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Research Article

Fixed point theorems under c-distance in ordered cone metric space

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Abstract

Recently, Cho et al. [Y. J. Cho, R. Saadati, S. H. Wang, Common fixed point theorems on generalized distance in ordered cone metric spaces, Comput. Math. Appl. 61 (2011) 1254-1260] defined the concept of the c-distance in a cone metric space and proved some fixed point theorems on c-distance. In this paper, we prove some new fixed point and common fixed point theorems by using the distance in ordered cone metric spaces.

Keywords : Cone metric space; Common fixed point; Fixed point; c-distance; Partially ordered set.

1 Introduction

THe Banach contraction principle is the most celebrated fixed point theorem [6]. Afterward, some various definitions of contractive mappings were introduced by other researchers and several fixed and common fixed point theorems were considered in [7, 10, 17, 19, 24]. Huang and Zhang [13] have introduced the concept of the cone metric space, replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive type mappings on cone metric spaces. Then, several fixed and common fixed point results in cone metric spaces were introduced in [2, 3, 9, 15, 23] and the references contained therein. Recently, also, the existence of fixed and common fixed points in partially ordered cone metric spaces was studied in [4, 5, 26].

In 1996, Kada et al. [18] defined the concept of w-distance in complete metric space. Later, many authors proved some fixed point theorems in complete metric spaces (see [1, 20, 21, 22]). Also, note that Saadati et al. [25] introduced a probabilistic version of the w-distance of Kada et al. in a Menger probabilistic metric space. Recently, Cho et al. [8], and Wang and Guo [28] defined a concept of the c-distance in a cone metric space, which is a cone version of the w-distance of Kada et al. and proved some fixed point theorems in ordered cone metric spaces. Then, Sintunavarat et al. [27] generalized the Banach contraction theorem on c-distance of Cho et al. [8]. Also, note that Dordević et al. in [12] proved some fixed point and common fixed point theorems under c-distance for contractive mappings in tvs-cone metric spaces.

The purpose of this work is to extend the Banach contraction principal [6] and Chatterjea contraction theorem [7] on c-distance of Cho et al. [8], and to prove some fixed point and common fixed point theorems in ordered cone metric spaces.

2 Preliminaries

First let us start by defining some basic definitions.

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Definition 2.1 ([11, 13]) *Let E be a real Banach space and 0 denote the zero element in E. A subset P of E is named a cone if and only if*
(a) P is closed, non-empty and $P \neq \{0\}$;
(b) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$ imply that $ax+by \in P$;
(c) if $x \in P$ and $-x \in P$, then $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is interior of P). If $intP \neq \emptyset$, the cone P is called solid. The cone P is named normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$0 \preceq x \preceq y \Longrightarrow \|x\| \leq k\|y\|.$$

The least positive number satisfying the above is called the normal constant of P.

Definition 2.2 ([13]) *Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subset E$. Suppose that the mapping $d : X \times X \to E$ satisfies:*
(d1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then, d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3 ([13]) *Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$.*
(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n\to\infty} d(x_n, x) = 0$.
(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$, and we write $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) If every Cauchy sequence in X is convergent, then X is called a complete cone metric space.

Lemma 2.1 ([13, 23]) *Let (X, d) be a cone metric space and P be a normal cone with normal constant k. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y \in X$. Then the following hold:*
(c_1) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.
(c_2) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.
(c_3) If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.
(c_4) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.
(c_5) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Lemma 2.2 ([4, 14]) *Let E be a real Banach space with a cone P in E. Then, for all $u, v, w, c \in E$, the following hold:*
(p_1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(p_2) If $0 \preceq u \ll c$ for each $c \in intP$, then $u = 0$.
(p_3) If $u \preceq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = 0$.
(p_4) Let $x_n \to 0$ in E, $0 \preceq x_n$ and $0 \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.
(p_5) If $0 \preceq u \preceq v$ and k is a nonnegative real number, then $0 \preceq ku \preceq kv$.
(p_6) If $0 \preceq u_n \preceq v_n$ for all $n \in N$ and $u_n \to u$, $u_n \to u$ as $n \to \infty$, then $0 \preceq u \preceq v$.

Definition 2.4 ([8, 28]) *Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following are satisfied:*
(q_1) $0 \preceq q(x, y)$ for all $x, y \in X$;
(q_2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;
(q_3) for all $n \geq 1$ and $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(q_4) for all $c \in E$ with $0 \ll c$, there exists $e \in E$ with $0 \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 2.1 ([8, 28]) *Each w-distance q in a metric space (X, d) is a c-distance (with $E = \mathbb{R}^+$ and $P = [0, \infty)$). But the converse does not hold. Therefore, the c-distance is a generalization of the w-distance.*

Example 2.1 ([8, 27, 28]) (1) *Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance.*
(2) *Let $E = \mathbb{R}$, $X = [0, \infty)$ and*

$$P = \{x \in E : x \geq 0\}.$$

Define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$

by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance.
(3) *Let $E = C^1_{\mathbb{R}}[0, 1]$ with the norm*

$$\|x\|= \|x\|_\infty+\|x'\|_\infty$$

and consider the cone

$$P = \{x \in E : x(t) \geq 0 \quad on \quad [0, 1]\}.$$

Also, let $X = [0,\infty)$ and define a mapping $d : X\times X \to E$ by $d(x, y) = |x-y|\psi$ for all $x, y \in X$, where $\psi : [0, 1] \to \mathbb{R}$ such that $\psi(t) = 2^t$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = (x + y)\psi$ for all $x, y \in X$. Then q is c-distance.
(4) *Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(w, y)$ for all $x, y \in X$, where $w \in X$ is a fixed point. Then q is a c-distance.*

Remark 2.2 **([8, 27, 28])** *From Example 2.1 $(1, 2, 4)$, we have three important results*
(i) Each cone metric d on X with a normal cone is a c-distance q on X.
(ii) For c-distance q, $q(x, y) = 0$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.
(iii) For c-distance q, $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.

Lemma 2.3 **([8, 27, 28])** *Let (X, d) be a cone metric space and let q be a c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are two sequences in P converging to 0. Then the following hold:*
(qp_1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbb{N}$, then $y = z$. Specifically, if $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$.
(qp_2) If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.
(qp_3) If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.
(qp_4) If $q(y, x_n) \preceq u_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

Remark 2.3 *Note that Dordević et al. [12] proved this theorem for a tvs-cone metric space, where tvs is a real Hausdorff topological vector space. Also, in Lemma 2.3 (qp_1) and (qp_2), set $u_n = v_n$ for $n \in \mathbb{N}$. Then, we get Lemma 2.12 of [8].*

Definition 2.5 **([4, 8])** *Let $(X, \sqsubseteq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \sqsubseteq gfx$ and $gx \sqsubseteq fgx$ hold for all $x \in X$.*

3 Main results

Our first result is the following theorem of Chatterjea type (see [7]) for c-distance in a cone metric space without normality condition of cone.

Theorem 3.1 *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following four conditions hold:*
$(t1)$ $\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$ and $\gamma(fx) \leq \gamma(x)$ for all $x \in X$;
$(t2)$ $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;
$(t3)$ for all $x, y \in X$ with $x \sqsubseteq y$,

$$q(fx, fy) \preceq \alpha(x)q(x, y) + \beta(x)q(x, fy) + \gamma(x)q(y, fx);$$

$(t4)$ for all $x, y \in X$ with $x \sqsubseteq y$,

$$q(fy, fx) \preceq \alpha(x)q(y, x) + \beta(x)q(fy, x) + \gamma(x)q(fx, y).$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.

Proof. If $fx_0 = x_0$, then x_0 is a fixed point of f and the proof is finished. Now, suppose that $fx_0 \neq x_0$. Since f is nondecreasing with respect to $\sqsubseteq$ and $x_0 \sqsubseteq fx_0$, we obtain by induction that

$$x_0 \sqsubseteq fx_0 \sqsubseteq \cdots \sqsubseteq f^n x_o \sqsubseteq f^{n+1}x_0 \sqsubseteq \cdots,$$

where $x_n = fx_{n-1} = f^n x_0$. Now, set $x = x_n$ and $y = x_{n-1}$ in $(t3)$, we have

$$\begin{aligned} & q(x_{n+1}, x_n) \qquad (3.1)\\ &= q(fx_n, fx_{n-1})\\ &\preceq \alpha(x_n)q(x_n, x_{n-1}) + \beta(x_n)q(x_n, x_n)\\ &\quad + \gamma(x_n)q(x_{n-1}, x_{n+1})\\ &\preceq \alpha(fx_{n-1})q(x_n, x_{n-1})\\ &\quad + \beta(fx_{n-1})[q(x_n, x_{n+1}) + q(x_{n+1}, x_n)]\\ &\quad + \gamma(fx_{n-1})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})]\\ &\preceq \alpha(x_{n-1})q(x_n, x_{n-1})\\ &\quad + (\beta + \gamma)(x_{n-1})q(x_n, x_{n+1})\\ &+ \beta(x_{n-1})q(x_{n+1}, x_n) + \gamma(x_{n-1})q(x_{n-1}, x_n)\\ &\preceq \cdots \preceq \alpha(x_0)q(x_n, x_{n-1})\\ &\quad + (\beta + \gamma)(x_0)q(x_n, x_{n+1})\\ &\quad + \beta(x_0)q(x_{n+1}, x_n) + \gamma(x_0)q(x_{n-1}, x_n). \end{aligned}$$

Similarly, set $x = x_n$ and $y = x_{n-1}$ in $(t4)$, we have

$$\begin{aligned} & q(x_n, x_{n+1}) \\ \preceq\ & \alpha(x_0)q(x_{n-1}, x_n) + \beta(x_0)q(x_n, x_{n+1}) \\ & + (\beta+\gamma)(x_0)q(x_{n+1}, x_n) \\ & + \gamma(x_0)q(x_n, x_{n-1}). \end{aligned} \tag{3.2}$$

Adding up (3.1) and (3.2), we have

$$\begin{aligned} & q(x_{n+1}, x_n) + q(x_n, x_{n+1}) \\ \preceq\ & (\alpha+\gamma)(x_0)[q(x_n, x_{n-1}) + q(x_{n-1}, x_n)] \\ & + (2\beta+\gamma)(x_0)[q(x_{n+1}, x_n) + q(x_n, x_{n+1})]. \end{aligned}$$

Set $v_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$. We get that

$$v_n \preceq (\alpha+\gamma)(x_0)v_{n-1} + (2\beta+\gamma)(x_0)v_n.$$

Thus, we have $v_n \preceq \lambda v_{n-1}$, where

$$\lambda = \frac{(\alpha+\gamma)(x_0)}{1-(2\beta+\gamma)(x_0)} < 1$$

by $(t2)$. By repeating the procedure, we get $v_n \preceq \lambda^n v_0$ for all $n \in \mathbb{N}$. Thus,

$$q(x_n, x_{n+1}) \preceq v_n \preceq \lambda^n[q(x_1, x_0) + q(x_0, x_1)]. \tag{3.3}$$

Let $m > n$, then it follows from (3.3) and $\lambda \in [0, 1)$ that

$$\begin{aligned} & q(x_n, x_m) \\ \preceq\ & q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots \\ & + q(x_{m-1}, x_m) \\ \preceq\ & (\lambda^n + \cdots + \lambda^{m-1})[q(x_1, x_0) + q(x_0, x_1)] \\ \preceq\ & \frac{\lambda^n}{1-\lambda}[q(x_1, x_0) + q(x_0, x_1)]. \end{aligned}$$

Lemma 2.3 implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x' \in X$ such that $x_n \to x'$ as $n \to \infty$. Continuity of f implies that $x_{n+1} = fx_n \to fx'$ as $n \to \infty$ and since the limit of a sequence is unique (by Lemma 2.1(c_2)), we get that $fx' = x'$. Thus, x' is a fixed point of f.
Now, suppose that $fz = z$. Then, $(t3)$ implies that

$$\begin{aligned} q(z, z) &= q(fz, fz) \\ &\preceq \alpha(z)q(z, z) + \beta(z)q(z, fz) \\ &\quad + \gamma(z)q(z, fz) \\ &= (\alpha+\beta+\gamma)(z)q(z, z). \end{aligned}$$

Since $(\alpha+\beta+\gamma)(z) < (\alpha+2\beta+2\gamma)(z)$ and $(\alpha+2\beta+2\gamma)(z) < 1$ (by $(t2)$), we get that $q(z, z) = 0$ by Lemma 2.2(p_3). This completes the proof.□

Corollary 3.1 *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exists $\alpha, \beta, \gamma > 0$ such that the following three conditions hold:*
$(t1)$ $\alpha + 2\beta + 2\gamma < 1$;
$(t2)$ for all $x, y \in X$ with $x \sqsubseteq y$,

$$q(fx, fy) \preceq \alpha q(x, y) + \beta q(x, fy) + \gamma q(y, fx);$$

$(t3)$ for all $x, y \in X$ with $x \sqsubseteq y$,

$$q(fy, fx) \preceq \alpha q(y, x) + \beta q(fy, x) + \gamma q(fx, y).$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.

Proof. We can prove this result by apply Theorem 3.1 with $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$.□

Our second result is the following theorem of Chatterjea type (see [7]) for c-distance in a cone metric space with a normal cone.

Theorem 3.2 *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete cone metric space and P be normal cone with normal constant k. Also, let q be a c-distance on X and $f : X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following five conditions hold:*
$(t1)$ $\alpha(fx) \le \alpha(x)$, $\beta(fx) \le \beta(x)$ and $\gamma(fx) \le \gamma(x)$ for all $x \in X$;
$(t2)$ $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;
$(t3)$ for all $x, y \in X$ with $x \sqsubseteq y$,

$$\begin{aligned} q(fx, fy) \preceq\ & \alpha(x)q(x, y) + \beta(x)q(x, fy) \\ & + \gamma(x)q(y, fx); \end{aligned}$$

$(t4)$ for all $x, y \in X$ with $x \sqsubseteq y$,

$$\begin{aligned} q(fy, fx) \preceq\ & \alpha(x)q(y, x) + \beta(x)q(fy, x) \\ & + \gamma(x)q(fx, y); \end{aligned}$$

$(t5)$ $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ for all $y \in X$ with $y \neq fy$.

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.

Proof. If $fx_0 = x_0$, then x_0 is a fixed point of f and the proof is finished. Now, suppose that

$fx_0 \neq x_0$. As in the proof of Theorem 3.1, we have

$$x_0 \sqsubseteq fx_0 \sqsubseteq \cdots \sqsubseteq f^n x_o \sqsubseteq f^{n+1}x_0 \sqsubseteq \cdots,$$

where $x_n = fx_{n-1} = f^n x_0$. Moreover, $\{x_n\}$ converges to a point $x' \in X$ and

$$q(x_n, x_m) \preceq \frac{\lambda^n}{1-\lambda}[q(x_1, x_0) + q(x_0, x_1)]$$

for all positive numbers with $m > n \geq 1$, where

$$\lambda = \frac{(\alpha+\gamma)(x_0)}{1-(2\beta+\gamma)(x_0)} < 1.$$

By (q_3), we get that

$$q(x_n, x') \preceq \frac{\lambda^n}{1-\lambda}[q(x_1, x_0) + q(x_0, x_1)]$$

for all $n \geq 1$. Since P is a normal cone with normal constant k, we have

$$\|q(x_n, x_m)\| \leq k(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\|$$

for all $m > n \geq 1$. In particular, we have

$$\|q(x_n, x_{n+1})\| \leq k(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\| \tag{3.4}$$

for all $n \geq 1$. Also, we have

$$\|q(x_n, x')\| \leq k(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\| \tag{3.5}$$

for all $n \geq 1$. Suppose that $x' \neq fx'$. Then by hypothesis, (3.4) and (3.5), we have

$$\begin{aligned}
0 &< \inf\{\|q(x, x')\| + \|q(x, fx)\| : x \in X\} \\
&\leq \inf\{\|q(x_n, x')\| + \|q(x_n, fx_n)\| : n \geq 1\} \\
&= \inf\{\|q(x_n, x')\| + \|q(x_n, x_{n+1})\| : n \geq 1\} \\
&\leq \inf\Big\{k(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\| \\
&+ k(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\| : n \geq 1\Big\} \\
&= 0.
\end{aligned}$$

which is a contradiction. Hence $x' = fx'$. Moreover, suppose that $fz = z$. Then, we have $q(z, z) = 0$ by the final part of the proof of Theorem 3.1. This completes the proof.□

Corollary 3.2 *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete cone metric space and P be normal cone with normal constant k. Also, let q be a c-distance on X and $f : X \to X$ be a non-decreasing mapping with respect to $\sqsubseteq$. Suppose that there exist $\alpha, \beta, \gamma > 0$ such that the following four conditions hold:*
$(t1)$ $\alpha + 2\beta + 2\gamma < 1$;
$(t2)$ for all $x, y \in X$ with $x \sqsubseteq y$,

$$q(fx, fy) \preceq \alpha q(x, y) + \beta q(x, fy) + \gamma q(y, fx);$$

$(t3)$ for all $x, y \in X$ with $x \sqsubseteq y$,

$$q(fy, fx) \preceq \alpha q(y, x) + \beta q(fy, x) + \gamma q(fx, y);$$

$(t4)$ $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ for all $y \in X$ with $y \neq fy$.

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.

Proof. We can prove this result by apply Theorem 3.2 with $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$.□

Our third result including two mappings and the existence of their common fixed point for c-distance in a cone metric space without normality condition of cone.

Theorem 3.3 *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a c-distance on X and $f, g : X \to X$ be two continuous and weakly increasing mappings with respect to $\sqsubseteq$. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following five conditions hold:*
$(t1)$ $\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$ and $\gamma(fx) \leq \gamma(x)$ for all $x \in X$;
$(t2)$ $\alpha(gx) \leq \alpha(x)$, $\beta(gx) \leq \beta(x)$ and $\gamma(gx) \leq \gamma(x)$ for all $x \in X$;
$(t3)$ $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;
$(t4)$ for all comparable $x, y \in X$,

$$\begin{aligned}
q(fx, gy) \preceq \alpha(x)q(x, y) + \beta(x)q(x, gy) \\
+ \gamma(x)q(y, fx);
\end{aligned}$$

$(t5)$ for all comparable $x, y \in X$,

$$\begin{aligned}
q(gy, fx) \preceq \alpha(x)q(y, x) + \beta(x)q(gy, x) \\
+ \gamma(x)q(fx, y).
\end{aligned}$$

Then f and g have a common fixed point. Moreover, if $fz = gz = z$, then $q(z, z) = 0$.

Proof. Let x_0 be an arbitrary point in X. We construct the sequence $\{x_n\}$ in X as follow:

$$x_{2n+1} = fx_{2n} \quad , \quad x_{2n+2} = gx_{2n+1}.$$

Since f and g are weakly increasing mappings, there exist $x_1, x_2, x_3 \in X$ such that

$$x_1 = fx_0 \sqsubseteq gfx_0 = gx_1 = x_2,$$
$$x_2 = gx_1 \sqsubseteq fgx_1 = fx_2 = x_3.$$

If we continue in this manner, then there exist $x_{2n+1} \in X$

$$x_{2n+1} = fx_{2n} \sqsubseteq gfx_{2n} = gx_{2n+1} = x_{2n+2}$$

and $x_{2n+2} \in X$

$$x_{2n+2} = gx_{2n+1} \sqsubseteq fgx_{2n+1} = fx_{2n+2} = x_{2n+3}$$

for $n = 0, 1, \cdots$. Thus,

$$x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$$

for all $n \geq 1$, that is $\{x_n\}$ is a nondecreasing sequence. Since $x_{2n} \sqsubseteq x_{2n+1}$ for all $n \geq 1$ and by using $(t4)$ for $x = x_{2n}$ and $y = x_{2n+1}$, we have

$$\begin{aligned}
&q(x_{2n+1}, x_{2n+2}) \\
&= q(fx_{2n}, gx_{2n+1}) \\
&\preceq \alpha(x_{2n})q(x_{2n}, x_{2n+1}) \\
&\quad + \beta(x_{2n})q(x_{2n}, gx_{2n+1}) \\
&\quad + \gamma(x_{2n})q(x_{2n+1}, fx_{2n}) \\
&= \alpha(gx_{2n-1})q(x_{2n}, x_{2n+1}) \\
&\quad + \beta(gx_{2n-1})q(x_{2n}, x_{2n+2}) \\
&\quad + \gamma(gx_{2n-1})q(x_{2n+1}, x_{2n+1}) \\
&\preceq \alpha(x_{2n-1})q(x_{2n}, x_{2n+1}) \\
&\quad + \beta(x_{2n-1})[q(x_{2n}, x_{2n+1}) \\
&\quad + q(x_{2n+1}, x_{2n+2})] \\
&\quad + \gamma(x_{2n-1})[q(x_{2n+1}, x_{2n+2}) \\
&\quad + q(x_{2n+2}, x_{2n+1})] \\
&= \alpha(fx_{2n-2})q(x_{2n}, x_{2n+1}) \\
&\quad + (\beta + \gamma)(fx_{2n-2})q(x_{2n+1}, x_{2n+2}) \\
&\quad + \beta(fx_{2n-2})q(x_{2n}, x_{2n+1}) \\
&\quad + \gamma(fx_{2n-2})q(x_{2n+2}, x_{2n+1}) \\
&\preceq \cdots \preceq (\alpha + \beta)(x_0)q(x_{2n}, x_{2n+1}) \\
&\quad + (\beta + \gamma)(x_0)q(x_{2n+1}, x_{2n+2}) \\
&\quad + \gamma(x_0)q(x_{2n+2}, x_{2n+1}).
\end{aligned}$$

Similarly, by using $(t5)$ for $x = x_{2n}$ and $y = x_{2n+1}$, we have

$$\begin{aligned}
q(x_{2n+2}, x_{2n+1}) \preceq\ & (\alpha + \beta)(x_0)q(x_{2n+1}, x_{2n}) \\
& + (\beta + \gamma)(x_0)q(x_{2n+2}, x_{2n+1}) \\
& + \gamma(x_0)q(x_{2n+1}, x_{2n+2}).
\end{aligned}$$

Adding up two previous relations, we have

$$\begin{aligned}
&q(x_{2n+2}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2}) \\
&\preceq (\alpha + \beta)(x_0)[q(x_{2n+1}, x_{2n}) + q(x_{2n}, x_{2n+1})] \\
&\quad + (\beta + 2\gamma)(x_0)[q(x_{2n+2}, x_{2n+1}) \\
&\quad + q(x_{2n+1}, x_{2n+2})].
\end{aligned}$$

Set $v_n = q(x_{2n+1}, x_{2n}) + q(x_{2n}, x_{2n+1})$ and $u_n = q(x_{2n+2}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2})$, we get that

$$u_n \preceq (\alpha + \beta)(x_0)v_n + (\beta + 2\gamma)(x_0)u_n.$$

Thus, we have

$$u_n \preceq \lambda v_n, \tag{3.6}$$

where

$$\lambda = \frac{(\alpha + \beta)(x_0)}{1 - (\beta + 2\gamma)(x_0)} \in [0, 1)$$

by $(t3)$. By a similar procedure, starting with $x = x_{2n+2}$ and $y = x_{2n+1}$, we have

$$v_{n+1} \preceq \lambda u_n. \tag{3.7}$$

From (3.6) and (3.7), we get that

$$v_{n+1} \preceq \lambda^2 v_n \quad , \quad u_n \preceq \lambda^2 u_{n-1},$$

for all $n \in \mathbb{N}$. Thus, $\{u_n\}$ and $\{v_n\}$ are two sequences converging to 0. Also, we have that $q(x_{2n}, x_{2n+1}) \preceq v_n$ and $q(x_{2n+1}, x_{2n+2}) \preceq u_n$ and it follows that $q(x_n, x_{n+1}) \preceq v_n + u_n$. On the other hand, it is easy to show that if $\{u_n\}$ and $\{v_n\}$ are two sequence in E converging to 0, then $\{u_n + v_n\}$ is a sequence converging to 0 (see [8, 12]). Lemma 2.3 implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x' \in X$ such that $x_n \to x'$ as $n \to \infty$. Continuity of f and g implies that $x_{2n+1} = fx_{2n} \to fx'$ and $x_{2n+2} = gx_{2n+1} \to gx'$ as $n \to \infty$ and since the limit of a sequence is unique (by Lemma 2.1(c_2)), we get that $fx' = x'$ and $gx' = x'$. Thus, x' is a common fixed point of f and g.

Suppose that $z \in X$ is any point satisfying $fz = gz = z$. Then, $(t4)$ implies that

$$\begin{aligned}
q(z, z) &= q(fz, gz) \\
&\preceq \alpha(z)q(z, z) + \beta(z)q(z, gz) \\
&\quad + \gamma(z)q(z, fz) \\
&\preceq (\alpha + \beta + \gamma)(z)q(z, z).
\end{aligned}$$

Since $(\alpha + \beta + \gamma)(z) < (\alpha + 2\beta + 2\gamma)(z)$ and $(\alpha + 2\beta + 2\gamma)(z) < 1$ for all $z \in X$ by $(t3)$, we get $q(z, z) = 0$ by property (p_3). This completes the proof.□

Corollary 3.3 *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a c-distance on X and $f, g : X \to X$ be two continuous and weakly increasing mappings with respect to $\sqsubseteq$. Suppose that there exist $\alpha, \beta, \gamma > 0$ such that the following three conditions hold:*
(t1) $(\alpha + 2\beta + 2\gamma) < 1$;
(t2) for all comparable $x, y \in X$,

$$q(fx, gy) \preceq \alpha q(x, y) + \beta q(x, gy) + \gamma q(y, fx);$$

(t3) for all comparable $x, y \in X$,

$$q(gy, fx) \preceq \alpha q(y, x) + \beta q(gy, x) + \gamma q(fx, y).$$

Then f and g have a common fixed point. Moreover, if $fz = gz = z$, then $q(z, z) = 0$.

Proof. We can prove this result by apply Theorem 3.3 with $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$.□

The next result including two mappings and the existence of their common fixed point for c-distance in a cone metric space with a normal cone.

Theorem 3.4 *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete cone metric space and P be normal cone with normal constant k. Also, let q be a c-distance on X and $f, g : X \to X$ be two weakly increasing mappings with respect to $\sqsubseteq$. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following seven conditions hold:*
(t1) $\alpha(fx) \le \alpha(x)$, $\beta(fx) \le \beta(x)$ and $\gamma(fx) \le \gamma(x)$ for all $x \in X$;
(t2) $\alpha(gx) \le \alpha(x)$, $\beta(gx) \le \beta(x)$ and $\gamma(gx) \le \gamma(x)$ for all $x \in X$;
(t3) $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;
(t4) for all comparable $x, y \in X$,

$$q(fx, gy) \preceq \alpha(x)q(x, y) + \beta(x)q(x, gy) + \gamma(x)q(y, fx);$$

(t5) for all comparable $x, y \in X$,

$$q(gy, fx) \preceq \alpha(x)q(y, x) + \beta(x)q(gy, x) + \gamma(x)q(fx, y)$$

(t6) $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ for all $y \in X$ with $y \ne fy$;
(t7) $\inf\{\|q(x, y)\| + \|q(x, gx)\| : x \in X\} > 0$ for all $y \in X$ with $y \ne gy$.
Then f and g have a common fixed point. Moreover, if $fz = gz = z$, then $q(z, z) = 0$.

Proof. The proof is similar to the Theorem 3.2. One can prove this theorem by using the Theorems 3.2 and 3.3.

Corollary 3.4 *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete cone metric space and P be normal cone with normal constant k. Also, let q be a c-distance on X and $f, g : X \to X$ be two weakly increasing mapping with respect to $\sqsubseteq$. Suppose that there exist mappings $\alpha, \beta, \gamma > 0$ such that the following five conditions hold:*
(t1) $\alpha + 2\beta + 2\gamma < 1$;
(t2) for all comparable $x, y \in X$,

$$q(fx, gy) \preceq \alpha q(x, y) + \beta q(x, gy) + \gamma q(y, fx);$$

(t3) for all comparable $x, y \in X$,

$$q(gy, fx) \preceq \alpha q(y, x) + \beta q(gy, x) + \gamma q(fx, y);$$

(t4) $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ for all $y \in X$ with $y \ne fy$;
(t5) $\inf\{\|q(x, y)\| + \|q(x, gx)\| : x \in X\} > 0$ for all $y \in X$ with $y \ne gy$.

Then f and g have a common fixed point. Moreover, if $fz = gz = z$, then $q(z, z) = 0$.

Proof. We can prove this result by apply Theorem 3.4 with $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$.

Example 3.1 *Let $E = \mathbb{R}$ and $P = \{x \in E : x \ge 0\}$. Also, let $X = [0, 1]$ and define a mapping $d : X \times X \to E$ by*

$$d(x, y) = |x - y|$$

for all $x, y \in X$. Then (X, d) is a cone metric space. Define a function $q : X \times X \to E$ by $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is c-distance (by example 2.1). Let an order relation $\sqsubseteq$ defined by

$$x \sqsubseteq y \Longleftrightarrow x \le y.$$

Also, let a mapping $f : X \to X$ defined by $f(x) = \frac{x^2}{4}$ for all $x \in X$.
Take mapping $\alpha(x) = \frac{x+1}{4}$, $\beta(x) = \frac{x}{8}$ and $\gamma(x) = 0$ for all $x \in X$. Observe that:
(1) $\alpha(fx) = \frac{1}{4}\Big(\frac{x^2}{4} + 1\Big) \le \frac{1}{4}\Big(x^2 + 1\Big) \le \alpha(x)$ for all $x \in X$.
(2) $\beta(fx) = \frac{x^2}{32} \le \frac{x^2}{8} \le \frac{x}{8} = \beta(x)$ for all $x \in X$.
(3) $\gamma(fx) = 0 \le 0 = \gamma(x)$ for all $x \in X$.

(4) $(\alpha + 2\beta + 2\gamma)(x) = \frac{x+1}{4} + \frac{2x}{8} = \frac{2x+1}{4} < 1$ *for all* $x \in X$.
(5) *For all comparable* $x, y \in X$ *with* $x \sqsubseteq y$, *we get*

$$\begin{aligned} q(fx, fy) &= \Big|\frac{x^2}{4} - \frac{y^2}{4}\Big| \leq \frac{|x+y||x-y|}{4} \\ &= \Big(\frac{x+y}{4}\Big)|x-y| \\ &\leq \Big(\frac{x+1}{4}\Big)|x-y| \\ &\leq \alpha(x)q(x,y) + \beta(x)q(x,fy) \\ &\quad + \gamma(x)q(y,fx) \end{aligned}$$

(6) *Similarly, we have*

$$\begin{aligned} q(fy, fx) &\leq \alpha(x)q(y,x) + \beta(x)q(fy,x) \\ &\quad + \gamma(x)q(fx,y) \end{aligned}$$

for all $x, y \in X$.
Moreover, f *is a nondecreasing and continuous mapping with respect to* $\sqsubseteq$. *Therefore, all the conditions of Theorem 3.2 are satisfied. Thus,* f *has a fixed point* $x = 0$ *and* $q(0, 0) = 0$.

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