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Numerical Solution of Fractional Control System by Haar-wavelet Operational Matrix Method

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Abstract

In recent years, there has been greater attempt to find numerical solutions of differential equations using wavelet's methods. The following method is based on vector forms of Haar-wavelet functions. In this paper, we will introduce one dimensional Haar-wavelet functions and the Haar-wavelet operational matrices of the fractional order integration. Also the Haar-wavelet operational matrices of the fractional order differentiation are obtained. Then we propose the Haar-wavelet operational matrix method to achieve the Haar-wavelet time response output solution of fractional order linear systems where a fractional derivative is defined in the Caputo sense. Using collocation points, we have a Sylvester equation which can be solve by Block Krylov subspace methods. So we have analyzed the errors. The method has been tested by a numerical example. Since wavelet representations of a vector function can be more accurate and take less computer time, they are often more useful.

Keywords : Fractional control system; Haar wavelet; Sylvester equation.

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1 Introduction

FRactional differential equations have generalized from integer order ones, which achieved by replacing integer order derivatives by fractional ones. In recent years, studies on application of the FDE in science has been attracting more attention [5, 22, 8, 27] and the reader may refer to [22, 8] for the theory and applications of fractional calculus. For instance, Bagley and Torvik formulated the motion of a rigid plate immersing in a Newtonian fluid [22, 8, 20]. It shows that the use of fractional derivatives for the mathematical modelling of viscoelastic materials is quite natulral [22]. It should be mentioned that the main reasons for the theoretical development are mainly the wide use of polymers in various fields of engineering [22]. Also in 1991, S. Westerlund suggested using fractional derivatives for the description of propagation of plane electromagnetic waves in an isotropic and homogeneous, lossy dielectric and in the paper on electrochemically polarizable media, published in 1993[22]. Caputo suggested the fractional-order version of the relationship between electric field and electric flux density [22].
Recently, fractional derivatives have been used to new applications in neural networks and control system [25].
A typical n-term linear non-homogeneous fractional order differential equation (FDE) in time

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domain can be described as the following form,

$$a_n(D_t^{\alpha_n}y(t)) + \cdots + a_1(D_t^{\alpha_1}y(t)) + \\ +a_0(D_t^{\alpha_0}y(t)) = u(t). \tag{1.1}$$

A fractional-order system described by n-term fractional differential Eq. (1.1) can be rewritten to the state-space representation in the form [11, 30]:

$$\begin{array}{c} {}_aD_t^{\beta}x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{array}. \tag{1.2}$$

For this reason the behavior of output in system 1.2 are useful.

Wavelets are mathematical tools that cut up data, functions or operators into different frequency components and then study each component with a resolution matching its scale. Much of the work on Haar functions was performed in the 1930s. In 1909, Haar discovered the simplest function that is called as Haar wavelet. The integral of Haar family called Haar operational matrix was derived by Chen and Hsiao [7] in 1997. Recently, Operational matrix method has became a very useful technique for solving fractional differential equations [28, 23, 6] and optimal control system [17].

In this paper, we will present Haar-wavelet time response of the fractional order system of the form

$$\begin{array}{c} {}_aD_t^{\alpha}x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \\ 0 \le t \le \eta, x(0) = \left(\begin{array}{c} \lambda_1, \lambda_2, \cdots, \lambda_n \end{array} \right)^T, \end{array} \tag{1.3}$$

with $0 < \alpha \le 1$, where A, B, C and D are $n \times n$, $n \times m$, $p \times n$ and $p \times m$ matrices respectively and $u(t)$ is an m-vector function.The rest of the paper has organized as follows: in Section 2 we recall some necessary definitions and theorems. The Function approximations and operational matrices is presented in Section 3. In Section 4, we express the method of the solution and the error analysis are study in Section 5. Section 6 contain one numerical example and Finally, the main conclusions are drawn in Section 7.

2 Preliminaries

In this Section, we present some basic definitions and properties of fractional calculus [22, 9, 26].

Definition 2.1 *A real function $f(x), x \ge 0$ is said to be in space $C_\mu, \mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in [0, \infty)$, and it is said to be in the space C_μ^m iff $f^m \in C_\mu, m \in N$.*

Definition 2.2 *The Riemann-Liouville fractional derivative of order α with respect to the variable x and with the starting point at $x = a$ is*

$$\begin{aligned} &{}_aD_t^{\alpha}f(x) \\ &= \tfrac{1}{\Gamma(-\alpha+m+1)} \tfrac{d^{m+1}}{dx^{m+1}} \textstyle\int_a^x (x-\tau)^{m-\alpha} f(\tau)d\tau, \end{aligned} \tag{2.1}$$

for $0 \le\ m \le \alpha < m+1$ and

$${}_aD_t^{\alpha}f(x) = \frac{d^{m+1}}{dx^{m+1}} f(x)$$

for $\alpha = m+1 \in N$.

Definition 2.3 *The Riemann-Liouville fractional integral of order α is*

$${}_aD_t^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)d\tau, \alpha > 0. \tag{2.2}$$

Definition 2.4 *The fractional derivative of $f(x)$ by means of Caputo sense is defined as*

$$\begin{aligned} &D_t^{\alpha}f(x) \\ &= \tfrac{1}{\Gamma(n-\alpha)} \textstyle\int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, \end{aligned} \tag{2.3}$$

where $n-1 < \alpha \le n, n \in N, x > 0, f \in C_{-1}^n$.

For the Caputo's derivative we have $D_t^{\alpha}C = 0$, C is a constant.

Definition 2.5 *(Fractional Derivative of a Vector) If $X(x) = (X_1(x) \cdots X_n(x))^T$ is a vector function, we define*

$$D_x^{\alpha}X(x) = \left(\begin{array}{c} D_x^{\alpha}X_1(x), \cdots, D_x^{\alpha}X_n(x) \end{array} \right)^T. \tag{2.4}$$

Definition 2.6 *The m-set of block-pulse functions is defined as:*

$$b_i(t) = \left\{ \begin{array}{ll} 1 & ; \frac{\eta i}{m} \le t \le \frac{\eta(i+1)}{m} \\ 0 & ; otherwise \end{array} \right. \tag{2.5}$$

for $i = 0, 1, 2, \cdots, m-1$.

The functions b_i are disjoint and orthogonal [9].

Definition 2.7 *(The Haar Wavelet Function) Let $[0,\eta)$ be an interval, we define $h_0(t)$ and $h_1(t)$ on $[0,\eta)$ as follows*

$$h_0(t) = \frac{1}{\sqrt{\eta}} \begin{cases} 1 & ;\ 0 \leq t < \eta, \\ 0 & ;\ otherwise, \end{cases}$$

$$h_1(t) = \frac{1}{\sqrt{\eta}} \begin{cases} 1 & ;\ 0 \leq t < \frac{\eta}{2}, \\ -1 & ;\ \frac{\eta}{2} \leq t < 1, \\ 0 & ;\ otherwise, \end{cases}$$

and for $i = 2^j + k$, $j \geqslant 0$, $0 \leq k \leq 2^j - 1$, we define

$$h_i(t) = \frac{2^{\frac{j}{2}}}{\sqrt{\eta}} h_1(2^j t - k).$$

The best way to understand wavelets is through a multi-resolution analysis. Given a function $f \in L_2(R)$ a multi-resolution analysis (MRA) of $L_2(R)$ produces a sequence of subspaces V_j, V_{j+1}, $\cdots$ such that the projections of f onto these spaces give finer and finer approximations of the function f as $j \to \infty$.

Definition 2.8 *(Multi-resolution Analysis(MRA)) A multi-resolution analysis of $L^2(R)$ is defined as a sequence of closed subspaces $V_j \subset L^2(R), j \in Z$ with the following properties*
i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots$.
ii) The spaces V_j satisfy $\bigcup_{j\in Z} V_j$ is dense in $L^2(R)$ and $\bigcap_{j\in Z} V_j = 0$.
iii) If $f(x) \in V_0, f(2^j x) \in V_j$, i.e. the spaces V_j are scaled versions of the central space V_0.
iv) If $f(x) \in V_0, f(2^j x - k) \in V_j$, i.e. all the V_j are invariant under translation.
v) There exists $\phi \in V_0$ such that $\phi(x-k); k \in Z$ is a Riesz basis in V_0.

The space V_j is used to approximate general functions by defining appropriate projection of these functions onto these spaces. Since the union of all the V_j is dense in $L^2(R)$, so it guarantees that any function in $L^2(R)$ can be approximated arbitrarily close by such projections. As an example the space V_j can be defined like

$$V_j = W_{j-1} \oplus V_{j-1} = W_{j-1} \oplus W_{j-2} \oplus V_{j-2}$$
$$= \cdots = \bigoplus_{i=0}^{j-1} W_i \oplus V_0$$

then the scaling function $h_0(x)$ generates an MRA for the sequence of spaces $\{V_j, j \in Z\}$ by translation and dilation as defined in definition 2.8. For each j the space W_j serves as the orthogonal complement of V_j in V_{j+1}. The space W_j include all the functions in V_{j+1} that are orthogonal to all those in V_j under some chosen inner product. The set of functions which form basis for the space W_j are called wavelets [13, 21].
The following theorem gives several equivalent statements which permit us to check if an orthonormal system is also a basis:

Theorem 2.1 *Given an orthonormal system $x_1, x_2, \cdots$ in E, the following are equivalent:*
i) The set of vectors $x_1, x_2, \cdots$ is an orthonormal basis for E.
ii) If $< x_i, y >= 0$ for $i = 1, 2, \cdots$, then $y = 0$, where $< x, y >$ is the inner product of x and y.
iii) $span(x_i)$ is dense in E, that is, every vector in E is a limit of a sequence of vectors in $span(x_i)$.
iv) For every y in E,

$$||y||^2 = \textstyle\sum_i |< x_i, y >|^2,$$

*which is called **Parsevals equality**.*
v) For every y_1 and y_2 in E,

$$< y_1, y_2 >= \textstyle\sum_i < x_i, y_1 > * < x_i, y_2 >,$$

*which is often called the **generalized Parsevals equality**.*

Proof. see [12].

Lemma 2.1 *Every characteristic function of the form $\chi_{[0,k/2n)(t)}$ is a finite linear combination of the $h_i(t)$.*

Proof. We will induct on n. Let P_n be the statement that for all integers k with $0 \leq k \leq 2^n - 1$ the characteristic function $\chi_{[0,\frac{k}{2^n}\eta)}(t)$ is finite linear combination of the h_i. P_0 is true, since $\chi_{[0,\eta)}(t) = \sqrt{(\eta)} h_0(t)$. Assume that P_n is true. We use this to show that $\chi_{[0,\frac{k}{2^{n+1}}\eta)}$ is a finite linear combination of h_i. We first do the case when $k \leq 2^n - 1$. Since P_n is true we have $\chi_{[0,\frac{k}{2^n}\eta)}(t) = \sum_i a_i h_i(t)$, but $\chi_{[0,\frac{k}{2^{n+1}}\eta)}(t) = \chi_{[0,\frac{k}{2^n}\eta)}(2t)$, thus we can write

$$\chi_{[0,\frac{k}{2^{n+1}}\eta)}(t) = \begin{cases} \sum_i a_i h_i(t) & ; t \leq \frac{\eta}{2} \\ 0 \ \ ; t > \frac{\eta}{2} \end{cases},$$

and therefore P_{n+1} is true. Now we need to take care of the case when $k > 2^n - 1$. We first observe that if $k > 2^n - 1$ then

$$\chi_{[0,\frac{k}{2^{n+1}}\eta)} = \chi_{[0,\frac{1}{2}\eta)} + \chi_{[\frac{1}{2}\eta,\frac{k}{2^{n+1}}\eta)},$$

We already know that $\chi_{[0,\frac{1}{2}\eta)}$ is a finite linear combination of the h_i, so we only need to show that $\chi_{[\frac{1}{2}\eta,\frac{k}{2^{n+1}}\eta)}$ is too. Observe that

$$\chi_{[\frac{1}{2}\eta,\frac{k}{2^{n+1}}\eta)}(t) = \chi_{[0,\frac{k-2^n}{2^n}\eta)}(2t-\eta),$$

applying the assumption that P_n is true for the above equality and the proof can be completed.

Theorem 2.2 *Any function $y(t) \in L^2[0,\eta)$ can be decomposed as*

$$y(t) = \sum_{i=0}^{\infty} c_i h_i(t), \tag{2.6}$$

where the coefficients c_i are determined by

$c_i = 2^j \int_0^\eta y(t)h_i(t)dt,\ i = 2^j + k,\ j \geqslant 0,\ 0 \leq k \leq 2^j - 1.$

Proof. Let $f \in L^2[0,\eta)$ such that $\int_0^\eta f(t)h_i(t)dt = 0$ for $i = 0, 1, \cdots$. therefore

$\int_0^{\frac{k}{2^n}\eta} f(t)dt = \int_0^\eta \chi_{[0,\frac{k}{2^n}\eta)}(t)f(t)dt = \int_0^\eta (\sum_i a_i h_i(t))f(t)dt = 0.$

The set of all numbers of the form $\frac{k}{2^n}\eta$ are dense in R and for evry $x \in R$ there is an increasing sequence x_i such that $x_i \to x$. This shows that $\int_0^x f(t)dt = 0$ for all $x \in [0,\eta)$ and therefore $f = 0$, so theorem 2.9 shows that $h_i, i = 0, 1, \cdots$ is a basis for $L^2[0,\eta)$■

Theorem 2.3 *Assume that $y(t) \in L^2(R)$ with the bounded first derivative on $(0,1)$ and $y_m(t) = \sum_{i=0}^{2^{m+1}} c_i h_i(t)$, then*

$$||y(t) - y_m(t)||^2$$

$$= \sum_{i=m}^{\infty} \sum_{j=m}^{\infty} c_i c_j \int_{-\infty}^{\infty} h_i(t)h_j(t)dt \tag{2.7}$$

$$\leq \frac{k}{7}c^2 2^{-\frac{3}{2}m} \quad ,$$

where $c = \int_0^1 |th_2(t)|dt$ and k is a constant and $||g(t)||= (\int_{-\infty}^{\infty} g^2(t)dt)$
12.

Proof. The error at J^{th} level may be defined as $|e_J(t)|= |y(t) - y_J(t)|= \sum_{i=2^{J+1}+1}^{\infty} c_i h_i(t)$ where $y_J(t) = \sum_{i=1}^{2^{J+1}} c_i h_i(t)$. Thus we have

$||e_J(t)||^2$

$$= \int_{-\infty}^{\infty} (\sum_{i=2^{J+1}+1}^{\infty} c_i h_i(t), \sum_{l=2^{J+1}+1}^{\infty} c_l h_l(t))dt$$

$$= \sum_{i=2^{J+1}+1}^{\infty} \sum_{l=2^{J+1}+1}^{\infty} c_i c_l \int_{-\infty}^{\infty} h_i(t)h_l(t)dt,$$

this shows that

$$||e_J(t)||^2 \leq \sum_{i=2^{J+1}+1}^{\infty} |c_i|^2. \tag{2.8}$$

But $|c_i| \leq c2^{-\frac{3i}{2}} \max(y'(\eta))$ where $c = \int_0^1 |th_2(t)|dt$ and $\eta \in (k2^{-j}, (k+1)2^{-j})$[14, 16]. Thus

$||e_J(t)||^2 \leq \sum_{i=2^{J+1}+1}^{\infty} kc^2 2^{-3i},$

where $|y'(t)| \leq k$ for all $t \in (0,1)$ and k is a positive constant. From the last relation we have

$||e_J(t)||^2 \leq kc^2 \frac{1}{7} 2^{-3m},$
or
$||e_J(t)|| \leq \sqrt{\frac{k}{7}} c2^{-\frac{3}{2}m}$ ■

In our survey, the fractional derivatives and fractional integrals have considered in the Caputo and Riemann-Liouville sense, respectively. For more details see [22, 3]. Let us consider the fractional differential equation

$$D_t^\alpha x(t) = Ax(t) + q(t), \tag{2.9}$$

with $0 < \alpha < 1$, an $N \times N$ matrix A, a given function $q : [0, h] \to \mathbb{C}^N$ and an unknown solution $x : [0, h] \to \mathbb{C}^N$. Two following theorems shows the form of the general solution of (2.9) where $E_\alpha(t)$ is the ***Mittag-Leffler function***.

Definition 2.9 *The Mittag-Leffler function with parameter α is given by*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},\ \Re(\alpha) > 0,\ z \in \mathbb{C}.$$

It is obvious that $E_\alpha(z) = e^z$ for $\alpha = 1$.

Theorem 2.4 *Let $\lambda_1, \cdots, \lambda_N$, be the eigenvalues of A and $u^{(1)}, \cdots, u^{(N)}$ be the corresponding eigenvectors. Then, the general solution of the homogeneous differential equation $D_t^\alpha x(t) = Ax(t)$, has the form*

$$x(t) = \sum_{l=1}^{N} c_l u^{(l)} E_\alpha(\lambda_l x^\alpha), \tag{2.10}$$

with certain constants $c_l \in \mathbb{C}$. The unique solution of this differential equation subject to the initial condition $x(0) = x_0$ is characterized by the linear system

$$x_0 = (u^{(1)}, \cdots, u^{(N)})(c_1, \cdots, c_N)^T. \tag{2.11}$$

Proof. see [10].

For the inhomogeneous boundary value problem we can state the following result.

Theorem 2.5 *The general solution of the boundary value problem (2.9) has the form $x = x_{hom} + x_{inhom}$ where x_{hom} is the general solution of the associated homogeneous problem and x_{inhom} is a particular solution of the inhomogeneous problem.*

Proof. see [10].

3 Function approximations and operational matrices

The series expansion of $y(t)$ in (2.6) contains an infinite terms. If $y(t)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(t)$ will be terminate at finite terms, that is

$$y(t) \simeq \sum_{i=0}^{m-1} c_i h_i(t) = C_m^T H_m(t), \tag{3.12}$$

where T indicates transposition and $C_m = \begin{pmatrix} c_0 & c_1 & \cdots & c_{m-1} \end{pmatrix}^T$ is the Haar coefficient vector of $y(t)$ and $H_m(t) = \begin{pmatrix} h_0(t) & h_1(t) & \cdots & h_{m-1}(t) \end{pmatrix}^T$ and $m = 2^j$. At collocation points $t_i = \frac{2i+1}{2m}$, $i = 0, 1, \cdots, m-1$, one can define $m \times m$ Haar matrix as

$$H_{m\times m} = \begin{pmatrix} H_m(t_0) & H_m(t_1) & \cdots & H_m(t_{m-1}) \end{pmatrix}.$$

Since $H_{m\times m}$ is singular [9, 26], the Haar coefficients $c_i, i = 0, 1, 2, \cdots, m-1$ can be also be determined by matrix inversion as follows

$$C_m^T = y_m H_{m\times m}^{-1}, \quad y_m = \begin{pmatrix} y(t_0) & y(t_1) & \cdots & y(t_{m-1}) \end{pmatrix}. \tag{3.13}$$

The integration of Haar function vector $H_m(t)$ is given by

$$\int_0^t H_m(s)ds \simeq P_{m\times m} H_m(t), \tag{3.14}$$

where $P_{m\times m}$ is the Haar wavelet operational matrix of integration [26] and is given by

$$P_{m\times m} = \frac{1}{2m}\begin{pmatrix} 2mP_{\frac{m}{2}\times\frac{m}{2}} & -H_{\frac{m}{2}\times\frac{m}{2}} \\ H^{-1}_{\frac{m}{2}\times\frac{m}{2}} & 0 \end{pmatrix}.$$

Also the Haar wavelets can be expand into m-set of block-pulse functions as

$$H_m(t) = H_{m\times m} B_m(t) \tag{3.15}$$

where the block-pulse function vector $B_m(t)$ is defined as $B_m(t) = \begin{pmatrix} b_1(t) & b_2(t) & \cdots & b_{m-1}(t) \end{pmatrix}^T$. Fractional integration of the block-pulse function vector is given as

$$(I^\alpha B_m)(t) = F^\alpha B_m(t), \tag{3.16}$$

where F^α is the block-pulse operational matrix of the fractional order integration [18]. The Haar wavelet operational matrix of fractional order integration can be derive as following [19],

$$(I^\alpha H_m)(t) = P^\alpha_{m\times m} H_m(t), \tag{3.17}$$

where $P^\alpha_{m\times m}$ is the Haar wavelet operational matrix of fractional order integration and can be obtained by substituting (3.15) and (3.16) in (3.17) as

$$P^\alpha_{m\times m} = H_{m\times m} F^\alpha H^{-1}_{m\times m}. \tag{3.18}$$

4 Method of solution

In this Section we consider the fractional order system . From 1.3 we find that

$$D^\alpha x_i(t) = A_i x(t) + B_i u(t). \tag{4.19}$$

Also from (3.12) we can approximate each $x_i(t)$ by

$$x_i(t) \simeq C_{i,m}^T H_m(t), \tag{4.20}$$

thus we can set

$$x(t) = C_x H_m(t), \tag{4.21}$$

where $C_x = \begin{pmatrix} C_{1,m}^T, C_{2,m}^T, \cdots, C_{n,m}^T \end{pmatrix}^T$,

similarly $u(t) = C_u H_m(t)$ where C_u can derived by (3.13), now we have

$$\begin{aligned} &(I^\alpha D^\alpha x_i)(t) = I^\alpha(A_i x + B_i u)(t) \\ \Rightarrow \quad & x_i(t) = A_i C_x (I^\alpha H_m)(t) + \\ & + B_i C_u (I^\alpha H_m)(t) + x_i(0), \end{aligned} \tag{4.22}$$

using (3.16), (3.17) and (3.18) in (4.22) we have

$$\begin{aligned} & x_i(t) \\ & = (A_i C_x + B_i C_u) H_{m\times m} F^\alpha H_{m\times m}^{-1} H_m(t) + \\ & + x_i(0), \end{aligned} \tag{4.23}$$

or in matrix form

$$\begin{aligned} & x(t) \\ & = (AC_x + BC_u) H_{m\times m} F^\alpha H_{m\times m}^{-1} H_m(t) + \\ & + x(0). \end{aligned} \tag{4.24}$$

Now by (4.21) and (4.24) we find that

$$\begin{aligned} & C_x H_m(t) \\ & = (AC_x + BC_u) H_{m\times m} F^\alpha H_{m\times m}^{-1} H_m(t) + \\ & + x(0), \end{aligned} \tag{4.25}$$

dispersing (4.25) by the collocation points t_i we can obtain

$$\begin{aligned} & C_x H_{m\times m} \\ & = (AC_x + BC_u) H_{m\times m} F^\alpha H_{m\times m}^{-1} H_{m\times m} \\ & + X_0, \end{aligned}$$

where $X_0 = \begin{pmatrix} x(0) & x(0) & \cdots & x(0) \end{pmatrix}$, thus we have

$$\begin{aligned} & C_x H_{m\times m} (H_{m\times m} F^\alpha)^{-1} - AC_x \\ & = BC_u + X_0 (H_{m\times m} F^\alpha)^{-1}, \end{aligned} \tag{4.26}$$

which is a Sylvester equation. This equation can be solve by Block Krylov subspace methods [24]. From above discussion we can response output of system as

$$y(t) = Cx(t) \simeq (CC_x + DC_u) H_m(t). \tag{4.27}$$

5 The error analysis

Theorem 5.1 *Assume that theorem* 2.8 *holds for $x_i(t); i = 1, 2, \cdots, n$, then we have*

$$||C_x H_m(t) - x(t)||^2 \leq n \frac{k}{7} c^2 2^{-\frac{3}{2}m}. \tag{5.28}$$

Proof: The error may be defined as

$$||v(t)|| = \left(\int_{-\infty}^{\infty} v^T(t) v(t)\right)^{\frac{1}{2}}$$

(v(t) is a column vector). So

$$\begin{aligned} & \|C_x H_m(t) - x(t)\|^2 \\ & = \| \begin{pmatrix} C_{1,m}^T H_m(t) & \cdots & C_{n,m}^T H_m(t) \end{pmatrix}^T - \\ & - \begin{pmatrix} \sum_{i=0}^\infty c_{i,1} h_i(t) & \cdots & \sum_{i=0}^\infty c_{i,n} h_i(t) \end{pmatrix}^T \|^2 \\ & = \| \sum_{i=2m}^{\infty} \begin{pmatrix} c_{i,1} h_i(t) & \cdots & c_{i,n} h_i(t) \end{pmatrix}^T \|^2 \\ & = \int_{-\infty}^{\infty} \Big(\sum_{i=2m}^{\infty} c_{i,1} h_i(t) \sum_{j=2m}^{\infty} c_{j,1} h_j(t) + \cdots + \\ & + \sum_{i=2m}^{\infty} c_{i,n} h_i(t) \sum_{j=2m}^{\infty} c_{j,n} h_j(t) \Big) \\ & = \sum_{l=1}^{n} \sum_{i=2m}^{\infty} \sum_{j=2m}^{\infty} c_{i,l} c_{j,l} \int_{-\infty}^{\infty} h_i(t) h_j(t) \end{aligned}$$

since $h_i^{,}s$ are orthonormal we can indicate

$$\|C_x H_m(t) - x(t)\|^2 = \sum_{l=1}^{n} \sum_{i=2m}^{\infty} c_{i,l}^2,$$

and since theorem 2.5 holds for each $x_i(t)$ we can write

$$\sum_{l=1}^{n} \sum_{i=2m}^{\infty} c_{i,l}^2 \leq \sum_{l=1}^{n} k_l c^2 \frac{1}{7} 2^{-3J},$$

so

$$\|C_x H_m(t) - x(t)\|^2 \leq \frac{(k_1 + \cdots + k_n)}{7} c^2 2^{-3J}.$$

■

From the above theorem, it is obvious that the error bound is inversely proportional to m. This ensures the convergence of the Haar wavelet approximation when m is increased.

Table 1: Haar wavelet numerical solution of example 6.1

	m=32		m=64		m=128		exact	
t	y˙1(t)	y˙2(t)	y˙1(t)	y˙2(t)	y˙1(t)	y˙2(t)	y˙1(t)	y˙2(t)
0.1	-4.9381	16.5918	-4.8503	16.6628	-4.7983	16.6632	-4.8341	16.6823
0.3	-4.3729	4.5574	-4.2037	3.5890	-4.2891	4.0012	-4.3060	4.0802
0.5	1.8293	-13.0698	1.7127	-13.1488	1.6339	-13.1337	1.5413	-13.0783
0.7	1.9040	1.6909	2.0606	0.9180	1.9617	1.5588	1.9400	1.7000
0.9	-4.7672	23.4380	-5.3226	24.0710	-5.3424	24.2568	-5.3226	24.2569

6 Examples

To demonstrate the efficiency and the practicability of the proposed method based on Haar wavelet operational matrix method, we consider the following example. In order to show the efficiency of method for solving system 1.3, we apply it to solve different types of fractional linear systems whose exact solutions are known. We use $\| \,.\, \|_2$ to compare exact and numerical solution.

Example 6.1 *In this example we consider a fractional system with three equations,*

$$D_t^{\alpha}x(t) = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & -9 \\ 3 & 6 & 1 \end{pmatrix} x(t) \tag{6.29}$$

$$y = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 3 \end{pmatrix} x(t),$$

$x(0) = \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix}, 0 \le t \le 1, 0 < \alpha \le 1.$ *The general solution of (6.29) according to theorem 2.13, is given by*

$$x(t) = c_1 u_1 E(\lambda_1 t^{\alpha}) + c_2 u_2 E(\lambda_2 t^{\alpha}) +$$

$$+ c_3 u_3 E(\lambda_3 t^{\alpha}), \tag{6.30}$$

$$y = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 3 \end{pmatrix} x(t),$$

where c_1, c_2, c_3 are constants and $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues and u_1, u_2, u_3 are corresponding eigenvectors of A. For $\alpha = 0.975$ and $m = 8$ we have

$$C_x = \begin{pmatrix} -2.8107 & -2.4681 & \cdots & -1.1778 \\ 4.1749 & 1.5822 & \cdots & 5.1485 \\ 1.2002 & 2.6363 & \cdots & 2.5816 \end{pmatrix}.$$

Thus from (4.27) we have

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \simeq C C_x H_8(t).$$

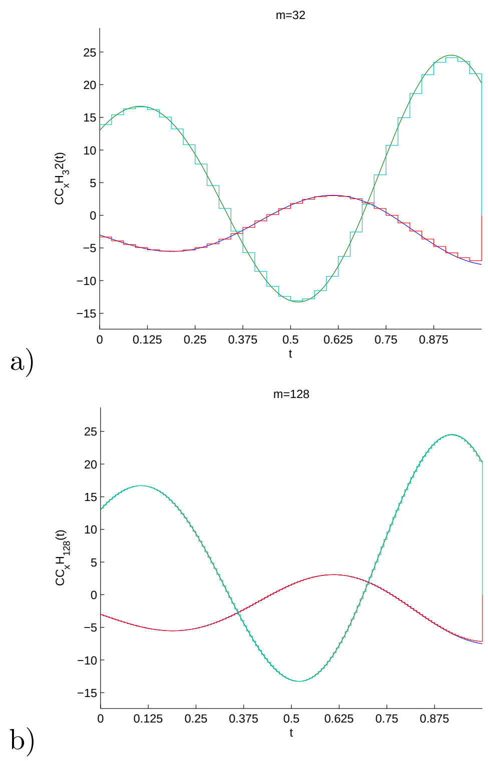
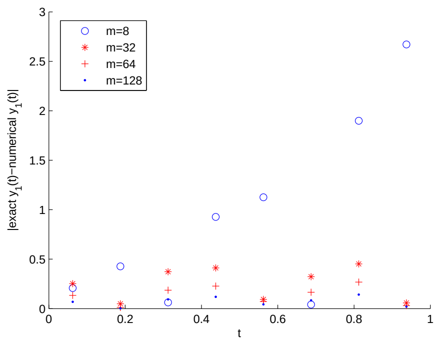
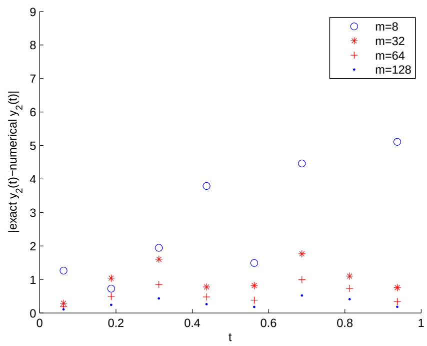


Figure 1: Output time response of example 6.1 by Haar representation for m=32, 128.

The Haar domain solution along with the actual solution are shown in Fig. 1.a, 1.b for $m = 32, 128$ respectively. Table 1 shows the nu-



a)



b)

Figure 2: Absolute errors of $y_1(t)$ and $y_2(t)$ at $t = \frac{1}{16}, \frac{3}{16}, ..., \frac{15}{16}$ for $m = 8, 32, 64, 128$.

merical solutions for different values of m. Also Fig. 2 shows the absolute errors for $y_1(t)$ and $y_2(t)$ respectively at the collocation points t_i for $m = 8, 32, 64, 128$. From Table 1, we see that we can achieve a good approximation for output with the exact solution by using $m = 64, 128$. The Haar domain solution along with the actual solution are shown in Fig. 1.a, 1.b for $m = 32, 128$ respectively. Table 1 shows the numerical solutions for different values of m. Also Fig. 2 shows the absolute errors for $y_1(t)$ and $y_2(t)$ respectively at the collocation points t_i for $m = 8, 32, 64, 128$. From Table 1, we see that we can achieve a good approximation for output with the exact solution by using $m = 64, 128$.

7 Conclusion

In this paper,we introduced Haar-wavelet operational matrix method to fractional control system. We translated the control system with initial condition into a Sylvester equation which can be solve by Block Krylov subspace methods. From Section 5 we found that the error bound is inversely proportional to m. This ensures the convergence of the Haar wavelet approximation when m is increased. An example presented in Section 6 and the results obtained are compared with exact solutions. Moreover if we use distributed order fractional derivative instead of fractional derivative, then what will be the form of operational matrix represented in Section 3?

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