Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 6, No. 2, 2014 Article ID IJIM-00499, 8 pages
Research Article

Application of iterative method for solving fuzzy Bernoulli equation under generalized H-differentiability

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Abstract

In this paper, the Picard method is proposed to solve the Bernoulli equation with fuzzy initial condition under generalized H-differentiability. The existence and uniqueness of the solution and convergence of the proposed method are proved in details. Finally an example shows the accuracy of this method.

Keywords : Bernoulli equation; Fuzzy-valued function; h-difference; Generalized differentiability; Picard method.

1 Introduction

As we know the fuzzy differential equations FDE are one of the important part of the fuzzy analysis theory that play major role in numerical analysis. For example, population models [5] , the golden mean [52], quantum optics and gravity [54], control chaotic systems [50, 61], medicine [11, 26]. Recently, some mathematicians have studied FDE [1, 2, 3, 4, 15, 19, 20, 21, 22, 23, 29, 34, 35, 36, 37, 44, 45, 47, 48, 58, 59, 60, 64, 65, 66, 57, 42, 24, 43, 31, 46, 16, 56, 25, 67, 41, 51, 63, 9]. The fuzzy partial differential equations FPDE are very important in mathematical models of physical, chemical, biological, economics and other fields. Some mathematicians have studied solution of FPDE by numerical methods [62, 8, 27, 39, 53, 68, 12, 13, 28]. The Bernoulli differential equation is named after the Switzerland nobleman Count Jakob Bernoulli (1655-1705). The book of Hoffman [18] contains the fundamental theories of Bernoulli equation, with applications to random processes, optimal control, and diffusion problems. Besides important engineering science applications that today are considered classical, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis, the newer applications include such areas as financial mathematics [14, 55, 40, 6]. In this work, we present the Picard method to solve the Bernoulli equation with fuzzy condition as follows:

$$\widetilde{u}'(t) = \widetilde{Q}(t)\widetilde{u}(t) + \widetilde{R}(t)\widetilde{u}^n(t), \quad 0 \leq t \leq T, T \in \mathbb{R}, \tag{1.1}$$

with fuzzy initial condition:

$$\widetilde{u}(0) = \widetilde{a}_0, \tag{1.2}$$

where $\widetilde{Q}(t)$, $\widetilde{R}(t)$ and $\widetilde{P}(t)$ are fuzzy functions and $\widetilde{a}_0$ is fuzzy constant value.
The structure of this paper is organized as follows: In Section 2, some basic notations and definitions in fuzzy calculus are brought. In Section 3, Eq.(1.1) and Eq.(1.2) are solved by Picard method. The existence and uniqueness of the solution and convergence of the proposed method are proved in Section 4 respectively. Finally, in Section 5, the accuracy of method by solving a nu-

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merical examples are illustrated and a brief conclusion is given in Section 6.

2 Basic concepts

Here basic definitions of a fuzzy number are given as follows, [32, 33, 49, 10, 17]

Definition 2.1 *An arbitrary fuzzy number $\widetilde{u}$ in the parametric form is represented by an ordered pair of functions $(\underline{u}, \overline{u})$ which satisfy the following requirements:*

(i) $\overline{u} : r \to \underline{u}(r) \in \mathbb{R}$ is a bounded left-continuous non-decreasing function over $[0,1]$,
(ii) $\underline{u} : r \to \overline{u}(r) \in \mathbb{R}$ is a bounded left-continuous non-increasing function over $[0,1]$,
(iii) $\underline{u}(r) \leq \overline{u}(r), \quad 0 \leq r \leq 1.$

Definition 2.2 *For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E$, we use the distance (Hausdorff metric) [30]*
$D(u(r), v(r)) = \max\{\sup_{r\in[0,1]}|\underline{u}(r) - \underline{v}(r)|, \sup |\overline{u}(r) - \overline{v}(r)|\}$,
and it is shown [7] that (E , D) is a complete metric space and the following properties are well known:

$$D(\widetilde{u}+\widetilde{w}, \widetilde{v}+\widetilde{w}) = D(\widetilde{u}, \widetilde{v}), \forall\ \widetilde{u}, \widetilde{v} \in E,$$
$$D(k\widetilde{u}, k\widetilde{v}) = \mid k \mid D(\widetilde{u}, \widetilde{v}),$$
$$\forall\ k \in \mathbb{R}, \widetilde{u}, \widetilde{v} \in E,$$
$$D(\widetilde{u}+\widetilde{v}, \widetilde{w}+\widetilde{e}) \leq D(\widetilde{u}, \widetilde{w}) + D(\widetilde{v}, \widetilde{e}),$$
$$\forall\ \widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{e} \in E.$$

Definition 2.3 *Consider x, $y \in E$. If there exists $z \in E$ such that $x = y + z$ then z is called the H- difference of x and y, and is denoted by $x \ominus y$. [19]*

Proposition 2.1 *If $f : (a, b) \to E$ is a continuous fuzzy-valued function then $g(x) = \int_a^x f(t)\ dt$ is differentiable, with derivative $g'(x) = f(x)$ [19].*

Definition 2.4 *(see [19]) Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. We say that f is generalized differentiable at x_0 (Bede-Gal differentiability), if there exists an element $f'(x_0) \in E$, such that:*

i) for all $h > 0$ sufficiently small, $\exists f(x_0+h) \ominus f(x_0), \exists f(x_0) \ominus f(x_0-h)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0-h)}{h}$$
$$= f'(x_0)$$

or

ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h), \exists f(x_0-h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0+h)}{-h} = \lim_{h\to 0} \frac{f(x_0-h) \ominus f(x_0)}{-h}$$
$$= f'(x_0)$$

or

iii) for all $h > 0$ sufficiently small, $\exists f(x_0+h) \ominus f(x_0), \exists f(x_0-h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0-h) \ominus f(x_0)}{-h}$$
$$= f'(x_0)$$

or

iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h), \exists f(x_0) \ominus f(x_0-h)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0+h)}{-h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0-h)}{h}$$
$$= f'(x_0)$$

Definition 2.5 *Let $f : (a, b) \to E$. We say f is (i)-differentiable on (a, b) if f is differentiable in the sense (i) of Definition (2.4) and similarly for (ii), (iii) and (iv) differentiability.*

Definition 2.6 *A triangular fuzzy number is defined as a fuzzy set in E, that is specified by an ordered triple $u = (a, b, c) \in R^3$ with $a \leq b \leq c$ such that $u(r) = [\underline{u}(r), \overline{u}(r)]$ are the endpoints of r-level sets for all $r \in [0,1]$, where $\underline{u}(r) = a+(b-a)r$ and $\overline{u}(r) = c-(c-b)r$. Here, $\underline{u}(0) = a, \overline{u}(0) = c, \underline{u}(1) = \overline{u}(1) = b$, which is denoted by $u(1)$. The set of triangular fuzzy numbers will be denoted by E.*

Definition 2.7 *(see [29]) The mapping $f : T \to E$ for some interval T is called a fuzzy process. Therefore, its r-level set can be written as follows:*

$$f(t)(r) = [\underline{f}(t, r), \overline{f}(t, r)],$$
$$t \in T, \quad r \in [0, 1].$$

Definition 2.8 *(see [29]) Let $f : T \to E$ be Hukuhara differentiable and denote $f(t)(r) = [\underline{f}(t,r), \overline{f}(t,r)]$. Then, the boundary function $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable (or Seikkala differentiable) and*

$$(f'(t))(r) = [\underline{f}'(t,r), \overline{f}'(t,r)], \quad t \in T, \quad r \in [0,1].$$

If f is (ii)-differentiable then

$$f'(t)(r) = [\overline{f}'(t,r), \underline{f}'(t,r)], \quad t \in T, \quad r \in [0,1].$$

3 Description of the method

In this section we are going to solve the Bernoulli equation with fuzzy initial condition under generalized H-differentiability.

To obtain the approximation solution of Eq.(1.1) , based on Definition (2.4) we have two cases as follows:
Case (1):
$\widetilde{u}'$ is (i)-differentiable, in this case we have,

$$\widetilde{u}(t) = \widetilde{a}_0 + \int_0^t \widetilde{Q}(s)\widetilde{u}(s)\, ds + \int_0^t \widetilde{R}(s)\widetilde{u}^n(s)\, ds. \tag{3.3}$$

Case (2):
$\widetilde{u}'$ is (ii)-differentiable, in this case we have,

$$\widetilde{u}(t) = \widetilde{a}_0 \ominus (-1)[\int_0^t \widetilde{Q}(s)\widetilde{u}(s)\, ds + \int_0^t \widetilde{R}(s)\widetilde{u}^n(s)\, ds]. \tag{3.4}$$

Now, we can write successive iterations (by using Picard method) as follows:

Case (1):

$$\begin{aligned} &\widetilde{u}_0(t) = \widetilde{a}_0, \\ &\widetilde{u}_{n+1}(t) = \widetilde{a}_0 + \int_0^t \widetilde{Q}(s)\widetilde{u}_n(s)\, ds + \int_0^t \widetilde{R}(s)\widetilde{u}_n^n(s)\, ds, \quad n \geq 0. \end{aligned} \tag{3.5}$$

Case (2):

$$\begin{aligned} &\widetilde{u}_0(t) = \widetilde{a}_0, \\ &\widetilde{u}_{n+1}(t) = \widetilde{a}_0 \ominus (-1)[\int_0^t \widetilde{Q}(s)\widetilde{u}_n(s)\, ds + \int_0^t \widetilde{R}(s)\widetilde{u}_n^n(s)\, ds], \quad n \geq 0. \end{aligned} \tag{3.6}$$

4 Existence and convergence analysis

In this Section we are going to prove the existence and uniqueness of the solution and convergence of the method by using the following assumptions.

$$D(\widetilde{u}^n(t), \widetilde{u}^{*^n}(t)) \leq MD(\widetilde{u}(t), \widetilde{u}^*(t)).$$

$$D(\widetilde{Q}(t), \widetilde{0}) \leq L_1,$$
$$D(\widetilde{R}(t), \widetilde{0}) \leq L_2.$$

Let,

$$\alpha = T(L_1 + ML_2).$$

Lemma 4.1 *If $\widetilde{u}, \widetilde{v}, \widetilde{w} \in E^n$ and $\lambda \in \mathbb{R}$, then,*
(i) $D(\widetilde{u} \ominus \widetilde{v}, \widetilde{u} \ominus \widetilde{w}) = D(\widetilde{v}, \widetilde{w})$,
(ii) $D(\ominus\lambda\widetilde{u}, \ominus\lambda\widetilde{v}) = \mid \lambda \mid D(\widetilde{u}, \widetilde{v})$.

Proof. (i): By the definition of D, we have,

$$D(\widetilde{u} \ominus \widetilde{v}, \widetilde{u} \ominus \widetilde{w}) = \max\{\sup_{r\in[0,1]} \mid \underline{u(r) - v(r)} - \underline{u(r) - w(r)} \mid, \sup_{r\in[0,1]} \mid \overline{u(r) - v(r)} - \overline{u(r) - w(r)} \mid\} =$$

$$\max\{\sup_{r\in[0,1]} \mid (\underline{u}(r) - \underline{v}(r)) - (\underline{u}(r) - \underline{w}(r) \mid,$$

$$\sup_{r\in[0,1]} \mid (\overline{u}(r) - \overline{v}(r)) - (\overline{u}(r) - \overline{w}(r)) \mid\} =$$

$$\max\{\sup_{r\in[0,1]} \mid \underline{w}(r) - \underline{v}(r) \mid,$$

$$\sup_{r\in[0,1]} \mid \overline{w}(r) - \overline{v}(r) \mid\} =$$

$$\max\{\sup_{r\in[0,1]} \mid \underline{v}(r) - \underline{w}(r) \mid,$$

$$\sup_{r\in[0,1]} \mid \overline{v}(r) - \overline{w}(r) \mid\} = D(\widetilde{v}, \widetilde{w}). \qquad \square$$

Proof. (ii):

$$D(\ominus\lambda\widetilde{u}, \ominus\lambda\widetilde{v})$$

$$= \max\{\sup_{r\in[0,1]} \mid \underline{\lambda u(r)} - \underline{\lambda v(r)} \mid, \sup_{r\in[0,1]} \mid \overline{\lambda u(r)} - \overline{\lambda v(r)} \mid\} =$$

$$\max\{\sup_{r\in[0,1]} \mid \underline{\lambda u(r)} - \underline{\lambda v(r)} \mid,$$

$$\sup_{r\in[0,1]} \mid \overline{\lambda u(r)} - \overline{\lambda v(r)} \mid\} = D(\lambda\widetilde{u}, \lambda\widetilde{v}) =$$

$$\mid \lambda \mid D(\widetilde{u}, \widetilde{v}). \qquad \square$$

Table 1: Numerical results for Example 5.1.

t	$(\underline{u}, n = 15, r = 0.4)$	$(\overline{u}, n = 15, r = 0.4)$
0.1	0.3439415	0.3724523
0.2	0.4047319	0.4432258
0.3	0.5164142	0.5572326
0.4	0.6324056	0.6671443
0.5	0.7228925	0.7551642
0.6	0.8337827	0.8534658

Table 2: Numerical results for Example 5.1.

x	$(\underline{u}, n = 13, r = 0.4)$	$(\overline{u}, n = 13, r = 0.4)$
0.1	0.4214517	0.4545184
0.2	0.5332087	0.5568476
0.3	0.6427321	0.6777483
0.4	0.7659784	0.7856309
0.5	0.8208448	0.8427544
0.6	0.8736665	0.8974806

Theorem 4.1 *Let $0 < \alpha < 1$, then Eq.(1.1), hase an unique solution and the solution $\widetilde{u}_n(t)$ obtained from the Eq.(3.4) using Picard method converges to the exact solution of the Eq.(1.1) when $\widetilde{u}'$ is (ii)-differentiable.*

Proof. *Let $\widetilde{u}$ and $\widetilde{u}^*$ be two different solutions of Eq.(1.1) then*

$$D(\widetilde{u}(t), \widetilde{u}^*(t)) = D(\widetilde{a}_0 \ominus (-1)[\int_0^t \widetilde{Q}(s)\widetilde{u}(s)\ ds+$$

$$\int_0^t \widetilde{R}(s)\widetilde{u}^n(s)\ ds], \widetilde{a}_0$$

$$\ominus(-1)[\int_0^t \widetilde{Q}(s)\widetilde{u}^*(s)\ ds+$$

$$\int_0^t \widetilde{R}(s)\widetilde{u}^{*^n}(s)\ ds]) =$$

$$D(\ominus(-1)[\int_0^t \widetilde{Q}(s)\widetilde{u}(s)\ ds + \int_0^t \widetilde{R}(s)\widetilde{u}^n(s)\ ds],$$

$$\ominus(-1)[\int_0^t \widetilde{Q}(s)\widetilde{u}^*(s)\ ds+$$

$$\int_0^t \widetilde{R}(s)\widetilde{u}^{*^n}(s)\ ds])$$

$$\leq TL_1(D(\widetilde{u}(t), \widetilde{u}^*(t)))+$$

$$TL_2M(D(\widetilde{u}(x,t), \widetilde{u}^*(x,t)))$$

$$= \alpha D(\widetilde{u}(t), \widetilde{u}^*(t)).$$

From which we get $(1-\alpha)D(\widetilde{u}(t), \widetilde{u}^*(t)) \leq 0$. Since $0 < \alpha < 1$, then $D(\widetilde{u}(t), \widetilde{u}^*(t)) = 0$. Implies $\widetilde{u}(t) = \widetilde{u}^*(t)$.

Also, we have

$$D(\widetilde{u}_{n+1}(t), \widetilde{u}(t)) \leq \alpha D(\widetilde{u}_n, \widetilde{u}).$$

Since, $0 < \alpha < 1$, then $D(\widetilde{u}_n(t), \widetilde{u}(t)) \to 0$ as $n \to \infty$. Therefore, $\widetilde{u}_n(t) \to \widetilde{u}(t)$. □

Remark 4.1 *The proof of other case is similar to the previous theorem.*

5 Numerical examples

In this section, we solve the fuzzy Bernoulli equation by using the Picard method. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm :

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relations (5) and (6).

Step 3. If $D(\widetilde{u}_{n+1}(t), \widetilde{u}_n(t)) < \varepsilon$ then go to step 4,

else $n \leftarrow n+1$ and go to step 2.

Step 4. Print $\widetilde{u}_n(t)$ as the approximate of the exact solution.

Example 5.1 *Consider the Riccati equation as follows:*

$$\widetilde{u}' = \widetilde{u}(t) + \widetilde{u}^3(t). \quad (5.7)$$

With fuzzy initial condition:

$$\widetilde{u}(0) = (0.15, 0.23, 0.36). \quad (5.8)$$

$$\varepsilon = 10^{-4},$$

$$\alpha = 0.568792.$$

Table 1 shows that the approximation solution of the fuzzy Bernoulli equation is convergent with 15 iterations by using the Picard method when u' is (i)-differentiable.

$$\alpha = 0.625643.$$

Table 2 shows that the approximation solution of the fuzzy Bernoulli equation is convergent with 13 iterations by using the Picard method when u' is (ii)-differentiable.

Acknowledgments

The author would like to express her sincere appreciation to the Department of Mathematics, Islamic Azad University, Qazvin Branch for their cooperation.

6 Conclusion

The fuzzy Bernoulli Equation is the most important and applicable equation in financial mathematics. Since solution of this equation under generalized H-differentiability has not been reviewed before. In this paper we are going to achieve the approximated solutions of this equation by using iterative method with the new and important definition in the fuzzy field, so proving the existence, solution singularity and method convergence on this equation is one of the new jobs that has been done on this paper. Up to now there was only reviewed of first definition of generalized H-differentiability on fuzzy Bernoulli equation and not in general case.

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