3c_1$ sited at $x = x_1$ and amplitude $3c_2$ sited at $x = x_2$. An interaction occurs when the larger is placed to the left of the smaller. We study such an interaction with $k_1 = k_2 = 0.5, x_1 = 15, x_2 = 35, c_1 = 1.0, c_2 = 0.5$ running the simulation using the range $0 \le x \le 40$ which can be seen in Fig. 11.

7 Conclusion

In this work, we have used the homotopy perturbation method (HPM) for standard KDV and RLW equations. The HPM has the capabilities to bereave the complicated differential equation models to number of simple iterative models once the effective initial guess satisfying the boundary conditions is made and leads to generic solutions in addition to their rapid convergence. The algorithm developed facilitates for less time consuming. Also even for smaller values desirable solutions are obtained. The results are comparable to analytic ones. The HPM has been observed to be powerful tool with its addition advantages and suitability for computer simulations over the existing tools having potential to be used in more complicated partial differential equation systems.

References

[1] S. Dhawan, S. Kapoor, S. Kumar, S. Rawat, *Contemporary review of techniques for the solution of nonlinear Burgers equation*, Journal of Computational Science, J. Comput. Science 3 (2012) 405-419.

[2] D.J. Korteweg , G. de Vries, *On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves*, Phil. Mag. J. Science 39 (1895) 422-443.

[3] A. C. Scott, Y. F. Chu and D. W. McLaughlin, *The Soliton: A New Concept In Applied Science*, IEEE Proc. 61 (1973) 1443-1452.

[4] A. C. Vliegenthart, *On finite difference method for Korteweg-de Vries equation*, J. Eng. Math. 5 (1971) 137-155.

[5] S. V. Vladimirov, M. Y. Yu, V. N. Tsytovich, *Recent advances in the theory of nonlinear surface waves*, Physics Reports 241 (1994) 1-63.

[6] D. H. Peregrine, *Calculations of the development of an undular bore*, J. Fluid Mech 25 (1966) 321-330.

[7] D. H. Peregrine, *Long waves on a beach*, J. Fluid Mech 27 (1967) 815-827.

[8] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, H. C. Morris, *Solitons and nonlinear wave equations*, New York: Academic Press (1982).

[9] Siraj-ul-Islam, A. J. Khattak, Ikram A. Tirmizi, *A meshfree method for numerical solution of KdV equation*, Engineering Analysis with Boundary Elements 32 (2008) 849-855.

[10] A. A. Soliman, M. H. Hussien, *Collocation solution for RLW equation with septic spline*, Applied Mathematics and Computation 161 (2005) 623-636.

[11] J. H. He, *Homotopy perturbation technique*, Comp. Meth. Appl. Mech. Engng 178 (1999) 257-262.

[12] J. H. He, *The homotopy perturbation method for nonlinear oscillators with discontinuities*, Appl. Math. Comput 151 (2004) 287-292.

[13] Sh. S. Behzadia, A.Yildirim, *A method to estimate the solution of a weakly singular nonlinear integro-differential equations by applying the Homotopy methods*, International Journal of Industrial Mathematics 4 (2012) 41-51.

[14] E. Babolian, A. R. Vahidi, Z. Azimzadeh, *An improvement to the homotopy perturbation method for solving integro-differential equations*, International Journal of Industrial Mathematics 4 (2012) 353-363.

[15] J. H. He, *A note on the homotopy perturbation method*, Thermal Science 14 (2010) 565-568.

[16] D. Grover, Virendra Kumar and Dinkar Sharma, *A Comparitive Study Of Numerical Techniques And Homotopy Perturbation Method For Solving Parabolic Equation And Non-Linear Equations*, International Journal for Computational Methods in Engineering Science & Mechanics (Accepted).

[17] S. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, PhD thesis, Shanghai Jiao Tong University, 1992.

[18] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman and Hall/CRC Press, Boca Raton, (2003).

[19] S. J. Liao, *Notes on the homotopy analysis method, some definitions and theorems*, Commun. in Nonlinear Sci. and Numer. Simulat 14 (2009) 983-997.

[20] S. J. Liao, *Topology and geometry for physicists*, Academic Press, Florida Press (1983).

[21] S. Abbasbandy, *Homotopy analysis method for heat radiation equations*, International Communications in Heat and Mass Transfer 34 (2007) 380-387.

[22] S. Abbasbandy, Y. Tan, S. J. Liao, *Newton-homotopy analysis method for nonlinear equations*, Applied Mathematics and Computation 188 (2007) 1794-1800.

[23] M. Sajid, T. Hayat, S. Asghar, *On the analytic solution of the steady flow of a fourth grade fluid*, Phy. Lett. A 355 (2006).

[24] E. Babolian, N. Dastani, *Numerical solutions of two-dimensional linear and nonlinear Volterra integral equation: Homotopy perturbation method and differential transform method*, International Journal of Industrial Mathematics 3 (2011) 157-167.

[25] Sushila Rathore, Devendra Kumar, Jagdev Singh, Sumit Gupta, *Homotopy Analysis Sumudu Transform Method for Nonlinear Equations*, International Journal of Industrial Mathematics 4 (2012) 301-314.

[26] S. Abbasbandy, *The application of homotopy analysis method to nonlinear equations arising in heat transfer*, Physics Letters A 360 (2006) 109-113.

S. Dhawan received Ph.D in Mathematics from National Institute of Technology Jalandhar India, in 2012, Post-doctoral fellow, National Board for Higher Mathematics, Department of Atomic Energy, Govt. of India. She is currently working as Assistant Professor in Mathematics at D.A.V. University Jalandhar, India. Her research interests are heat transfer, fluid dynamics, FEM, Wavelets and related areas.

D. Grover is working as Assitant Professor at Graphic Era University Dehradun, India. He has been working in several research fields on computational , numerical analysis techniques. He is member of various National/International scientific bodies. He has (co)authored many research papers in the Journals of high international repute.

S. Kumar is Professor, Department of Mathematics, Dr. B. R. Ambedkar National Institute of Technology, Jalandhar, India. He received M.Sc. degree in Mathematics in 1973, from Indian Institute of Technology, Kanpur, India and Ph.D. degree in 1981, from Indian Institute of Technology, Delhi, India. His research interests are Numerical Analysis: Finite element methods, Wavelet methods for Partial differential equations.