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Research Article

Singular constrained linear systems

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Abstract

In the linear system $Ax = b$ the points x are sometimes constrained to lie in a given subspace S of column space of A. Drazin inverse for any singular or nonsingular matrix, exist and is unique. In this paper, the singular consistent or inconsistent constrained linear systems are introduced and the effect of Drazin inverse in solving such systems is investigated. Constrained linear system arise in electrical network theory.

Keywords : Singular matrix; Drazin inverse; Constrained systems; Bott-Duffin inverse.

1 Introduction

Linear system of equations play major role in various areas of sciences and engineering such as fuzzy mathematics [1], linear regression [15], etc. There are different forms of linear system of equations for different purposes such as fuzzy linear system [16, 17], fully fuzzy linear system [8, 19], dual linear interval equations [11]. A linear system is consistent if it has a solution, and inconsistent otherwise. In the linear system $Ax = b$ the points x are sometimes constrained to lie in a given subspace S of column space of A. Such system is called a constrained linear system. By means of complex representation of a quaternion matrix, Jianga el al. [9] study the relationship between the solutions of the quaternion equality constrained least squares problem and that of complex equality constrained least squares problem, and obtain a newtechnique of finding a solution of the quaternion equality constrained least squares problem.

The constrained linear system

$$Ax + y = b, \quad x \in L, \quad y \in L^{\perp}$$

with given matrix $A \in C^{n\times n}$, $b \in C^n$ and a subspace L of C^n is converted to be the equivalent linear system of equations [2, 12]. A constrained linear system that is equivalent to a singular linear system is called a singular constrained linear system. Constrained linear system arise in electrical network theory [2, 12]. Electrical network has useful in many applications [22]. In [2, 7], an electrical network is described topologically in terms of its graph consisting of nodes and branches, and electrically in terms of its currents and voltages.

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Unlike the case of the nonsingular matrix, which has a single unique inverse for all purposes, there are different generalized inverses for different purposes [2, 5, 21]. Drazin inverse is one of the generalized inverses and in solving consistent or inconsistent singular linear system has been used [2, 14]. There are many methods for solving linear system using generalized inverses [2, 12]. Minimization of quadratic forms using the Drazin-inverse solution is investigated in [20]. Bott-Duffin inverse is another generalized inverse that in solving constrained linear system was applied [4]. On the consistency of singular constrained systems is discussed in [12].

In this paper, singular constrained linear system is introduced and solving singular constrained linear system using Drazin inverse is investigated. In Section 2, we recall some preliminaries. Section 3 is on the singular constrained system. Solving singular constrained linear system is investigated in Section 4. In Section 5 some numerical example, are given, followed by a suggestion and concluding remarks in Section 6.

2 Preliminaries

We first present some preliminaries and basic definitions which are needed in this paper. For more details, we refer the reader to [2, 5].

Definition 2.1 *Let $A \in C^{n\times n}$. We say the nonnegative integer number k to be the index of matrix A and is denoted by $ind(A)$, if k is the smallest nonnegative integer number such that*

$$rank(A^{k+1}) = rank(A^k).$$

Definition 2.2 *The system*

$$Ax + y = b, \quad x \in L, \quad y \in L^{\perp}, \tag{2.1}$$

with given matrix $A \in C^{n\times n}$, $b \in C^n$, and a subspace L of C^n where $L^{\perp}$ is the orthogonal complement of L is called a constrained linear system.

Theorem 2.1 *([2]) The consistency of (2.1) is equivalent to the consistency of the following system*

$$(AP_L + P_{L^{\perp}})z = b, \tag{2.2}$$

and that $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution of (2.2) if and only if

$$x = P_L z, \quad y = P_{L^{\perp}} z = b - AP_L z,$$

where P_L orthogonal projector on L, $L^{\perp}$ is called the orthogonal complement of L and z is a solution of (2.2).

Definition 2.3 *Let $A \in C^{n\times n}$ and let L be a subspace of C^n. If $(AP_L + P_{L^{\perp}})$ is nonsingular, the Bott-Duffin inverse of A with respect to L, denoted by $A^{(-1)}_{(L)}$, is defined by*

$$A^{(-1)}_{(L)} = P_L(AP_L + P_{L^{\perp}})^{-1}.$$

Some properties of $A^{(-1)}_{(L)}$ are collected in [2].

Definition 2.4 *Let $A \in C^{n\times n}$, with $ind(A) = k$. The matrix X of order n is the Drazin inverse of A ,denoted by A^D , if X satisfies the following conditions*

$$AX = XA, XAX = X, A^kXA = A^k.$$

Theorem 2.2 *([5]) A^Db is a solution of*

$$Ax = b, k = ind(A), \tag{2.3}$$

if and only if $b \in range(A^k)$, and A^Db is an unique solution of (2.3) provided that $x \in range(A^k)$.

Theorem 2.3 *([5]) Let $A \in C^{n\times n}$, with $ind(A) = k$, $rank(A^k) = r$. We may assume that the Jordan normal form of A has the form as follows*

$$A = P \begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix} P^{-1},$$

where P is a nonsingular matrix, D is a nonsingular matrix of order r , and N is a nilpotent matrix that $N^k = 0$. Then we can write the Drazin inverse of A in the form

$$A^D = P \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

When $ind(A) = 1$, it is obvious that $N = 0$.

3 Singular constrained linear system

In this Section, singular constrained linear system is introduced and some results on the singular constrained linear system, are given.

Theorem 3.1 *The constrained linear system*

$$Ax + y = b, \quad x \in L, \quad y \in L^{\perp},$$

wherein A is a nonsingular matrix may be equivalent to a singular linear system.

Proof. From [6], we have

$$rank(AP_L) \leq min\{rank(A), rank(P_L)\}.$$

We know that (AP_L) is a singular matrix. From [6], we have

$$rank(AP_L + P_{L^{\perp}}) \leq rank(AP_L) + rank(P_{L^{\perp}}).$$

Therefore it is possible that

$$(AP_L) + rank(P_{L^{\perp}}),$$

be a singular matrix.

Corollary 3.1 *The constrained linear system*

$$Ax + y = b, \quad x \in L, \quad y \in L^{\perp},$$

wherein A is a singular matrix may be equivalent to the linear system

$$(AP_L + P_{L^{\perp}})z = b,$$

wherein $(AP_L + P_{L^{\perp}})$ is a nonsingular matrix.

Definition 3.1 *The system*

$$Ax + y = b, \quad x \in L, \quad y \in L^{\perp}, \tag{3.4}$$

with given matrix $A \in C^{n\times n}$, $b \in C^n$, and a subspace L of C^n, while $(AP_L + P_{L^{\perp}})$ be a singular matrix is called a singular constrained linear system.

Theorem 3.2 *The singular constrained system (3.4) has a set of solutions if and only if*

$$rank[AP_L + P_{L^{\perp}}] = rank[AP_L + P_{L^{\perp}}|b].$$

Proof. We assume that the system (3.4) is equivalent to the following singular linear system

$$(AP_L + P_{L^{\perp}})z = b. \tag{3.5}$$

If $rank[AP_L + P_{L^{\perp}}] = rank[AP_L + P_{L^{\perp}}|b]$ we know the singular linear system (3.5) has solution. From [10], the singular linear system (3.5) has a set of solutions.

Conversely, suppose the system (3.5) is solvable i.e.

$$rank[AP_L + P_{L^{\perp}}] = rank[AP_L + P_{L^{\perp}}|b].$$

From [10], the singular linear system (3.5) has a set of solutions.

Definition 3.2 *The singular constrained linear system (3.4) is called a singular consistent constrained system while*

$$rank[AP_L + P_{L^{\perp}}] = rank[AP_L + P_{L^{\perp}}|b],$$

inconsistent otherwise.

Corollary 3.2 *The equivalent singular linear system of any singular inconsistent constrained system has a set of least-squares solutions.*

Definition 3.3 *Let the singular inconsistent constrained system*

$$Ax + y = b, \quad x \in L, \quad y \in L^{\perp},$$

is equivalent to the following singular linear system

$$(AP_L + P_{L^{\perp}})z = b.$$

The system

$$(AP_L + P_{L^{\perp}})^k(AP_L + P_{L^{\perp}})z = (AP_L + P_{L^{\perp}})^k b$$

is called indicial equations wherein k is the index of matrix $(AP_L + P_{L^{\perp}})$.

4 Solving singular constrained linear systems

In this Section, the effect of Drazin inverse in solving singular constrained linear system is investigated.

Theorem 4.1 *Let the singular consistent constrained linear system (3.4) be equivalent to the singular linear system*

$$(AP_L + P_{L^\perp})z = b, \tag{4.6}$$

wherein

$$k = ind(AP_L + P_{L^\perp}).$$

A member of set of solutions of the system (4.6) is

$$(AP_L + P_{L^\perp})^D b,$$

if and only if $b \in R((AP_L + P_{L^\perp})^k)$.

Proof. By Definition 3.1 the matrix $(AP_L+P_{L^\perp})$ is singular. Therefore, By [10], the system (4.6) has a set of solutions and from [5], $(AP_L+P_{L^\perp})^D b$ is a member of set of solutions of the system (4.6).

Theorem 4.2 *Let the singular inconsistent constrained linear system (3.4) be equivalent to the singular linear system*

$$(AP_L + P_{L^\perp})z = b, \tag{4.7}$$

wherein

$$k = ind(AP_L + P_{L^\perp}).$$

A solution of the system (4.7) is

$$((AP_L + P_{L^\perp})^k (AP_L + P_{L^\perp}))^D (AP_L + P_{L^\perp})^k b.$$

Proof. According to [18, 13], and properties of the Drazin inverse [3, 5], in order to obtain the Drazin inverse the projection method solves consistent or inconsistent singular linear system (4.7) through solving the following indicial equations

$$(AP_L + P_{L^\perp})^k (AP_L + P_{L^\perp})z = (AP_L + P_{L^\perp})^k b.$$

Indicial equations is a consistent singular linear system [13], and give a solution using Drazin inverse . Therefore by from [5], z is equal to

$$((AP_L + P_{L^\perp})^k (AP_L + P_{L^\perp}))^D (AP_L + P_{L^\perp})^k b.$$

5 Numerical examples

In this Section, the effect of Drazin inverse in solving singular constrained linear system is illustrated.

Example 5.1 *The singular consistent constrained linear system*

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix},$$

with subspace

$$L = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} |, \quad a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \forall a, b \in R\},$$

is equivalent to the singular linear system

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}.$$

By Theorem 2.3 we have

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P,$$

$$P = \begin{pmatrix} 1 & 1 & -3 \\ 1 & 1 & -2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore by Theorem 2.2 we get

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}.$$

Thus,

$$x = P_L z = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 0 \\ \frac{5}{2} \end{pmatrix},$$

and

$$y = P_{L^\perp} z = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}.$$

Example 5.2 *The singular inconsistent constrained linear system*

$$\begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

with subspace

$$L = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} |, \quad a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \forall a, b \in R\},$$

is equivalent to the inconsistent singular linear system

$$\begin{pmatrix} -1 & -3 & -2 \\ 2 & 4 & 2 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The indicial equations is

$$\begin{pmatrix} -1 & -3 & -2 \\ 2 & 4 & 2 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -3 & -2 \\ 2 & 4 & 2 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & -3 & -2 \\ 2 & 4 & 2 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

then

$$\begin{pmatrix} -1 & -3 & -2 \\ 2 & 4 & 2 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -13 \\ 16 \\ -11 \end{pmatrix},$$

and

$$\begin{pmatrix} -1 & -3 & -2 \\ 2 & 4 & 2 \\ -2 & -3 & -1 \end{pmatrix} = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P,$$

$$P = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \\ 2 & 3 & 2 \end{pmatrix}.$$

Therefore by theorem4.2 we can get

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -1 & -3 & -2 \\ 2 & 4 & 2 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} -13 \\ 16 \\ -11 \end{pmatrix} = \begin{pmatrix} -13 \\ 16 \\ -11 \end{pmatrix}.$$

Thus,

$$x = P_L z = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -13 \\ 16 \\ -11 \end{pmatrix} = \begin{pmatrix} -12 \\ 16 \\ -12 \end{pmatrix},$$

and

$$y = P_{L^\perp} z = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

6 Conclusion and suggestions

The transformation $P_L(AP_L + P_{L^\perp})^{-1}$ was introduced and studied by Bott and Duffin who called it the constrained inverse of A. The Drazin inverse of the singular matrix $(AP_L+P_{L^\perp})$ in solving singular constrained linear system is used. Solving the constrained fuzzy linear system $Ax+y = b$ with given crisp matrix $A \in C^{n\times n}$ and fuzzy vector b is suggested.

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