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Planarity of Intersection Graph of submodules of a Module

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Abstract

Let R be a commutative ring with identity and M be an unitary R-module. The intersection graph of an R-module M, denoted by $\Gamma(M)$, is a simple graph whose vertices are all non-trivial submodules of M and two distinct vertices N_1 and N_2 are adjacent if and only if $N_1 \cap N_2 \neq 0$. In this article, we investigate the concept of a planar intersection graph and maximal submodules of an R-module. In particular, we show that if $\Gamma(M)$ is a planar graph, then $M \cong M_1 \oplus M_2$ for a multiplication R-module M with $|Max(M)| \neq 1$.

Keywords : Interval methods; Multiplication modules; Planar Graph; Module Theory; Torsion Graphs.

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1 Introduction

It is well known that graph is a very useful tool to model problems originated in all most all areas of our life. In this article, we concentrate our discussion on intersection graphs. Let $S = \{S_i : i \in I\}$ be an arbitrary family of sets. The intersection graph $\Gamma(S)$ of S is the graph whose vertices are S_i, $i \in I$ and there is an edge between two distinct vertices S_i and S_j if and only if $S_i \cap S_j \neq \emptyset$. It is more interesting to study the intersection graphs $\Gamma(S)$ when the elements of S have an algebraic structure. These studies allow us to obtain characterization and representation of the classes of algebraic structure in terms of graphs and vice versa.

Let R be a commutative ring with identity and M be a unitary R-module. The idea of the intersection graph of semigroups was introduced by Bosak in [5]. Inspired by his work, Csákány and Pollák in [8], studied the graph of subgroups of a finite group. The intersection graph of ideals of a ring, was considered by Chakrabarty, Ghosh, Mukherjee and Sen in [7]. Recently, Akbari, Tavallaee and Khaiashi in [1], introduced and investigated the intersection graph of submodules of a module.

In this paper, we investigate the concept of intersection graph of a module. The intersection graph of an R-module M, denoted by $\Gamma(M)$, is defined to be the undirected simple graph whose vertices are all non-trivial submodules of M and two distinct vertices are adjacent if and only if the corresponding submodules of M have nonzero intersection. This study helps to illuminate the structure of M, for example, if $\Gamma(M)$ is a planar graph, then M is both Noetherian and Artinian.

Recall that a simple graph is finite if its vertices set is finite, and we use the symbol $|\Gamma(M)|$ to denote the number of vertices in graph $\Gamma(M)$. Also, a graph G is connected if there is a path between any two distinct vertices. The distance, $d(x, y)$ between connected vertices x, y is the length of the shortest path from x to y, ($d(x, y) = \infty$ if there is no such path). An isolated vertex is a

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vertex that has no edges incident to it. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n for the complete graph with n vertices. The complement $\overline{G}$ of G is the graph with vertex set $V(\overline{G}) = V(G)$, and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. The complement of a complete graph is the null graph. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A remarkably simple characterization of planar graphs was given by Kuratowski in [5], p.153. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

An R-module M is a multiplication module if for every R-submodule K of M there is an ideal I of R such that $K = IM$. Note that $I \subseteq [N : M]$, hence $N = IM \subseteq [N : M]M \subseteq N$. So $N = [N : M]M$. An R-module M is called a cancellation module if $IM = JM$ for any ideals I and J of R implies that $I = J$. Also, an R-module M is a weak-cancellation module if $IM = JM$ for any ideals I and J of R implies that $I + Ann(M) = J + Ann(M)$. Finitely generated multiplication modules are weak cancellation, Theorem 3 [2]. Let P be a maximal ideal of R. An R-module M is called P-torsion if for each $m \in M$ there exists $p \in P$ such that $(1 - p)m = 0$. On the other hand, M is called P-cyclic if there exists $x \in M$ and $q \in P$ such that $(1 - q)M \subseteq Rx$. Theorem 1.2 [6] showed that an R-module M is multiplication if and only if for every maximal ideal P of R either M is P-torsion or P-cyclic.

In this paper, we study the number of maximal and minimal prime submodule of multiplication modules. It is shown that if $\Gamma(M)$ is a planar graph, then $|Max(M)| \leq 4$ and $|Min(M)| \leq 4$. Also, we show that, if M is a multiplication R-module with $|Max(M)| \neq 1$ and $\Gamma(M)$ is a planar graph, then $M \cong M_1 \oplus M_2$.

Throughout the paper, $Max(M)$ is a set of the maximal submodules H of M, we use symbol $|Max(M)|$ to denote the number of maximal submodule of M. As a consequence of Theorem 2.5 [6], for any non-zero multiplication R-module $Max(M) \neq \emptyset$. Also, $Min(M)$ is a set of the minimal prime submodules N of M. let $J(R)$ be the Jacobson radical of R and

$$J(M) := \cap_{H \in Max(M)} H.$$

We follow standard notation and terminology from graph theory [5] and module theory [3].

2 Planar intersection graph

This section is concerned with some basic and important results in the theory of planar torsion graphs over a module.

Lemma 2.1 *Let M be an R-module. If $\Gamma(M)$ is a planar graph, then M is both Noetherian and Artinian.*

Proof. Let $N_1 \subset N_2 \subset N_3 \subset N_4 \subset N_5 \subset \ldots$ be a chine of nontrivial proper submodule of M. Then vertices N_i, $1 \leq i \leq 5$ form K_5 as an induced subgraph, which is a contradiction. So every chain of nontrivial proper submodule of M is stationary. Therefore M is both Noetherian and Artinian.

Lemma 2.2 *Let M be a multiplication R-module and N be a prime submodule of M. If $\bigcap_{i=1}^{n} N_i \subseteq N$, where N_i be a submodule of M, then there is $1 \leq i \leq n$ such that $N_i \subseteq N$.*

Proof. Let $\bigcap_{i=1}^{n} N_i \subseteq N$, where N_i be a submodule of M. Then $[N_1 : M][N_2 : M] \ldots [N_n : M]M \subseteq N$. Since N is a prime submodule of M, there is $1 \leq i \leq n$ such that $[N_i : M] \subseteq [N : M]$. Therefore $N_i \subseteq N$.

Lemma 2.3 *Let M be a Q-cyclic R-module for all maximal ideal Q of R. Then $[N : M]$ is a prime ideal of R for any proper submodule N of M if and only if $[N : M]M$ is a prime submodule of M.*

Proof. Let$[N : M]$ be a prime ideal of R. Clearly $[N : M]M$ is a proper submodule of M. Suppose $ax \in [N : M]M$ such that $a \notin [N : M]$, for some $a \in R$ and $x \in M$. Let $k = \{r \in R | rx \in [N : M]M\}$. If $k \neq R$, then there is a maximal ideal Q of R such that $k \subseteq Q$. Since M is a Q-cyclic R-module, $(1 - q)M \subseteq Rm$ for some $q \in Q$ and $m \in M$. Hence $(1 - q)ax \in (1 - q)[N : M]M \subseteq [N : M]m$. So $(1 - q)x = sm$ and $(1 - q)ax = \alpha m$ for some $s \in R$ and $\alpha \in [N : M]$. Thus $(\alpha - sa)m = 0$. It is clear that $(1 -$

$q)ann(m) \subseteq Ann(M)$. Therefore $(1-q)(\alpha - sa) \in Ann(M) \subseteq [N : M]$. Then $(1-q)sa \in [N : M]$. Hence $(1-q) \in k \subseteq Q$, which is a contradiction. This contradiction implies that $k = R$ and so $x \in [N : M]M$. Therefore $[N : M]M$ is a prime submodule of M.

Conversely, let N be a prime submodule of M. Thus $[N : M]$ is a proper ideal of R. Suppose $st \in [N : M]$. So $sM \subseteq N$ or $tM \subseteq N$. Therefore $[N : M]$ is a prime ideal of R.

Theorem 2.1 *Let M be a Q-cyclic R-module for all maximal ideal Q of R. If $\Gamma(M)$ is a planar graph, then $|Min(M)| \leq 3$.*

Proof. Let $\Gamma(M)$ be a planar graph. Suppose $|Min(M)| \geq 4$ and $N_1, N_2 \dots N_4$ be distinct minimal submodules of M, such that $N_1 \cap N_2 \cap N_3 = 0$. Then $[N_1 : M][N_2 : M][N_3 : M]M \subseteq N_4$. Hence $[N_1 : M][N_2 : M][N_3 : M] \subseteq [N_4 : M]$. It is clear that $[N_4 : M]$ is a prime ideal of R. So $[N_i : M]M \subseteq [N_4 : M]M \subseteq N_4$, for some $1 \leq i \leq 3$. By Lemma 2.3, $[N : M]M$ is a prime submodule of M. Also, since N is a minimal prime submodule of M, $[N : M]M = N$. Therefore $N_i = N_4$ for some $1 \leq i \leq 3$, which is a contradiction. Hence $N_1 \cap N_2 \cap N_3 \neq 0$. Therefore, vertices $N_1 \cap N_2$, $N_1 \cap N_3$, $N_2 \cap N_3$, N_1, N_2 and N_3 form K_6 as an induced subgraph, which is a contradiction. Consequently $|Min(M)| \leq 3$.

Corollary 2.1 *Let M be a multiplication R-module. If $\Gamma(M)$ is a planar graph, then $\bigcap_{N \in Min(M)} N = 0$.*

Proposition 2.1 *Let M be a multiplication R-module. If $\Gamma(M)$ is a planar graph, then $1 \leq |Max(M)| \leq 3$.*

Proof. Let $\Gamma(M)$ be a planar graph. Suppose $|Max(M)| \geq 4$ and $H_1, H_2, \dots H_4$ be distinct maximal submodules of M, such that $H_1 \cap H_2 \cap H_3 = 0$. Then $H_1 \cap H_2 \cap H_3 \subseteq H_4$. Sine every maximal submodule of multiplication modules is prime, by Lemma 2.2, $H_i \subseteq H_4$, for some $1 \leq i \leq 3$. But H_i is a maximal submodule of M implies that $H_i = H_4$ for some $1 \leq i \leq 3$, which is a contradiction, hence $H_1 \cap H_2 \cap H_3 \neq 0$. Therefore, vertices $H_1 \cap H_2$, $H_1 \cap H_3$, $H_2 \cap H_3$, H_1, H_2 and H_3 form K_6 as an induced subgraph, which is a contradiction. Consequently $1 \leq |Max(M)| \leq 3$.

Corollary 2.2 *Let M be a multiplication R-module. If $\Gamma(M)$ is a planar graph, then $J(M) = 0$.*

Proposition 2.2 *Let $M = M_1 \times M_2$ be an R-module. Then $\Gamma(M)$ is planar if and only if $\Gamma(M_1)$ or $\Gamma(M_2)$ is empty and another is null.*

Proof. Let $\Gamma(M)$ be a planar graph. Suppose that $\Gamma(M_1)$ and $\Gamma(M_2)$ are not empty. So there exist nontrivial proper submodules N_1 of M_1 and N_2 of M_2. Therefore $0 \times N_2$, $0 \times M_2$, $N_1 \times M_2$, $N_1 \times N_2$ and $M_1 \times N_2$ form K_5 as an induced subgraph, which is a contradiction. Hence one of $\Gamma(M_1)$ or $\Gamma(M_2)$ is empty. Let $\Gamma(M_2)$ be empty. Now we show that $\Gamma(M_1)$ is null. If N_1 is a proper nontrivial submodule of M_1 such that it is adjacent to H_1 for some $H_1 \in V(\Gamma(M_1))$, then $N_1 \cap H_1 \neq 0$. So $N_1 \times 0$, $H_1 \times 0$, $M_1 \times 0$, $N_1 \times M_2$ and $H_1 \times M_2$ form K_5 as an induced subgraph, which is a contradiction. This contradiction implies that $\Gamma(M_1)$ is null.

Corollary 2.3 *$\Gamma(M_1 \times M_2 \times M_3)$ is planar if and only if M_i is a simple R_i-module for $i \in \{1, 2, 3\}$.*

Proof. Let $\Gamma(M_1 \times M_2 \times M_3)$ be a planar graph and M_1 not simple. So there exists $0 \neq N < M_1$. Then $N \times M_2 \times M_3$, $0 \times M_2 \times M_3$, $N \times M_2 \times 0$, $M_1 \times M_2 \times 0$ and $M_1 \times 0 \times M_3$ form K_5 as an induced subgraph of $\Gamma(M)$, which is a contradiction. Therefore M_i is a simple R_i-module for $i \in \{1, 2, 3\}$.

Proposition 2.3 *Let M be a multiplication R-module with $|Max(M)| = 3$. If $\Gamma(M)$ is planar, then $M \cong M_1 \oplus M_2$ where M_1 and M_2 are simple.*

Proof. Let $|Max(M)| = 3$ and H_i, $1 \leq i \leq 3$ be distinct maximal submodules of M. By Corollary 2.2, $H_1 \cap H_2 \cap H_3 = 0$. If $H_2 \cap H_3 = 0$, then $H_2 \cap H_3 \subseteq H_1$ and by Lemma 2.2, $H_1 = H_2$ or $H_1 = H_3$, which is a contradiction. Hence $M = H_1 \oplus H_2 \cap H_3$. By Proposition 2.2, one of $\Gamma(H_1)$ or $\Gamma(H_2 \cap H_3)$ is null another is empty. Suppose that $\Gamma(H_1)$ be null. If H_1 is not a simple submodule of M. Then there is a nontrivial submoddule N_1 of H_1 such that N_1 is adjacent to $N_1 + H_2 \cap H_3$. So $\Gamma(H_1)$ is not null, which is a contradiction. Thus H_1 and $H_2 \cap H_3$ are simple.

Lemma 2.4 *Let M be a faithful finitely generated multiplication R-module. Then $J(R)M = J(M)$.*

Proof. Let M be a faithful finitely generated multiplication R-module and H be a maximal submodule of M. By Theorem 3.1 of [6], $hM \neq M$ for all maximal ideal h of M. Also, by Theorem 2.5 of [6], $H = hM$ for some maximal ideal h of M. On the other hand by Theorem 1.6 of [6], $J(M) = \cap_{H \in Max(M)} H = \cap_{h \in Max(R)}(hM) = (\cap_{h \in Max(R)} h)M = J(R)M$

Theorem 2.2 *Let M be a faithful multiplication R-module with $|Max(M)|= 2$. Then $\Gamma(M)$ is a planar graph if and only if $M \cong [H_1 : M]^4M \oplus [H_2 : M]^4M$ such that $\Gamma([H_1 : M]^4M)$ or $\Gamma([H_1 : M]^4M)$ is empty another is null, where H_1, H_2 are maximal submodule of M.*

Proof. Let H_1 and H_2 be distinct maximal submodules of M. Suppose that $[H_1 : M]^4M + [H_2 : M]^4M \neq M$. By Theorem 2.5 of [6], there is a maximal submodule H of M such that $[H_1 : M]^4M + [H_2 : M]^4M \subseteq H$. Since $|Max(M)|= 2$, we have $H = H_1$ or $H = H_2$. It follows that $[H_1 : M]^4M \subseteq H_2$ or $[H_2 : M]^4M \subseteq H_1$. Thus $H_1 = H_2$, which is a contradiction. So $M = [H_1 : M]^4M + [H_2 : M]^4M$. Assume $[H_1 : M]^4M \cap [H_2 : M]^4M \neq 0$. Hence $H_1 \cap H_2 \neq 0$. On the other hand By Theorem 1.6 [6], $[H_1 : M]^iM \cap [H_2 : M]^iM = ([H_1 : M]^i \cap [H_2 : M]^i)M$, for all positive integer i. Since M is a cyclic faithful multiplication module, by Lemma 2.4, we have $J(R)M = J(M)$. Now Nakayama's lemma follows that $([H_1 : M]^4 \cap [H_2 : M]^4)M \subset \ldots \subset ([H_1 : M] \cap [H_2 : M])M \subset H_1$. Hence $\Gamma(M)$ contains an induced subgraph K_5, which is a contradiction. Therefore $[H_1 : M]^4M \cap [H_2 : M]^4M = 0$. Consequently $M \cong [H_1 : M]^4M \oplus [H_2 : M]^4M$ and by Proposition 2.2, the result follows.

Proposition 2.4 *Let M be a multiplication R-module with $|Max(M)|= 1$. If $\Gamma(M)$ is a planar graph, then $|M|\leq 5$ or $[H : M]^5M = 0$ where H is a maximal submodule of M.*

Proof. Suppose M be a faithful multiplication R-module. If $\Gamma(M)$ is a planar graph, then by Lemma 2.1, M is finitely generated and by Lemma 2.4, R is a local ring with unique maximal ideal $[H : M]$. By Nakayama's lemma, we have $[H : M]^iM \neq [H : M]^jM$ for all positive integer $i \neq j$. Since $\Gamma(M)$ is a planar graph, then $[H : M]^5M = 0$. If M is not faithful, then $\Gamma(M)$ is a complete graph. Hence $|M|\leq 5$.

Now we obtain the central results of this section.

Corollary 2.4 *Let M be a multiplication R-module with $|Max(M)|\neq 1$. If $\Gamma(M)$ is a planar graph, then $M \cong M_1 \oplus M_2$.*

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