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Research Article

Fixed point theorem for non-self mappings and its applications in the modular space

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Abstract

In this paper, based on [A. Razani, V. Rakočević and Z. Goodarzi, Nonself mappings in modular spaces and common fixed point theorems, Cent. Eur. J. Math. 2 (2010) 357-366.] a fixed point theorem for non-self contraction mapping T in the modular space X_ρ is presented. Moreover, we study a new version of Krasnoseleskii's fixed point theorem for $S+T$, where T is a continuous non-self contraction mapping and S is continuous mapping such that $S(C)$ resides in a compact subset of X_ρ, where C is a nonempty and complete subset of X_ρ, also C is not bounded. Our result extends and improves the result announced by Hajji and Hanebally [A. Hajji and E. Hanebaly, Fixed point theorem and its application to perturbed integral equations in modular function spaces, Electron. J. Differ. Equ. 2005 (2005) 1-11]. As an application, the existence of a solution of a nonlinear integral equation on $C(I, L^\varphi)$ is presented, where $C(I, L^\varphi)$ denotes the space of all continuous function from I to L^φ, L^φ is the Musielak-Orlicz space and $I = [0, b] \subset \mathbb{R}$. In addition, the concept of quasi contraction non-self mapping in modular space is introduced. Then the existence of a fixed point of these kinds of mapping without Δ_2-condition is proved. Finally, a three step iterative sequence for non-self mapping is introduced and the strong convergence of this iterative sequence is studied. Our theorem improves and generalized recent know results in the literature.

Keywords : Modular space; Non-self mappings; Quasi contraction; Krasnoseleskii's fixed point theorem; Integral equation.

1 Introduction

The notion of modular space, as a generalization of a metric space, was introduced by Nakano [11] in 1950 in connection with the theory of order spaces and generalized by Musielak and Orlicz [10] in 1959. These spaces were developed following the successful theory of Orlicz spaces, which replaces the particular, integral form of the nonlinear functional, which controls the growth of members of the space, by an abstractly given functional with some good properties(see [8]). In 1974, Ćirić [2], introduced quasi-contraction mappings and proved the existence of fixed point for these kind of mappings in complete metric spaces. Fixed point theorems in modular spaces, generalizing the classical Banach fixed point theorem in metric spaces, have been studied extensively.
In 2005, Hajji et al. [4] presented a modular version of Krasnosel'skii fixed point theorem, for a ρ-contraction and a ρ-completely continuous

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mapping.
In 2010, Razani et al. [13], study a common fixed point theorem for non-self contraction mapping in the modular space.
In 2014 Azizi et al. [1], study the modular version of Krasnosel'skii fixed point theorem for $S+T$, where T is a ρ-expansive mapping and the image of B under S i.e. $S(B)$ resides in a compact subset of X_ρ, where B is a subset of X_ρ.
In 2015 Moradi et al. [9], introduce a new nonlinear iterative algorithms in the modular spaces. They study the convergence of generated iterative sequences by this algorithms. Moreover, they introduce a new double sequence iteration and prove these sequences convergence strongly to a fixed point of ρ-quasi contraction mapping.

Here, in Sections 2 and 3 based on [12] and [13], some fixed point theorems for contraction and quasi contraction non-self mappings in modular spaces are proved. Using the same argument as [1] and [4], the existence of solution of a nonlinear integral equation is studied and an example is presented to guarantee our results, in Section 4. Finally in Section 5, according to [9] we study an iterative algorithm for non-self mapping in modular space.

Due to this, we recall the following definitions and theorems (see [1], [5], [6], [8], [12] and [13]).

Definition 1.1 *Let X be an arbitrary vector space over $K = (\mathbb{R}$ or $\mathbb{C})$.*
a) A functional $\rho : X \longrightarrow [0,\infty]$ is called modular if:
i) $\rho(x) = 0$ iff $x = 0$.
ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha|= 1$, for all $x \in X$.
iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all $x, y \in X$.
If iii) is replaced by:
iii)′ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $\alpha,\ \beta \geq 0$, $\alpha + \beta = 1$, for all $x, y \in X$, then the modular ρ is called a convex modular.
b) A modular ρ defines a corresponding modular space, i.e. the space X_ρ given by:

$$X_\rho = \{x \in X\ |\rho(\alpha x) \to 0\ as\ \alpha \to 0\}.$$

c) If ρ is convex modular, the modular X_ρ can be equipped with a norm called the Luxemburg norm defined by:

$$\|x\|_\rho = \inf\{\alpha > 0; \quad \rho(\frac{x}{\alpha}) \leq 1\}.$$

Remark 1.1 *Note that ρ is an increasing function. Suppose that $0 < a < b$, then property (iii) with $y = 0$, shows that $\rho(ax) = \rho(\frac{a}{b}(bx)) \leq \rho(bx)$.*

Definition 1.2 *Let X_ρ be a modular space. Then we have the following*
a) A sequence $(x_n)_{n\in}$ in X_ρ is said to be:
i) ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$.
ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
b) X_ρ is ρ-complete if every ρ-Cauchy sequence is ρ-convergent.
c) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n)_{n\in} \subset B$ and $x_n \to x$ then $x \in B$.
d) A subset $B \subset X_\rho$ is called ρ-bounded if $\delta_\rho(B) = \sup \rho(x - y) < \infty$ for all $x, y \in B$, where $\delta_\rho(B)$ is called the ρ-diameter of B.
e) ρ has the Fatou property if:

$$\rho(x - y) \leq \liminf \rho(x_n - y_n),$$

whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
f) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

Definition 1.3 A function $f : X_\rho \to X_\rho$ is called ρ-continuous, if $\rho(x_n - x) \to 0$, then $\rho(f(x_n) - f(x)) \to 0$.

Using the same argument as in [13], we have the following definition.

Definition 1.4 Assume X_ρ is a modular space. For $x, y \in X_\rho$, we write

$$seg[x, y] = \{z \in X_\rho : \quad z = (1 - t)x + ty, \quad 0 \leq t \leq 1\}. \tag{1.1}$$

Now, we recall a remark as follow:

Remark 1.2 *If $u \in X_\rho$ and $z_0 = (1-t_0)x+t_0y \in seg[x, y]$, $0 \leq t_0 \leq 1$, then*

$$\begin{aligned} &\rho(u - z_0) \\ = \ &\rho((1 - t_0)u + t_0u - (1 - t_0)x - t_0y) \\ \leq \ &(1 - t_0)\rho(u - x) + t_0\rho(u - y) \\ \leq \ &\max\{\rho(u - x), \rho(u - y)\}. \end{aligned}$$

2 Fixed point theorem for non-self mappings

In this section, by using the same argument as in [12] and [13], some fixed point theorems for non-self mappings in modular spaces are proved. In order to do this, a lemma and a remark are presented as follows:

Lemma 2.1 *Let X_ρ be a modular space, and C a nonempty ρ-closed subset of X_ρ and ∂C the ρ-boundary of C. If $x \in C$ and y is not in C, there is $z \in \partial C$ such that*

$$z \in \partial C \cap seg[x, y].$$

Proof. Let us define

$$t_0 = \sup\{t : z = (1-t)x + ty \in C, 0 \le t \le 1\}.$$

Now, $z = (1-t_0)x + t_0 y \in \partial C \cap seg[x, y]$.

Remark 2.1 *Let X_ρ be a modular space, C a nonempty and ρ-closed subset of X_ρ and ∂C the ρ-boundary of C. Suppose $T : C \to X_\rho$ and $T(\partial C) \subset C$. If $x \equiv x_1 \in C$, we construct a sequence $\{x_n\}$ of points in C as follows: Suppose $T(x_1)$ is given. If $T(x_1)$ is in C, there is $x_2 \in C$ such that $x_2 = T(x_1)$. If $T(x_1)$ is not in C, by Lemma 2.1 there is $x_2 \in \partial C$ such that $x_2 \in \partial C \cap seg[x_1, T(x_1)]$. Hence, by induction, one can construct a sequence $\{x_n\}$ of points in C as follows. If $T(x_n) \in C$, then $x_{n+1} = T(x_n)$ for some $x_{n+1} \in C$; if $T(x_n)$ is not in C, then by Lemma 2.1, $x_{n+1} \in \partial C$ such that*

$$x_{n+1} \in \partial C \cap seg[x_n, T(x_n)].$$

We call a sequence $\{x_n\}$, T-chain of x, and set $C(x) = \{x_n \cup T(x_n)\}$.

Theorem 2.1 *(Schauder's fixed point theorem [3]) Let $(X, \|.\|)$ be a Banach space and $K \subset X$ is a nonempty, closed and convex subset. Suppose the mapping $S : K \longrightarrow K$ is continuous and $S(K)$ resides in a compact subset of X. Then S has at least one fixed point in K.*

Theorem 2.2 *Let X_ρ be a modular space where ρ is convex and satisfy the Δ_2-condition and the Fatou property. Let C be a nonempty and ρ-complete subset of X_ρ, $T : C \to X_\rho$ and $T(\partial C) \subset C$. Suppose T satisfy the following condition: There exists $c, \lambda \in \mathbb{R}^+$ such that $c > 1$ and $\lambda \in (0, 1)$ also for every $x, y \in C$,*

$$\rho(c(Tx - Ty)) \le \lambda\rho(x - y).$$

Let x_1, x_n and Tx_n be as in the Remark 2.1. Then there exists a unique fixed point of T.

Proof. $\{T(x_n)\}$ and $\{x_n\}$ are ρ-Cauchy sequences. First, we prove

$$x_{n+1} \ne T(x_n) \Rightarrow x_n = T(x_{n-1}). \tag{2.2}$$

Suppose the contrary $x_n \ne T(x_{n-1})$. Then $x_n \in \partial C$. Since $T(\partial C) \subset C$ then $T(x_n) \in C$, hence $x_{n+1} = T(x_n)$. Thus (2.2) is proved.
Now, we prove $\{x_n\}$ and $\{T(x_n)\}$ are ρ-Cauchy sequences.

Case 1. Let for all $n \in \mathbb{N}$, $T(x_n) \in C$ then $x_{n+1} = T(x_n)$ and

$$\begin{aligned} \rho(c(T(x_n) - T(x_{n-1}))) \le & \ \lambda\rho(x_n - x_{n-1}) \\ \le & \ \cdots \\ \le & \ \lambda^{n-1}\rho(x - Tx), \end{aligned}$$

Case 2. If $x_{n+1} \ne T(x_n)$, then $x_n = T(x_{n-1})$ and

$$x_{n+1} \in seg[x_n, T(x_n)] = seg[T(x_{n-1}), T(x_n)],$$

therefore

$$\begin{aligned} & \rho(c(x_{n+1} - x_n)) \\ = & \ \rho(c(x_{n+1} - T(x_{n-1}))) \\ \le & \ \max\{0, \rho(c(T(x_n) - T(x_{n-1})))\} \\ \le & \ \lambda\rho(x_n - x_{n-1}). \end{aligned}$$

Therefore by *Case 1* and *Case 2*, we have,

$$\begin{aligned} \rho(c(x_{n+1} - x_n)) \le & \ \rho(c(T(x_n) - T(x_{n-1}))) \\ \le & \ \lambda^{n-1}\rho(x - Tx), \end{aligned}$$

since $\lambda \in (0, 1)$ then $\rho(c(x_{n+1} - x_n)) \longrightarrow 0$. Also by Δ_2- condition $\rho(x_{n+1} - x_n) \longrightarrow 0$.
Again $\{x_n\}$ and $\{T(x_n)\}$ are ρ-Cauchy sequences. If not, then there exists an $\varepsilon > 0$ and two sequences of integers $\{n(s)\}$, $\{m(s)\}$, with, $n(s) > m(s) \ge s$, such that

$$\rho(T(x_{n(s)}) - T(x_{m(s)})) \ge \varepsilon \quad for \quad s = 1, 2, \cdots. \tag{2.3}$$

We can assume that

$$\rho(T(x_{n(s)-1}) - T(x_{m(s)})) < \varepsilon. \tag{2.4}$$

In order to show this, suppose $n(s)$ is the smallest number exceeding $m(s)$ for which (2.3) holds and

$$\textstyle\sum_s = \{n \in \mathbb{N} | \exists m(s) \in \mathbb{N}; \rho(T(x_n) - T(x_{m(s)})) \ge \varepsilon \text{ and } n > m(s) \ge s\}.$$

Obviously $\sum_s \ne \phi$ and since $\sum_s \subset \mathbb{N}$, then by well ordering principle, the minimum element of

$\sum_s$ is denoted by $n(s)$, and clearly (2.4) holds. Now

$$\begin{array}{rl} & \rho(c(T(x_{n(s)}) - T(x_{m(s)}))) \\ \leq & \lambda\rho(T(x_{n(s)-1}) - T(x_{m(s)-1})), \end{array}$$

moreover

$$\begin{array}{rl} & \rho(T(x_{n(s)-1}) - T(x_{m(s)-1})) \\ \leq & \rho(c(T(x_{n(s)-1}) - T(x_{m(s)}))) \\ & +\rho(\alpha(T(x_{m(s)}) - T(x_{m(s)-1}))), \end{array}$$

where $\alpha \in \mathbb{R}^+$ is the conjugate of c. By using Δ_2-condition

$$\rho(\alpha(T(x_{m(s)}) - T(x_{m(s)-1}))) \longrightarrow 0,$$

therefore

$$\begin{array}{rl} \varepsilon \leq & \rho(c(T(x_{n(s)}) - T(x_{m(s)}))) \\ \leq & \lambda\rho(c(T(x_{n(s)-1}) - T(x_{m(s)-1}))) \\ \leq & \lambda\varepsilon, \end{array}$$

which is a contradiction. Therefore, by Δ_2-condition $\{x_n\}$ and $\{T(x_n)\}$ are ρ-Cauchy sequences.

Since $\{x_n\} \subseteq C$ and C is a ρ-complete subset of X_ρ, then $\lim_{n\longrightarrow\infty} x_n = w \in C$. We show $\lim_{n\longrightarrow\infty} T(x_n) = w$. For each $m \in \mathbb{N}$,

$$\rho(w - T(x_m)) \leq \liminf_m \rho(x_n - T(x_m)).$$

Thus $\lim_{m\longrightarrow\infty} \rho(w - T(x_m)) = 0$, i.e., $\lim_{n\longrightarrow\infty} T(x_n) = w$. Also

$$\begin{array}{rl} \rho(w - Tw) \leq & \liminf_n \rho(T(x_n) - T(w)) \\ \leq & \lambda\rho(x_n - w). \end{array}$$

Since $\lambda < 1$, $\rho(w - Tw) = 0$ or $T(w) = w$. Let z and w are two arbitrary fixed point of T. Then

$$\begin{array}{rl} \rho(c(z-w)) = & \rho(c(Tz - Tw)) \\ \leq & \lambda\rho(z-w) \\ \leq & \lambda\rho(c(z-w)), \end{array}$$

which implies $\rho(c(z-w)) = 0$; therefore $z = w$.

In 2005, Hajji [4] proved a modular version of Krasnoseleskili's fixed point theorem for self mapping $T : C \longrightarrow C$, where C is a convex, closed and bounded subset of X_ρ. We prove a modular version of Krasnoseleskili's fixed point theorem for non-self mapping, where C is not bounded.

Theorem 2.3 *Let X_ρ be a modular space where ρ is convex and satisfy the Δ_2-condition and the Fatou property. Let C be a nonempty and ρ-complete subset of X_ρ. Suppose T and S satisfy the following conditions:*
(I) $T : C \to X_\rho$ and $T(\partial C) \subset C$ also there exist $c, \lambda \in \mathbb{R}^+$ such that $c > 1$ and $\lambda \in (0,1)$, for every $x, y \in C$

$$\rho(c(Tx - Ty)) \leq \lambda\rho(x - y). \tag{2.5}$$

(II) $S : C \to X_\rho$ and $S(\partial C) \subset C$ is a ρ-continuous and $S(C)$ resided in a ρ-compact subset of X_ρ.
(III) $T(C)+S(C) \subset C$ and $T(\partial C)+S(\partial C) \subset C$. Then there exists a point $w \in C$ with $Tw + Sw = w$.

Proof. Let $z \in C$, then the mapping $T + Sz : C \longrightarrow X_\rho$ satisfies the assumptions of Theorem 2.2, therefore the equation $Tx + Sz = x$ has unique solution $x = \Lambda(Sz) \in C$. Then it follows that for any $z \in C$, there exists $x \in C$ such that $(I - T)x = Sz$. Operator $I - T$ is injective, because, if z, w in C, such that $(I-T)z = (I-T)w$, then by inequality (2.5), $z = w$. Therefore $\Lambda(Sw) = (I-T)^{-1}Sw$ for all $w \in C$ there exists. We consider the mapping $\Lambda S : C \longrightarrow C$ by $w \longrightarrow \Lambda(Sw)$. We show ΛS is ρ-continuous. Let $\{x_n\} \subset C$ be ρ-continuous to $x \in C$. Since S is ρ-continuous mapping then $\rho(Sx_n - Sx) \longrightarrow 0$. We consider the sequence defined by $\Lambda(w_n) = (I-T)^{-1}(w_n)$ and $\Lambda(w_n) - T\Lambda(w_n) = w_n$ where $w_n = Sx_n$ and $w = Sx$. Also

$$\begin{array}{rl} & \rho(\Lambda(w_n) - \Lambda(w)) \\ \leq & \rho(\alpha(w_n - w)) + \rho(c(T\Lambda(w_n) - T\Lambda(w))) \\ \leq & \rho(\alpha(w_n - w)) + \lambda\rho(\Lambda(w_n) - \Lambda(w)), \end{array}$$

where $\alpha \in \mathbb{R}^+$ is the conjugate of c. We have,

$$(1-\lambda)\rho(\Lambda(w_n) - \Lambda(w)) \leq \rho(w_n - w),$$

then $\rho(\Lambda(w_n) - \Lambda(w)) \longrightarrow 0$ as $n \longrightarrow \infty$, hence $\Lambda : S(C) \longrightarrow C$ is ρ-continuous mapping. Since S is ρ-continuous mapping then $\Lambda S : C \longrightarrow C$ is also ρ-continuous and by Δ_2-condition ΛS is $\|.\|_\rho$-continuous, by (II), $\Lambda S(C)$ resided in a ρ-compact subset of X_ρ. Then by using Theorem 2.1, there exists a $w \in C$ such that $w = \Lambda(Sw)$ and $Tw + Sw = w$.

3 Quasi contraction non-self mappings

Recently, Khamsi [7] study quasi contraction mapping in modular spaces. Here, we consider quasi contraction non-self mappings in modular space and generalize fixed point theorems of Ćirić [2], and Ume [14] in modular spaces. In this section, based on [13], some fixed point theorems for quasi contraction non-self mappings without Δ_2-condition are proved in modular spaces.

Theorem 3.1 *Let X_ρ be a modular space, where ρ is convex and satisfies the Fatou property. Suppose C is a nonempty ρ-complete subset of X_ρ, $T : C \to X_\rho$ and $T(\partial C) \subset C$, also for every $x, y \in C$, $\rho(Tx - Ty) \leq M_\omega(x, y)$, where*

$$\begin{aligned} M_\omega(x,y) &= \max\{\omega_1[\rho(x-y)], \\ &\quad \omega_2[\rho(x-Tx)], \omega_3[\rho(y-Ty)], \\ &\quad \omega_4[\rho(x-Ty)], \omega_5[\rho(y-Tx)]\}, \end{aligned}$$

and $\omega_i : [0,\infty) \to [0,\infty)$, $i = 1, 2, \cdots, 5$ is a nondecreasing semicontinuous function from the right, such that $\omega_i(r) < r$, for $r > 0$, and $\lim_{r\to\infty}[r - \omega_i(r)] = \infty$.
Let x_1, x_n and Tx_n be as in the Remark 2.1. If $\delta_\rho(C(x)) < \infty$, then x_n and Tx_n are ρ-convergent sequences with the same limit, say $w \in C$. Moreover, if $\rho(w - T(w)) < \infty$, then w is a unique fixed point of T, i.e. $T(w) = w$.

Proof. Let x_1, x_n and Tx_n be as in the Remark 2.1. We prove $\{T(x_n)\}$ and $\{x_n\}$ are ρ-Cauchy sequences.

$$x_{n+1} \neq T(x_n) \Rightarrow x_n = T(x_{n-1}).$$

To prove this, let us consider

$$A_n = (\bigcup_{i=0}^{n-1} x_i)\bigcup(\bigcup_{i=0}^{n-1} T(x_i)),$$

and $a_n = \delta_\rho(A_n)$. We prove

$$a_n = \max\{\rho(x_0 - T(x_i)) : 0 \leq i \leq n-1\}. \quad (3.6)$$

If $a_n = 0$, then $x_0 = T(x_0)$ and x_0 is a fixed point of T. Suppose that $a_n > 0$. To prove (3.6), three cases are considered.

Case 1. Suppose $a_n = \rho(x_i - T(x_j))$ for some $0 \leq i, j \leq n-1$.

(1.I) Now, if $i \geq 1$ and $x_i = T(x_{i-1})$ for some $k \in \{1, 2, \cdots, 5\}$

$$\begin{aligned} a_n &= \rho(x_i - T(x_j)) \\ &= \rho(T(x_{i-1}) - T(x_j)) \\ &\leq M_\omega(a_n) \\ &< a_n, \end{aligned}$$

and this is a contradiction. Hence $i = 0$.
(1.II) If $i \geq 1$ and $x_i \neq T(x_{i-1})$, we have $i \geq 2$ and $x_{i-1} = T(x_{i-2})$. Hence $x_i \in seg[T(x_{i-2}), T(x_{i-1})]$, and for some $k \in \{1, 2, \cdots, 5\}$

$$\begin{aligned} a_n = &\ \rho(x_i - T(x_j)) \\ \leq &\ \max\{\rho(T(x_{i-2}) - T(x_j)), \\ &\quad \rho(T(x_{i-1}) - T(x_j)) \\ \leq &\ \max\{M_\omega(x_{i-2}, x_j), M_\omega(x_{i-1}, x_j)\} \\ \leq &\ \omega_k(a_n) \\ < &\ a_n, \end{aligned}$$

and this is a contradiction.

Case 2. Suppose $a_n = \rho(x_i - x_j)$ for some $0 \leq i, j \leq n-1$.
(2.I) If $x_i = T(x_{i-1})$, then (2.I) reduces to (1.I).
(2.II) If $x_i \neq T(x_{i-1})$ then $x_{i-1} = T(x_{i-2})$ and

$$x_i \in \partial C \cap seg[T(x_{i-2}), T(x_{i-1})],$$

hence

$$\begin{aligned} a_n = &\ \rho(x_i - x_j) \\ \leq &\ \max\{\rho(x_j - T(x_{i-2})), \\ &\quad \rho(x_j - T(x_{i-1}))\}, \end{aligned}$$

and (2.II) reduces to (1.II).

Case 3. If $a_n = \rho(T(x_i) - T(x_j))$ then *Case 3* reduces to (1.I). Thus (3.6) is proved. Let

$$B_n = (\bigcup_{i=n}^{\infty} x_i)\bigcup(\bigcup_{i=n}^{\infty} T(x_n)),$$

and

$$\begin{aligned} b_n = &\ \delta_\rho(B_n) \\ = &\ \sup_{j\leq n} \rho(x_n - T(x_j)), \end{aligned}$$

where $n = 2, 3 \cdots$. Note that b_n is defined, because $\delta_\rho(C(x)) < \infty$. We have, two cases:
If $x_n = T(x_{n-1})$, then for each $j \leq n$ and some $k \in \{1, 2, \cdots, 5\}$,

$$\begin{aligned} b_n = &\ \rho(x_n - T(x_j)) \\ = &\ \rho(T(x_{n-1}) - T(x_j)) \\ \leq &\ \omega_k(b_{n-1}). \end{aligned}$$

If $x_n \neq T(x_{n-1})$, then for each $n \geq 1$ and $j \geq n$, for some $k \in \{1, 2, \cdots, 5\}$,

$$\begin{aligned} b_n = \ & \rho(x_n - T(x_j)) \\ \leq \ & \max\{\rho(T(x_{n-2}) - T(x_j)), \\ & \rho(T(x_{n-1}) - T(x_j))\} \\ \leq \ & \omega_k(b_{n-2}). \end{aligned}$$

Thus, there exists a subsequence β_n of b_n and $k \in \{1, 2, \cdots, 5\}$ such that for each n

$$\beta_n \leq \omega_k(b_{n-2}), n = 2, 3, \cdots.$$

Since b_n is a positive and decreasing sequence, then $\lim_n b_n = \lim_n \beta_n = b$. We prove $b = 0$, otherwise $b \leq \omega_k(b) < b$ and this is a contradiction. Then $\{x_n\}$ and $\{T(x_n)\}$ are two ρ-Cauchy sequences. Since $\{x_n\} \subset C$ and C is a ρ-complete subset of X_ρ, we conclude $\lim_n x_n = w \in C$.
Now, we prove $\lim_n T(x_n) = w$. For each $m \in \mathbb{N}$,

$$\begin{aligned} \rho(w - T(x_m)) \leq \ & \liminf_n \rho(x_n - T(x_m)) \\ \leq \ & b(m). \end{aligned}$$

Thus $\lim_m \rho(w - T(x_m)) = 0$, i.e., $\lim T(x_n) = w$. We prove $Tw = w$. If not, i.e., $Tw \neq w$ then

$$\begin{aligned} & \rho(Tw - T(x_n)) \\ \leq \ & \max\{\omega_1[\rho(w - x_n)], \omega_2[\rho(w - Tw)], \\ & \omega_3[\rho(x_n - T(x_n))], \omega_4[\rho(w - T(x_n))], \\ & \omega_5[\rho(x_n - T(w))]\}. \end{aligned}$$

By taking limit when $n \to \infty$,

$$\begin{aligned} & \rho(Tw - w) \\ \leq \ & \max\{\omega_1[\rho(w - w)], \omega_2[\rho(w - Tw)], \\ & \omega_3[\rho(w - w)], \omega_4[\rho(w - w)], \\ & \omega_5[\rho(w - T(w))]\}. \end{aligned}$$

Hence, for some $k \in \{1, 2, \cdots, 5\}$,

$$\begin{aligned} \rho(Tw - w) \leq \ & \omega_k(\rho(Tw - w)) \\ < \ & \rho(Tw - w), \end{aligned}$$

and this is a contradiction. Hence $Tw = w$. In order to prove the uniqueness, suppose w^* is a fixed point of T in C such that $w \neq w^*$, then

$$\begin{aligned} & \rho(w^* - w) \\ = \ & \rho(Tw^* - Tw) \\ \leq \ & \max\{\omega_1[\rho(w - w)], \omega_2[\rho(w^* - Tw^*)], \\ & \omega_3[\rho(w - Tw)], \omega_4[\rho(w^* - Tw)], \\ & \omega_5[\rho(w - Tw^*)]\}. \end{aligned}$$

Then

$$\begin{aligned} & \rho(w^* - w) \\ \leq \ & \max\{\omega_1[\rho(w - w)], \omega_2[\rho(w^* - w^*)], \\ & \omega_3[\rho(w - w)], \omega_4[\rho(w^* - w)], \\ & \omega_5[\rho(w - w^\star)]\}. \end{aligned}$$

Hence, for some $k \in \{1, 2, \cdots, 5\}$

$$\rho(w^* - w) \leq \omega_k(\rho(w - w^*)) < \rho(w - w^*),$$

and this is a contradiction, thus $w = w^*$.

Corollary 3.1 *Let X_ρ be a modular space, where ρ is convex and satisfies the Fatou property. Let C be a nonempty and ρ-complete subset of X_ρ, $T : C \to X_\rho$ and $T(\partial C) \subset C$. Suppose T satisfies the following condition:*
There exists a constant $\lambda \in (0, 1)$ such that for every $x, y \in C$

$$\rho(Tx - Ty) \leq \lambda M(x, y),$$

where

$$\begin{aligned} & M(x, y) \\ = \ & \max\{\rho(x - y), \rho(x - Tx), \\ & \rho(y - Ty), \rho(x - Ty), \rho(y - Tx)\}. \end{aligned}$$

Let x_1, x_n and $T(x_n)$ be as in the Remark 2.1. If $\delta_\rho(C(x)) < \infty$, then $\{x_n\}$ and $\{Tx_n\}$ are ρ-convergent sequences with the same limit, say $w \in C$. Moreover, if $\rho(w - T(w)) < \infty$, then w is a unique fixed point of T, i.e. $T(w) = w$.

Proof. We construct sequences $\{x_n\}$ and $\{T(x_n)\}$ as the same as the previous theorem and similar to that

$$x_{n+1} \neq T(x_n) \Rightarrow x_n = T(x_{n-1}).$$

Again $\{T(x_n)\}$ and $\{x_n\}$ are ρ-Cauchy sequences. Now, let

$$\begin{aligned} & B(n, k) = \{x_j, T(x_j) : n \leq j \leq n + k\} \\ & B(n) = \{x_j, T(x_j) : n \leq j\} \\ & b(n, k) = \sup\{\rho(x - y) : x, y \in B(n, k)\} \\ & b(n) = \sup\{\rho(x - y) : x, y \in B(n)\}. \end{aligned}$$

Note that $b(n, k)_{k \to \infty} \uparrow b(n)$ and $b(n) \downarrow$. Hence, $b = \lim n \to \infty b(n) \geq 0$ exists. By same argument Theorem 3.1, $b = 0$ and therefore $\{x_n\}$ and $\{T(x_n)\}$ are ρ-Cauchy sequences. Since C is a ρ-complete subset of X_ρ, then $\lim_{n \longrightarrow \infty} x_n = w \in C$. One can prove $\lim_{n \longrightarrow \infty} T(x_n) = w$. For each $m \in \mathbb{N}$,

$$\begin{aligned} \rho(w - T(x_m)) \leq \ & \liminf_n \rho(x_n - T(x_m)) \\ \leq \ & b(m). \end{aligned}$$

Thus $\lim_m \rho(w - T(x_m)) = 0$, i.e., $\lim T(x_n) = w$. Note that

$$\begin{aligned} & M(x_n, w) \\ = \ & \max\{\rho(x_n - w), \rho(x_n - T(x_n)), \\ & \rho(w - Tw), \rho(x_n - Tw), \rho(w - T(x_n))\} \\ \leq \ & \max\{\rho(x_n - w), b(n), \\ & \rho(w - Tw), \rho(x_n - Tw), \rho(w - T(x_n))\}. \end{aligned}$$

Now

$$\begin{aligned} \rho(w - Tw) \leq \ & \liminf_n \rho(T(x_n) - T(w)) \\ \leq \ & \lambda \max\{0, \rho(w - Tw)\}. \end{aligned}$$

Since $\lambda < 1$, $\rho(w - Tw) = 0$ or $T(w) = w$. If w^* is any fixed point of T in C such that $\rho(w - w^*) < \infty$, then

$$\rho(w - w^*) = \rho(Tw - Tw^*) \leq \lambda\rho(w - w^*),$$

which implies $\rho(w - w^*) = 0$ or $w = w^*$.

4 An integral equation in modular function space

In this section, using the same argument as in [1], we study the following integral equation:

$$\begin{aligned} u(t) \ &= f(t, u(t)) + \\ & \sum_{i=1}^n g_i(t, u(t)) \int_0^t \lambda(t, s)\Lambda_i(s, u(s))ds \\ & + \sum_{j=1}^n h_j(t, u(t)) \int_0^t \Omega_j(t, s, u(s))ds, \end{aligned} \tag{4.7}$$

where L^φ, is the Musielak-Orlicz space and $I = [0, b] \subset \mathbb{R}$. $C(I, L^\varphi)$ denote the space of all ρ-continuous function from I to L^φ with the modular $\sigma(u) = \sup_{t\in I}\|u(t)\|_\rho$. Also $C(I, L^\varphi)$ is a real vector space. If ρ is a convex modular, then σ is a convex modular. Also, if ρ satisfies the Fatou property and Δ_2-condition, then σ satisfies the Fatou property and Δ_2-condition (see [4]).
Suppose B is a ρ-closed and convex subset of L^φ. We consider the following hypotheses:

(1) $f : I \times B \longrightarrow L^\varphi$ is a $\|.\|_\rho$-contractive mapping, that is, there exists constant $q \in \mathbb{R}^+$ such that $q < 1$ and for all $u, v \in B$

$$\|f(t, u) - f(t, v)\|_\rho \leq q\|u - v\|_\rho.$$

Also for $t \in I$, $f(t, .) : B \longrightarrow L^\varphi$ is $\|.\|_\rho$-continuous and f is onto.

(2) g_i are functions from $I \times B$ into L^φ, for $i = 1, ..., n$ such that $g_i(t, .) : B \longrightarrow L^\varphi$, for $i = 1, ..., n$ are $\|.\|_\rho$-continuous and there exist $a_i \geq 0$ such that

$$\|g_i(t, u) - g_i(t, v)\|_\rho \leq a_i\|u - v\|_\rho,$$

for $i = 1, ..., n$, and for all $t \in I$ and $u, v \in B$. Also for $u \in B$, $t \longrightarrow g_i(t, u)$ are nondecreasing on I and for $t \in I$, $u \longrightarrow g_i(t, u)$ are nondecreasing on B for $i = 1, ..., n$.

(3) Λ_i, are functions from $I \times B$ into L^φ, for $i = 1, \cdots, n$ such that $\Lambda_i(t, .) : B \longrightarrow L^\varphi$ are $\|.\|_\rho$-continuous and $t \longrightarrow \Lambda_i(t, u)$ are measurable for every $u \in B$. Also, there exist functions $\beta_i \in L^1(I)$ and nondecreasing continuous functions $\gamma_i : [0, \infty) \longrightarrow (0, \infty)$ such that

$$\|\Lambda_i(t, u)\|_\rho \leq \beta_i(t)\gamma_i(\|u\|_\rho),$$

for all $t \in I$ and $u \in B$. Also for $t \in I$, $u \longrightarrow \Lambda_i(t, u)$ are nondecreasing on B.

(4) h_j are functions from $I \times B$ into L^φ, for $j = 1, ..., n$, such that $h_j(t, .) : B \longrightarrow L^\varphi$ are $\|.\|_\rho$-continuous and there exist $\acute{a}_j \geq 0$ such that

$$\|h_j(t, u) - h_j(t, v)\|_\rho \leq \acute{a}_j\|u - v\|_\rho,$$

for $j = 1, ..., n$, for all $t \in I$ and $u, v \in B$. Also for $u \in B$, $t \longrightarrow h_j(t, u)$ are nondecreasing on I and for $t \in I$, $u \longrightarrow h_j(t, u)$ are nondecreasing on B.

(5) Ω_j are functions from $I \times I \times B$ into L^φ, for $j = 1, ..., n$ such that $\Omega_j(t, s, .) : u \longrightarrow \Omega_j(t, s, u)$ are $\|.\|_\rho$-continuous on B for almost all $t, s \in I$ and $s \longrightarrow \Omega_j(t, s, u)$ are measurable for every $u \in B$. Also, there exist nondecreasing continuous functions $\acute{\beta}_j, \acute{\gamma}_j : I \longrightarrow [0, \infty)$ such that

$$\lim_{t\longrightarrow\infty} \acute{\beta}_j(t) \int_0^t \acute{\gamma}_j(s)ds = 0,$$

and

$$\|\Omega_j(t, s, u)\|_\rho \leq \acute{\beta}_j(t)\acute{\gamma}_j(s),$$

for all $t, s \in I$, $s \leq t$ and $u \in B$.

(6) There exist measurable functions $\eta_j : I \times I \times I \longrightarrow \mathbb{R}^+$ such that

$$\|\Omega_j(t, s, u) - \Omega_j(r, s, u)\|_\rho \leq \eta_j(t, r, s),$$

for all $t, r, s \in I$ and $u \in B$, also $\lim_{t\longrightarrow r} \int_0^b \eta_j(t, r, s)ds = 0$.

(7) $\|\Omega_j(t,s,u)-\Omega_j(t,s,v)\|_\rho\leq \|u-v\|_\rho$ for all $t,s\in I$ and $u,v\in B$, $j=1,\cdots,n$.

(8) λ is function from $I\times I$ into $\mathbb{R}^+$. For each $t\in I$, $\lambda(t,s)$ is measurable on $[0,t]$. Also for $s\in I$, $t\longrightarrow\lambda(t,s)$ is nondecreasing on I. $\overline{\lambda(t)}=esssup|\lambda(t,s)|$ is bounded on $[0,b]$ and $k=\sup|\overline{\lambda(t)}|$. The map $\lambda(.,s):t\longrightarrow\lambda(t,s)$ is continuous from I to $L^\infty(I)$.

Theorem 4.1 *Suppose that the condition (1)-(8) are satisfied and L^φ satisfy the Δ_2-condition and there exists $r\geq 0$ such that for all $t,s\in I$,*

$$\int_0^t\beta_i(s)ds<\frac{r}{2n(a_ir+d_i)kb}\int_0^t\frac{1}{\gamma_i(r)}ds,$$

and

$$\int_0^t\acute{\gamma}_j(s)ds\leq\frac{r}{2n(\acute{a}_jr+\acute{d}_j)\acute{\beta}_j(b)},$$

where $d_i:=\sup\{\|g_i(t,u)\|_\rho,t\in I,u\in B\}$, for $i=1,\cdots,n$ and $\acute{d}_j:=\sup\{\|h_j(t,u)\|_\rho,t\in I,u\in B\}$ for $j=1,\cdots,n$ and also $\sup\{\|f(t,u)\|_\rho,t\in I,u\in B\}\leq r$. Then integral equation (4.7) has at least one solution $u\in C(I,L^\varphi)$.

Proof. Now consider the operators,

$$Tu(t)=f(t,u(t)),$$

and

$$\begin{aligned}&Su(t)\\=&\textstyle\sum_{i=1}^n g_i(t,u(t))\int_0^t\lambda(t,s)\Lambda_i(s,u(s))ds\\&+\textstyle\sum_{j=1}^n h_j(t,u(t))\int_0^t\Omega_j(t,s,u(s))ds.\end{aligned}$$

We show T and S satisfy the hypotheses of Theorem 2.3. By conditions, T and S are well defined on $C(I,B)$. Define,

$$A=\{u\in C(I,B);\|u(t)\|_\rho\leq r\text{ for all }t\in I\},$$

then A is a nonempty, $\|.\|_\rho$-bounded, $\|.\|_\rho$-closed and convex subset of $C(I,B)$. Next, we prove that $Su(t)\in A$, for $u\in A$, we have

$$\begin{aligned}&\|Su(t)\|_\rho\\=&\|\textstyle\sum_{i=1}^n g_i(t,u(t))\int_0^t\lambda(t,s)\Lambda_i(s,u(s))ds\\&+\textstyle\sum_{j=1}^n h_j(t,u(t))\int_0^t\Omega_j(t,s,u(s))ds\|_\rho\\\leq&\textstyle\sum_{i=1}^n\|(g_i(t,u(t))-g_i(t,0)+g_i(t,0))\\&\times\textstyle\int_0^t\lambda(t,s)\Lambda_i(s,u(s))ds\|_\rho\\&+\textstyle\sum_{j=1}^n\|(h_i(t,u(t))-h_i(t,0)\\&+h_i(t,0))\textstyle\int_0^t\Omega_j(t,s,u(s))ds\|_\rho\\\leq&\textstyle\sum_{i=1}^n(a_ir+d_i)k\int_0^t\beta_i(s)\gamma_i(r)ds\\&+\textstyle\sum_{j=1}^n(\acute{a}_jr+\acute{d}_j)\int_0^t\acute{\beta}_j(t)\acute{\gamma}_j(s)ds\\\leq&\ r.\end{aligned}$$

We show S is $\|.\|_\rho$-equicontinuous. Let $u\in A$, for $i,\cdots,n$

$$\begin{aligned}&\|g_i(t,u(t))\textstyle\int_0^t\lambda(t,s)\Lambda_i(s,u(s))ds-\\&\quad g_i(\tau,u(\tau))\textstyle\int_0^\tau\lambda(\tau,s)\Lambda_i(s,u(s))ds\|_\rho\\=&\|g_i(t,u(t))\textstyle\int_0^t\lambda(t,s)\Lambda_i(s,u(s))ds\pm\\&\quad g_i(t,u(t))\textstyle\int_0^t\lambda(\tau,s)\Lambda_i(s,u(s))ds\\&\quad\pm g_i(\tau,u(\tau))\textstyle\int_0^t\lambda(\tau,s)\Lambda_i(s,u(s))ds\\&\quad-g_i(\tau,u(\tau))\textstyle\int_0^\tau\lambda(\tau,s)\Lambda_i(s,u(s))ds\|_\rho\\\leq&\|g_i(t,u(t))(\textstyle\int_0^t\lambda(t,s)\Lambda_i(s,u(s))ds-\\&\quad\textstyle\int_0^t\lambda(\tau,s)\Lambda_i(s,u(s))ds)\|_\rho\\&\quad+\|(g_i(t,u(t))-g_i(\tau,u(\tau)))\\&\quad\textstyle\int_0^t\lambda(\tau,s)\Lambda_i(s,u(s))ds\|_\rho\\&\quad+\|g_i(\tau,u(\tau))\textstyle\int_\tau^t\lambda(\tau,s)\Lambda_i(s,u(s))ds\|_\rho,\end{aligned}$$

since

$$\begin{aligned}&\|g_i(t,u(t))(\textstyle\int_0^t\lambda(t,s)\Lambda_i(s,u(s))ds-\\&\quad\textstyle\int_0^t\lambda(\tau,s)\Lambda_i(s,u(s))ds)\|_\rho\\=&\|g_i(t,u(t))(\textstyle\int_0^t(\lambda(t,s)\\&\quad-\lambda(\tau,s))\Lambda_i(s,u(s))ds)\|_\rho\\\leq&\|(g_i(t,u(t))-g_i(t,0)+g_i(t,0))\\&\quad\times(\textstyle\int_0^t(\lambda(t,s)-\lambda(\tau,s))\Lambda_i(s,u(s))ds)\|_\rho\\\leq&(a_ir+d_i)|\lambda(t,0)-\lambda(\tau,0)|_{L_\infty}\textstyle\int_0^t\beta_i(s)\gamma_i(r)ds\\\leq&\tfrac{r}{2nk}|\lambda(t,0)-\lambda(\tau,0)|_{L_\infty},\end{aligned}$$

and

$$\begin{aligned}&\|(g_i(t,u(t))-g_i(\tau,u(\tau)))\textstyle\int_0^t\lambda(\tau,s)\\&\quad\Lambda_i(s,u(s))ds\|_\rho\\\leq&\|(g_i(t,u(t))-g_i(\tau,u(\tau)))k\textstyle\int_0^t\beta_i(s)\gamma_i(r)ds\|_\rho\\\leq&\tfrac{r}{2n(a_ir+d_i)}(\|g_i(t,u(t))-g_i(t,u(\tau))\|_\rho\\&\quad+\|g_i(\tau,u(\tau))-g_i(t,u(\tau))\|_\rho)\\\leq&\tfrac{r}{2n(a_ir+d_i)}(a_i\|u(t)-u(\tau)\|_\rho+d_i),\end{aligned}$$

and

$$\begin{aligned}&\|g_i(\tau,u(\tau))\textstyle\int_\tau^t\lambda(\tau,s)\Lambda_i(s,u(s))ds\|_\rho\\=&\|(g_i(\tau,u(\tau))-g_i(\tau,0)\\&\quad+g_i(\tau,0))\textstyle\int_\tau^t\lambda(\tau,s)\Lambda_i(s,u(s))ds\|_\rho\\\leq&(a_ir+d_i)k\textstyle\int_\tau^t\beta_i(s)\gamma_i(r)ds\\\leq&\tfrac{r}{2nb}|t-\tau|.\end{aligned}$$

By equation (4.7),

$$\begin{aligned}&\|h_j(t,u(t))\textstyle\int_0^t\Omega_j(t,s,u(s))ds\\&\quad-h_j(\tau,u(\tau))\textstyle\int_0^\tau\Omega_j(\tau,s,u(s))ds\|_\rho\\\leq&\|h_j(\tau,u(\tau))(\textstyle\int_0^t\Omega_j(t,s,u(s))ds\\&\quad-\textstyle\int_0^\tau\Omega_j(\tau,s,u(s))ds)\|_\rho\\&\quad+\|(h_j(t,u(t))-h_j(\tau,u(\tau)))\\&\qquad\textstyle\int_0^t\Omega_j(t,s,u(s))ds\|_\rho,\end{aligned}$$

since

$$\begin{aligned}&\|h_j(\tau,u(\tau))(\textstyle\int_0^t\Omega_j(t,s,u(s))ds\\&\quad-\textstyle\int_0^\tau\Omega_j(\tau,s,u(s))ds)\|_\rho\\&\leq(\acute{a}_jr+\acute{d}_j)\textstyle\int_0^b\eta(t,\tau,s)ds,\end{aligned}$$

and

$$\begin{array}{ll} & \|(h_j(t,u(t)) - h_j(\tau,u(\tau))) \\ & \quad \int_0^t \Omega_j(t,s,u(s))ds\|_\rho \\ \leq & \frac{\acute{a}_j r}{2n(\acute{a}_j r+\acute{d}_j)}\|u(t)-u(\tau)\|_\rho, \end{array}$$

then $S(A)$ is $\|.\|_\rho$-equicontinuous. By using the Arzela-Ascoli Theorem, S is a $\|.\|_\sigma$-compact mapping.
Finally, we show that S is $\|.\|_\sigma$-continuous. Let $u, v \in A$, for $i = 1, \cdots, n$

$$\begin{array}{ll} & \|g_i(t,u(t)) \int_0^t \lambda(t,s)\Lambda_i(s,u(s))ds \\ & \quad -g_i(t,v(t)) \int_0^t \lambda(t,s)\Lambda_i(s,v(s))ds\|_\rho \\ \leq & \|(g_i(t,u(t)) - g_i(t,v(t))) \\ & \quad \times \int_0^t \lambda(t,s)\Lambda_i(s,u(s))ds\|_\rho \\ & \quad +\|g_i(t,v(t)) \int_0^t \lambda(t,s)(\Lambda_i(s,u(s)) \\ & \quad -\Lambda_i(s,v(s)))ds\|_\rho \\ \leq & \frac{ra_i}{2n(a_ir+d_i)}\|u(t)-v(t)\|_\rho \\ & \quad +(a_ir+d_i)k\int_0^t\|u(s)-v(s)\|_\rho ds \\ \leq & \frac{ra_i}{2n(a_ir+d_i)}\|u-v\|_\sigma \\ & \quad +(a_ir+d_i)kb\|u-v\|_\sigma. \end{array}$$

and

$$\begin{array}{ll} & \|h_j(t,u(t)) \int_0^t \Omega_j(t,s,u(s))ds \\ & \quad -h_j(t,v(t)) \int_0^t \Omega_j(t,s,v(s))ds\|_\rho \\ \leq & \|(h_j(t,u(t)) - h_j(t,v(t))) \\ & \quad \times \int_0^t \Omega_j(t,s,u(s))ds\|_\rho \\ & \quad +\|h_j(t,v(t))(\int_0^t \Omega_j(t,s,u(s))ds \\ & \quad -\int_0^t \Omega_j(t,s,v(s))ds)\|_\rho \\ \leq & \frac{\acute{a}_j r}{2n(\acute{a}_j r+\acute{d}_j)}\|u-v\|_\sigma+(\acute{a}_j r \\ & \quad +\acute{d}_j)b\|u-v\|_\sigma. \end{array}$$

Therefore by Theorem 2.3, $u \in A$ is a solution of equation (4.7).

Example 4.1 *Let $E = (0,\infty)$, define modular $\rho : X \longrightarrow [0,\infty)$ as follows $\rho(u) = \int_0^\infty |u(x)|^{x+1}dx$, where X is the set of measurable function $u : E \longrightarrow \mathbb{R}$. Let M be the set of ρ-continuous function on E, such that $0 \leq u(x) \leq \frac{1}{4}$. Therefore M is ρ-closed subset of X_ρ.*
$C(I, M)$ denote the space of all ρ-continuous function from I to M with the modular $\varphi(u) = \sup_{t\in I} \rho(u(t))$, where $I = [0, b]$.
Now, consider the nonlinear integral equation

$$u(t) = \Phi(u(t)) + \int_0^t \Lambda(s)\Psi(u(s))ds, \quad (4.8)$$

where $u \in C(I, M)$. One can assume the following conditions are satisfied:

(1) $\Phi : M \longrightarrow X_\rho$ defined by

$$\Phi(u(t)) = \begin{cases} u(t-1) & if \quad t \geq 1, \\ 0, \quad if & t \in [0,1). \end{cases} \quad (4.9)$$

(2) $\Psi : M \longrightarrow X_\rho$ is a continuous function. Also there exists a constant $m \in \mathbb{R}^+$ and nondecreasing continuous function $\beta : [0,\infty) \longrightarrow (0,\infty)$ such that $\rho(\Psi(u)) \leq m\beta(\rho(u))$ for all $u \in M$.

(3) Λ be a function from I into $\mathbb{R}^+$ and for each $t \in I$, $\Lambda(t)$ is measurable on $[0, t]$. Also we consider $r = \sup_{t\in I}|\Lambda(t)| \leq 1$.

Let there exists a constant $k \geq 0$ such that for all $t \in I$; $m < \frac{k}{\beta(k)br}$ and $\sup\{\rho(\Phi(u(t))), t \in I\} \leq k$. Let

$$B = \{u \in C(I, M); \rho(u(t)) \leq k \text{ for all } t \in I\},$$

then B is a nonempty, ρ-bounded, ρ-closed and convex subset of $C(I, M)$. We consider $Tu(t) = \Phi(u(t))$ and $Su(t) = \int_0^t \Lambda(s)\Psi(u(s))ds$. For all $u, v \in M$,

$$\begin{array}{ll} & \rho(\Phi(u(t)) - \Phi(v(t))) \\ = & \int_0^\infty |\Phi(u(t)) - \Phi(v(t))|^{t+1}dt \\ = & \int_1^\infty |u(t-1) - v(t-1)|^{t+1}dt \\ = & \int_0^\infty |u(t) - v(t)|^{t+1}|u(t) - v(t)|dt \\ \leq & \frac{1}{4}\rho(u-v), \end{array}$$

therefore $\varphi(\Phi(u) - \Phi(v)) \leq \frac{1}{4}\varphi(u - v)$. Also for $u \in B$,

$$\begin{array}{rl} \rho(Su(t)) = & \rho(\int_0^t \Lambda(s)\Psi(u(s))ds) \\ \leq & rmb\beta(k) \\ \leq & k, \end{array}$$

then $S(B) \subset B$. Also $S(B)$ is φ-bounded and by Δ_2-condition $\|.\|_\varphi$-bounded. We show $S(B)$ is ρ-equcantinuous. For $t, \tau \in I$, such that $t > \tau$,

$$\begin{array}{rl} \rho(Su(t) - Su(\tau)) \leq & \rho(\int_\tau^t \Lambda(s)\Psi(u(s))ds) \\ \leq & rm\beta(k)|t-\tau|. \end{array}$$

By using the Arzela-Ascoli theorem, S is a φ-compact mapping. By condition (2), S is ρ-continuous. Therefore by Theorem 2.3, $S + T$ have a fixed point $u \in B$ with $Tu + Su = u$; i.e., u is a solution to (4.8).

5 Iterative sequence for non-self mapping

Let X be a Banach space. Mann iteration process is

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,$$

where T maps X into itself. If B is a proper subset of the real Banach space X and T maps B into X, then the sequence Mann may not be well defined. Therefore if $h : X \to B$ be retraction, Mann iteration process becomes

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n hT x_n,$$

where $x_1 \in B$.
In this section, we study an iterative sequence for non-self mapping in modular space.

Let X_ρ be a modular space and B a nonempty subset of X_ρ. A subset B of X_ρ is called retract of X_ρ if there exists a continuous map $h : X_\rho \to B$ such that $hx = x$ for all $x \in B$. A map $h : X_\rho \to B$ is called a retraction if $h^2 = h$.

Definition 5.1 *A non-self mapping T is called ρ-asymptotically nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\longrightarrow\infty} k_n = 1$such that*

$$\rho(T(hT)^n x - T(hT)^n y) \leq k_n \rho(x-y),$$

for all $x, y \in B$, and $n \geq 1$.

We need the following Lemma.

Lemma 5.1 *[15] Assume $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in real number such that

(I) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$.

(II) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\Sigma_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Theorem 5.1 *Let X_ρ be a ρ-complete modular space where ρ is convex and satisfies the Δ_2-condition. Let B be a nonempty ρ-closed and convex subset of X_ρ. Let $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ be real sequences in $(0,1)$. We consider $\{x_n\}_{n\geq 0}$ generated from an arbitrary $x_0 \in B$ by*

$$\begin{cases} z_n = h((1-\gamma_n)x_n + \gamma_n T(hT)^n x_n) \\ y_n = h((1-\beta_n)x_n + \beta_n T(hT)^n z_n), \\ x_{n+1} = h((1-\alpha_n)x_n + \alpha_n T(hT)^n y_n). \end{cases} \tag{5.10}$$

Suppose that the following conditions hold:

(I) *Let $T : B \longrightarrow X_\rho$ be a continuous non-self ρ-asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\longrightarrow\infty} k_n = 1$.*

(II) $\sum_{n\geq 0} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$.

(III) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

(IV) $0 < \liminf_{n\longrightarrow\infty} \gamma_n \leq \limsup_{n\longrightarrow\infty} \gamma_n < 1$.

(V) *h is a retraction from X_ρ to B.*

(VI) $F(T) \neq \emptyset$.

Then for any $w \in F(T)$, $\lim_{n\longrightarrow\infty} \rho(x_n - w) = 0$ and $\lim_{n\to\infty} \rho(x_n - Tx_n) = 0$.

Proof. For any $w \in F(T)$, by equation (5.10)

$$\begin{aligned} & \rho(x_{n+1} - w) \\ = \; & \rho(h((1-\alpha_n)x_n + \alpha_n T(hT)^n y_n) - w) \\ = \; & \rho((1-\alpha_n)(x_n - w) \\ & +\alpha_n T(hT)^n (y_n - w)) \\ \leq \; & (1-\alpha_n)\rho(x_n - w) + \alpha_n k_n \rho(y_n - w) \\ = \; & (1-\alpha_n)\rho(x_n - w) + \alpha_n k_n \rho(h((1-\beta_n)x_n \\ & +\beta_n T(hT)^n z_n) - w) \\ \leq \; & (1-\alpha_n)\rho(x_n - w) \\ & +\alpha_n k_n[(1-\beta_n)\rho(x_n - w) \\ & +\beta_n k_n \rho(z_n - w)] \\ \leq \; & (1-\alpha_n)\rho(x_n - w) \\ & +\alpha_n k_n[(1-\beta_n)\rho(x_n - w) \\ & +\beta_n k_n \rho(h((1-\gamma_n)x_n \\ & +\gamma_n T(hT)^n x_n) - w)] \\ \leq \; & (1-\alpha_n)\rho(x_n - w) \\ & +\alpha_n k_n[(1-\beta_n)\rho(x_n - w) \\ & +\beta_n k_n[(1-\gamma_n)\rho(x_n - w) \\ & +\gamma_n k_n \rho(x_n - w)]]. \end{aligned}$$

Therefore

$$\begin{aligned} & \rho(x_{n+1} - w) \\ \leq \; & [(1-\alpha_n) + \alpha_n k_n (1-\beta_n) \\ & +\alpha_n \beta_n k_n^2 (1-\gamma_n) \\ & +\alpha_n \beta_n \gamma_n k_n^3] \rho(x_n - w). \end{aligned}$$

By Lemma 5.1, so $\lim_{n\to\infty}\alpha_n = 0$, shows $\rho(x_n - w) \to 0$ as $n \to \infty$. Since T is a ρ-continuous then $\lim_{n\to\infty}\rho(Tx_n - Tw) = 0$, also by

$$\rho(\frac{x_n - Tx_n}{2}) \leq \rho(x_n - w) + \rho(Tx_n - w),$$

we have $\lim_{n\to\infty}\rho(x_n - Tx_n) = 0$.

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