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Research Article

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Dynamical Control of Computations Using the Family of Optimal Two-point Methods to Solve Nonlinear Equations

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Abstract

One of the considerable discussions for solving the nonlinear equations is to find the optimal iteration, and to use a proper termination criterion which is able to obtain a high accuracy for the numerical solution. In this paper, for a certain class of the family of optimal two-point methods, we propose a new scheme based on the stochastic arithmetic to find the optimal number of iterations in the given iterative solution and obtain the optimal solution with its accuracy. For this purpose, a theorem is proved to illustrate the accuracy of the iterative method and the CESTAC[1][§]method and CADNA[2][¶] library are applied which allows us to estimate the round-off error effect on any computed result. The classical criterion to terminate the iterative procedure is replaced by a criterion independent of the given accuracy (ϵ) such that the best solution is evaluated numerically, which is able to stop the process as soon as a satisfactory informational solution is obtained. Some numerical examples are given to validate the results and show the efficiency and importance of using the stochastic arithmetic in place of the floating-point arithmetic.

Keywords : Stochastic arithmetic; CESTAC method; CADNA library; Two-point methods; Nonlinear equations.

1 Introduction

Any iterative root-finding method, based on the evaluation of a function and its derivatives, makes sense only while absolute values of functions do not exceed the precision limit ϵ of the employed computer arithmetic. The second important limitation concerns the number of iterations, which must be finite. For this reason, before starting any iterative process, it is necessary to define in advance a stopping criterion. Suppose the nonlinear equation

$$f(x) = 0, \tag{1.1}$$

In order to solve Eq. (1.1) by an iterative method, one can use the common strategy to stop the iterations. For a given tolerance $\epsilon > 0$,

$$\begin{array}{l} 1.\ |x_n - x_{n-1}| < \epsilon, \\ 2.\ |f(x_n)| < \epsilon, \end{array} \tag{1.2}$$

where $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} x_n = x.$$

In some situations serious problems in connection to these criteria can appear. First, if ϵ is

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[§1]Controle et Estimation Stochastique des Arrondis de Calculs

[¶2]Control of Accuracy and Debugging for Numerical Application

very small, the inequalities (1.2) will never be satisfied since the rounding error will produce the increase or oscillation of the value on the left-hand side of the inequalities (1.2) before they are fulfilled. Second, we usually do not possess information about the behavior of the sequence of approximations x_n near the root. For this reason, the criteria (1.2) are not always reliable. If the stopping criteria (1.2) are used, then the number of significants digits that are common to corresponding entries of x_n and x cannot be specified. Another problem is to choose the value ϵ. When ϵ is chosen too large, then the iterative process is stopped too soon, and consequently the approximate solution has a poor accuracy. On the contrary, when ϵ is chosen too small, it is possible, due to the numerical instabilities, that many useless iterations are performed without improving the accuracy of the solution [24]. The aim of this paper is to obtain the optimal iteration and optimal solution. Furthermore, the useless iterations are eliminated.
The basic idea of the CESTAC method is to replace the usual floating-point arithmetic with a random arithmetic. Consequently, each result appears as a random variable. This approach leads toward two concepts: stochastic numbers and stochastic arithmetic.
In recent years, CESTAC method used to validate many problems in mathematics and physic such as interpolation polynomials [2], ill-condition functions [3], numerical integration [1, 4], linear algebra [23, 24] and others [5, 6, 7, 8, 9, 10, 25, 26]. In this work, we implement the algorithm of finding the root of Eq. (1.1) based on the CESTAC method by applying the iterative process presented by Petkovic' as a family of two-point methods [22].

This paper is organized as follows. In section 2, a brief description of stochastic round-off analysis, the CESTAC method and the CADNA library are described . In section 3, a theorem is proved in order to show that the significant digits common between x_{n+1} and x_n are almost equal to the significant digits common between x_n and the exact solution x. In section 4, some numerical examples are given which are computed by using the stochastic arithmetic and the CESTAC method.

2 Preliminaries

When the floating-point arithmetic is replaced by the stochastic arithmetic, one can therefore define a new number, called stochastic number. In this section, we present the main definitions and properties of this arithmetic. For more details see[10, 14, 15, 16, 17].

Definition 2.1 *We define the set S of stochastic numbers as the set of Gaussian random variables. We denote an element $X \in S$ by $X = (\mu, \sigma^2)$, where μ is the mean value of X and σ its standard deviation. If $X \in S$ and $X = (\mu, \sigma^2)$, there exists λ_β, depending only on β, such that*

$$P(X \in [\mu - \lambda_\beta \sigma, \mu + \lambda_\beta \sigma]) = 1 - \beta$$

$I_{\beta,x} = [\mu - \lambda_\beta \sigma, \mu + \lambda_\beta \sigma]$ is a confidence interval of μ at $(1-\beta)$. An upper bound to the number of significant digits common to μ and each element of $I_{\beta,x}$ is

$$C_{\beta,x} = log_{10}(\frac{|\mu|}{\lambda_{\beta,\sigma}}).$$

The following definition is the modelling of the concept of informatical zero proposed in [25]:

Definition 2.2 *$X \in S$ is a stochastic zero if and only if*

$$C_{\beta,X} \leq 0 \quad or \quad X = (0, 0).$$

The stochastic arithmetic can be used in scientific codes to serve

(i) during the run of a scientific code, to estimate the accuracy of an numerical result, to detect the numerical instabilities, and to check the branching.

(ii) to eliminate the programming expedients that are absolutely unfounded, such as those used, for example, in the termination criteria of iterative methods, and replace them by criteria that directly reflect the mathematical condition that must be satisfied at the solution.

The aim of the CESTAC method [25, 27, 28, 29, 30], based on this probabilistic approach, is to estimate the effect of propagation of the round-off errors on every computed result obtained with the floating point arithmetic. It consists in making the round-off errors propagate in different ways in order to distinguish between a stable part of mantissa (considered as the significant one) and an unstable part (nonsignificant). The different propagations are obtained by changing randomly the last bit of the mantisa of each intermediate computed result. In this way, a random arithmetic is generated. Then, by running the program several times in parallel, a sample of the different values for each intermediate result is obtained. The mean value defines the computed value and Student's test estimates its accuracy [7]. It has been proved [8, 9] that, under certain regularity conditions, every computed result R obtained with the CESTAC method can be modelled by

$R = r + \sum_{i=1}^{n} u_i(d).2^{-p}.z_i,$

where $u_i(d)$ are constants depending only on the data d, p is the number of bits of the mantissa and the $z_i^{'}s$ are independent identically distributed and centered random variables.The number of arithmetical operations is n and r is the exact mathematical result.

Consequently, each computed result can be modelled by a Gaussian random variable centered on the exact mathematical result. Its mean value is estimated from a sample using Student's test. So, in practice, the use of the CESTAC method consists in

(i) Running in parallel N times ($N = 2$ *or* 3) the program with this new arithmetic. Consequently, for each result R of any floating-point arithmetic operation, a set of N computed results $R_i, i = 1, 2, \ldots, N$, is obtained.

(ii) Taking the mean value $\bar{R} = \frac{1}{N}\sum_{i=1}^{N} R_i$ of the R_i as the computed result.

(iii) Using the Student distribution to estimate a confidence interval for R, and then compute the number $C_{\bar{R},r}$ of significant digits of $\overline{R}$, defined by

$$C_{\overline{R},r} = log_{10}\left(\frac{\sqrt{N}|\bar{R}|}{s.\tau_\beta}\right) \quad with \quad s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(R_i - \bar{R})^2,$$

where τ_β is the value of Student distribution for $N-1$ degrees of freedom and a probability level $1-\beta$.

A computed result R using the CESTAC method is an informatical zero, denoted by @.0, if and only if $\bar{R} = 0$ *or* $C_{\bar{R},r} \leq 0$.[26].

2.1 The CADNA library

CADNA is a library for programs written in FORTRAN77, FORTRAN90, or in C++ which allows the computation using stochastic arithmetic by automatically implementing the CESTAC method. CADNA is able to estimate the accuracy of the computed results, and to detect numerical instabilities occurring during the run. To use the CADNA library, it suffices to place the instruction USE CADNA at the top of the initial FORTRAN or C++ source code and to replace the declarations of the real type by the stochastic type and to change some statements such as printing statements. During the run, as soon as a numerical anomaly (for example, appearance of the informatical zero in a computation or a criterion) occurs, a message is written in a special file called Cadna-stability-f90.lst. The user must consult this file after the program has run. If it is empty, this means the program has been run without any problem, that it has accordingly been validated, and that the results have been given with their associated accuracy. If it contains messages, the user, using the debugger associated with the compiler, will find the instructions that are the cause of these numerical anomalies, and must reflect in order to correct them if necessary. The program execution time using the CADNA library is only multiplied by a factor 3, which is perfectly acceptable in view of the major advantage offered, i.e., the validation of programs. CADNA is also able to estimate the influence of data errors on the result provided by the computer [6].

3 Main idea

In this section, a family of two-point methods, proposed in [22] is considered. Petkovic' assumed that a real-weight function g and its derivatives g' and $g''$are continuous in the neighborhood of 0, and suggested the following two-step iterative method for solving Eq. (1.1).

3.1 Algorithm

For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f^{'}(x_n) \neq 0,$$

$$x_{n+1} = y_n - g(t_n)\frac{f(y_n)}{f'(x_n)}, \quad t_n = \frac{f(y_n)}{f(x_n)}, \quad n = 1,2,3,\dots \tag{3.3}$$

The weight function g in (3.3) has to be determined so that the two-point method (3.3) attains the optimal order four using only three function evaluations:
$f(x_n), f'(x_n), and\ f(x_n - \frac{f(x_n)}{f'(x_n)})$.
We now study the convergence analysis of Algorithm *3.1*.

Lemma 3.1 *[22] Let $\alpha \in I_f \subset D$ be a simple zero of a real single-valued function $f : D \subset \mathcal{R} \to \mathcal{R}$ possessing a certain number of continuous derivatives in the neighborhood of $\alpha \in I_f$, where I_f is an open interval. Let g be a function satisfying $g(0) = 1, g'(0) = 2$ and $|g''(0)|< \infty$. If x_0 is sufficiently close to α, then the order of convergence of the family of two-step methods (3.3) is four and the error relation*

$$e_{n+1} = [c_2^3(5-g''(0)/2)-c_2c_3]e_n^4+O(e_n^5), \tag{3.4}$$

holds, where
$c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 1,2,3,\dots and \quad e_n = x_n - \alpha.$

We give some special cases of the two-point family (3.3) of methods. This family produces a variety of new methods as well as some existing optimal two-point methods which appear as special cases. The chosen function g in the subsequent examples satisfies the conditions $g(0) = 1, g'(0) = 2$ *and* $|g''(0)|< \infty$, given in Lemma (3.1).
For g given by

$$g(t) = \frac{1+\beta t}{1+(\beta-2)t}, \quad \beta \in \mathcal{R} \tag{3.5}$$

we obtain Kings fourth-order family of two-point methods. Recall that Kings family produces as special cases Ostrowskis method ($\beta = 0$), Kou-Li-Wangs method ($\beta = 1$) and Chuns method ($\beta = 2$).
As mentioned in [1, 2, 11], to correctly quantify the accuracy of a computed result, one must estimate the number of its exact significant digits, The number of significant digits that are common to the computed result and the exact result. Therefore, we need the following definition:

Definition 3.1 *Let a and b be two real numbers, the number of significant digits that are common to a and b, denoted by $C_{a,b}$ can be defined by*

$$for a \neq b, C_{a,b} = \log_{10}|\frac{a+b}{2(a-b)}|, \tag{3.6}$$

for all real number a, $C_{a,a} = +\infty$.

Theorem 3.1 *Let $\alpha \in I_f \subset D$ be a simple zero of a real single-valued function $f : D \subset \mathcal{R} \to \mathcal{R}$ possessing a certain number of continuous derivatives in the neighborhood of $\alpha \in I_f$, where I_f is an open interval. Let g be a function satisfying $g(0) = 1, g'(0) = 2$ and $|g''(0)|< \infty$, and so suppose that algorithm 3.1 is a convergent iterative method to the exact solution α of the nonlinear equation (1.1). If x_0 is sufficiently close to α then*

$$C_{x_{n+1},x_{n+2}} - C_{x_{n+1},\alpha} = -log_{10}|1 - [c_2^3(5-g''(0)/2) - c_2c_3]^4 e_n^{12}| + O(e_n^4). \tag{3.7}$$

Proof. Let $e_n = x_n - \alpha$. According to (3.4), we get

$$x_{n+1} - x_{n+2} = (x_{n+1}) - \alpha - (x_{n+2} - \alpha)$$
$$= [c_2^3(5-g''(0)/2) - c_2c_3]e_n^4 + O(e_n^5)$$
$$-[c_2^3(5-g''(0)/2) - c_2c_3]e_{n+1}^4 + O(e_{n+1}^5)$$
$$= [c_2^3(5-g''(0)/2) - c_2c_3](e_n^4 - e_{n+1}^4)+$$
$$O(e_n^5) = [c_2^3(5-g''(0)/2) - c_2c_3]$$
$$(e_n^4 - ([c_2^3(5-g''(0)/2) - c_2c_3]e_n^4+$$
$$O(e_n^5))^4) + O(e_n^5)$$
$$= [c_2^3(5-g''(0)/2) - c_2c_3]$$
$$(e_n^4 - [c_2^3(5-g''(0)/2) - c_2c_3]^4 e_n^{16}) + O(e_n^5)$$
$$= [c_2^3(5-g''(0)/2) - c_2c_3]e_n^4$$
$$(1 - [c_2^3(5-g''(0)/2) - c_2c_3]^4 e_n^{12}) + O(e_n^5).$$

hence,

$$x_{n+1} - x_{n+2} = [c_2^3(5 - g''(0)/2) - c_2c_3]e_n^4$$

$$(1 - [c_2^3(5 - g''(0)/2) - c_2c_3]^4 e_n^{12}) + O(e_n^5). \tag{3.8}$$

Furthermore, from(3.8)

$$\frac{x_{n+1}+x_{n+2}}{2(x_{n+1}-x_{n+2})} = \frac{x_{n+1}}{x_{n+1}-x_{n+2}} - \frac{1}{2} =$$

$$\frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]}\times$$

$$\frac{1}{e_n^4(1-[c_2^3(5-g''(0)/2)-c_2c_3]^4e_n^{12})+O(e_n^5)} + O(1)$$

$$= \frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4}\times$$

$$\frac{1}{(1-[c_2^3(5-g''(0)/2)-c_2c_3]^4e_n^{12})(1+O(e_n))} + O(1)$$

$$= \frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4}\times$$

$$\frac{1}{(1-[c_2^3(5-g''(0)/2)-c_2c_3]^4e_n^{12})} + O(1).$$

Also,

$$\frac{x_{n+1}+\alpha}{2(x_{n+1}-\alpha)} = \frac{x_{n+1}}{x_{n+1}-\alpha} - \frac{1}{2} =$$

$$\frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4+O(e_n^5)} + O(1)$$

$$= \frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4(1+O(e_n))} + O(1)$$

$$= \frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4} + O(1).$$

Example 3.1 *In this example, the solution of the equation* $f_3(x) = x^2e^{x^2} - sin^2x + 3cosx + 5 = 0$ *is considered. The results are obtained by using Algorithm 4.1 and* $x_0 = -2$. *The optimal value of the root in the optimal iteration with different* β *based on the tables 7-9, is* $x = -0.120764782713091E + 001$.

Therefore, according to definition (2),

$$C_{x_{n+1},x_{n+2}} = log_{10}|\frac{x_{n+1}-x_{n+2}}{2(x_{n+1}-x_{n+2})}|$$

$$= log_{10}(|\frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4}\times$$

$$\frac{1}{(1-[c_2^3(5-g''(0)/2)-c_2c_3]^4e_n^{12})}(1+O(e_n^4))|) \tag{3.9}$$

$$= log_{10}|\frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4}|-$$

$$log_{10}|1 - [c_2^3(5 - g''(0)/2) - c_2c_3]^4e_n^{12}|$$

$$+O(e_n^4).$$

and

$$C_{x_{n+1},\alpha} = log_{10}|\frac{x_{n+1}-r}{2(x_{n+1}-\alpha)}|=$$

$$log_{10}|\frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4} + O(1)|$$

$$= log_{10}|\frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4}(1 + O(e_n^4))|$$

$$= log_{10}|\frac{x_{n+1}}{[c_2^3(5-g''(0)/2)-c_2c_3]e_n^4}|+O(e_n^4). \tag{3.10}$$

Finally, from Eqs. (3.9) and (3.10) the desired relation is obtained.

$$C_{x_{n+1},x_{n+2}} - C_{x_{n+1},\alpha} =$$

$$-log_{10}|1 - [c_2^3(5 - g''(0)/2) - c_2c_3]^4e_n^{12}|$$

$$+O(e_n^4). \quad \square$$

According to (3.7), the iterative method based on the algorithm (3.1) is convergent to the exact solution α of Eq.(1.1), when n increases, then e_n tends to zero, and the term $log_{10}|1-[c_2^3(5-g''(0)/2)-c_2c_3]^4e_n^{12}|$ is neglected. In this case, the significant digits common between x_{n+2} and x_{n+1} are almost equal to the significant digits common between x_{n+1} and the exact value α. We increase n until $x_{n+2} - x_{n+1}$ has not any significant digit.

4 Numerical Examples

In this section, the implementation of the CESTAC method is tested via CADNA library and C++ code on Linux operating system based on the following algorithm by solving some nonlinear equations mentioned in [20, 21, 22].

4.1 Algorithm

1. type (double st) The list of the real variables.

2. call cadna-init(-1)

3. $n = 1$

3. $cin >> x_0$

5. do

6. {

Table 1: Numerical solution of $f_1(x) = 0, with\ \beta = 0$.

n	x_n	$\|x_{n+1} - x_n\|$	$\|x_n - \alpha\|$
1	0.528846724699768E+001	0.71153275300231E+000	0.87971367157795E-001
2	0.537654469146678E+001	0.8807744446910E-001	0.10607731130E-003
3	0.537643861387768E+001	0.10607758909E-003	0.27779E-009
4	0.537643861415547E+001	0.27779E-009	@.0
5	0.537643861415547E+001	@.0	@.0

Table 2: Numerical solution of $f_1(x) = 0, with\ \beta = 1$.

n	x_n	$\|x_{n+1} - x_n\|$	$\|x_n - \alpha\|$
1	0.527099763131590E+001	0.72900236868409E+000	0.10544098283957E+000
2	0.537701709743707E+001	0.1060194661211E+000	0.5784832281597E-003
3	0.537643861298361E+001	0.57848445346E-003	0.117186E-008
4	0.537643861415548E+001	0.11718E-008	0.4E-014
5	0.537643861415547E+001	0.3552713678800E-014	@.0
6	0.537643861415547E+001	@.0	@.0

Table 3: Numerical solution of $f_1(x) = 0, with\ \beta = 2$.

n	x_n	$\|x_{n+1} - x_n\|$	$\|x_n - \alpha\|$
1	0.524407267487363E+001	0.755927325126364E+000	0.13236593928184E+000
2	0.537979058607988E+001	0.13571791120624E+000	0.335197192439995E-002
3	0.537643861897791E+001	0.335196710196594E-002	0.4822434E-008
4	0.537643861415546E+001	0.482244E-008	0.1E-013
5	0.537643861415547E+001	0.1E-013	@.0
6	0.537643861415547E+001	@.0	@.0

Table 4: Numerical solution of $f_2(x) = 0, with\ \beta = 0$.

n	x_n	$\|x_{n+1} - x_n\|$	$\|x_n - \alpha\|$
1	0.310649704076435E+001	0.606497040764350E+000	0.1013310835370E-001
2	0.309636393249552E+001	0.101331082688211E-001	0.84883E-010
3	0.309636393241064E+001	0.84883E-010	@.0
4	0.309636393241064E+001	@.0	@.0

Table 5: Numerical solution of $f_2(x) = 0, with\ \beta = 1$.

n	x_n	$\|x_{n+1} - x_n\|$	$\|x_n - \alpha\|$
1	0.321814332423935E+001	0.718143324239350E+000	0.12177939182870E+000
2	0.309640862033848E+001	0.12173470390086E+000	0.44687927840E-004
3	0.309636393250528E+001	0.44687833204E-004	0.94635E-010
4	0.309636393241064E+001	0.946358547082581E-010	@.0
5	0.309636393241064E+001	@.0	@.0

7. $y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0$

8. $x_{n+1} = y_n - g(t_n)\frac{f(y_n)}{f'(x_n)},$
$t_n = \frac{f(y_n)}{f(x_n)}, \quad n = 1, 2, 3, \dots$

9. $cout << "Root =", strp(x_{n+1})$

10. $n = n + 1$

11. }

12. while $((x_{n+1} - x_n)! = 0\ )$

Table 6: Numerical solution of $f_2(x) = 0, with\ \beta = 2$.

n	x_n	$\|x_{n+1} - x_n\|$	$\|x_n - \alpha\|$
1	0.312922939028678E+001	0.629229390286787E+000	0.32865457876141E-001
2	0.309636394018446E+001	0.328654501023182E-001	0.7773823E-008
3	0.309636393241064E+001	0.7773823E-008	@.0
4	0.309636393241064E+001	@.0	@.0

Table 7: Numerical solution of $f_3(x) = 0, with\ \beta = 0$.

n	x_n	$\|x_{n+1} - x_n\|$	$\|x_n - \alpha\|$
1	-0.146601672470482E+001	0.53398327529517E+000	0.25836889757390E+000
2	-0.121065373036711E+001	0.25536299433771E+000	0.3005903236197E-002
3	-0.120764782716232E+001	0.300590320478E-002	0.31407E-010
4	-0.120764782713091E+001	0.31408E-010	@.0
5	-0.120764782713091E+001	@.0	@.0

Table 8: Numerical solution of $f_3(x) = 0, with\ \beta = 1$.

n	x_n	$\|x_{n+1} - x_n\|$	$\|x_n - \alpha\|$
1	-0.160806013242408E+001	0.391939867575914E+000	0.400412305293166E+000
2	-0.127827653059560E+001	0.329783601828476E+000	0.70628703464690E-001
3	-0.120779975346507E+001	0.70476777130532E-001	0.151926334157E-003
4	-0.120764782713092E+001	0.15192633415E-003	0.377475828372553E-014
5	-0.120764782713091E+001	0.377475828372553E-014	@.0
6	-0.120764782713091E+001	@.0	@.0

Table 9: Numerical solution of $f_3(x) = 0, with\ \beta = 2$.

n	x_n	$\|x_{n+1} - x_n\|$	$\|x_n - \alpha\|$
1	-0.164394851878018E+001	0.356051481219817E+000	0.436300691649263E+000
2	-0.131999253140248E+001	0.323955987377698E+000	0.112344704271564E+000
3	-0.120911270975562E+001	0.110879821646854E+000	0.1464882624710E-002
4	-0.120764782719474E+001	0.1464882560882E-002	0.63828E-010
5	-0.120764782713091E+001	0.63828E-010	@.0
6	-0.120764782713091E+001	@.0	@.0

13. cadna-end().

The function "Strp" in the output instruction shows only the significant digits of the value. The successive values x_n are computed and at each iteration. When $x_{n+1} - x_n = @.0$, then x_{n+1} and x_n are equal stochastically. The computations of the sequence x_n are stopped when for an index like n_{opt}, the number of common significant digits in the difference between x_{nopt} and x_{n+1} become zero. In this case, one can say, before n_{opt}th iteration, $x_{n+1} - x_n$ has exact significant digits. But, the computations after n_{opt} iteration are useless. In other words, the number of iteration in n_{opt} has been optimized. Also, according to theorem (3.1), the significant digits of the last approximation x_n are in common with the value of the exact solution α. Therefore, x_n is an approximation of α. Let us now present the examples and the results which obtained from the CADNA library. The computations have been performed on a personal computer in double precision.

Example 4.1 *In this example, the solution of the equation* $f_1(x) = x^2 sin^2 x + e^{x cos x sin x} - 18 = 0$ *is considered. The results are obtained by using Algorithm 4.1 and* $x_0 = 6$*, in the stochastic arithmetic. The optimal number of iterations are shown in the Tables 1, 2 and 3 which are 5 for* β*=0 and 6 for* β*=1,2 and the optimal computed* *value at optimal iteration with different β based on the tables, is $x = 0.537643861415547E + 001$.*

Example 4.2 *In this example, the solution of the equation $f_2(x) = sinx - e^{-x} = 0$ is considered. The results are obtained by using Algorithm 4.1 and $x_0 = 2.5$. The optimal value of the root in the optimal iteration with $\beta = 0, 1, 2$ at the optimal number of iteration $n = 4, 5, 4$ respectively based on the tables 4,5 and 6, is $x = 0.309636393241064E + 001$.*

5 Conclusion

In this work, by using the CESTAC method based on the stochastic arithmetic, the family of optimal two-point methods to approximate the root of Eq. (1.1) was applied and the results of the proposed algoeithm was validated step by step. We obtained the optimal number of iterations (n_{opt}) of the two-point methods such that x_n is the best approximation of the exact root. Also, a theorem was proved to show the accuracy of the method and an algorithm based on the CADNA library was persented to determine the implemention of the CESTAC method to solve the given nonlinear equation. This approach can be done on any other iterative scheme to provide a relible way in order to find the optimal solution. Consequently, by using the optimal termination criterion which uses the computational zero, the iterative process is stopped correctly and computaion time is saved, because many useless operations and iterations are not performed.

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