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Research Article

A nonlinear model for common weights set identification in network Data Envelopment Analysis

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Abstract

In the Data Envelopment Analysis (DEA) the efficiency of the units can be obtained by identifying the degree of the importance of the criteria (inputs-outputs).In DEA basic models, challenges are zero and unequal weights of the criteria when decision-making units are evaluated. One of the strategies applied to deal with these problems is using common weights of the each input/output in all decision making units (DMUs). In practice the DMUs are containing intermediate process. However, these units are considered as a black box in DEA basic models, disregarding internal process. This was the main reason network data envelopment analysis was introduced. On the other hand, similar challenges mentioned for DEA, zero and unequal weights of the criteria, exist for network structures as well. This paper suggests a common set of the weights for network structures to deal with the above problems using nonlinear models, for general case. Also some numerical examples using proposed models are presented.

Keywords : Network Data Envelopment Analysis (NDEA); Decision Making Units (DMU); Efficiency; Epsilon; Assurance Value.

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1 Introduction

Data envelopment analysis (DEA), introduced by Charnes et al.[1], is an important tool for measuring the efficiency of decision making units (DMUs). This basic model evaluate a set of unites in which similar inputs applied to produce similar outputs. This method allocates weights to both input and output indicators to maximize the relative efficiency of each evaluated unit. The determined weights for each indicator are calculated in the best form for all the DMUs, in which they may vary from unit to unit. Charnes et al. classified set of controls on weights as following:

i) Direct analysis rejecting or assuming zero weight and eliminating some factors (ε).

ii) Ignoring decision maker's ideas.

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iii) Considering a relative importance of some factors by decision maker.
iv) Discarding the number of DMUs when the number of indicators is more than the number of DMUs under evaluation.

The common weights approach in DEA, initially introduced in 1990 by Cook et al.[3], and Roll et al.[4] in 1991, is known as one of the accurate approaches for evaluating all DMUs considering a unique weights for all the DMUs. Other researchers adopted some of the strategies for reaching common set of the weights. For instance, Roll et al.[4] obtained the common weights by narrowing the range of the weights and reducing domain using weighted average of the weights $\bar{U}_r = \frac{\sum_j E_j u_{rj}}{\sum_j E_j}$ and $\bar{V}_i = \frac{\sum_j E_j v_{ij}}{\sum_j E_j}$, in which E_j is the efficiency of DMU_j. In 1995, Doyle [5] considered the optimized average weights of all DMUs as the common weights. In 2013, Hosseinzadeh et al.[6] used multi-objective programming (MOP) method to attain the common weights. In 2005, Kao and Hung [7] proposed the best common weights for the two-stage network model using the calculated efficiency scores of DEA model and the shortest distance function.

Usually in the evaluating DMUs there are internal process with their own input and outputs. In some cases an internal output can be an input for another internal process or it can be the main output of the unit. In these system the output may get affected by the internal process and ignoring these internal process will result on inaccurate outcome. For the first time, in 1996, Fre and Grosskopf [8] called these units as networked structure units.

Network structures are generally classified into series, parallel, and general groups.The structures or units in which the internal processes are connected in series mode, known as a series network model. Kao and Hwangs, in 2008 [9], Fukuyama and Webers in 2010 [10], and Tone and Tsutsuis in 2009 [11] have studied series network models. Parallel network models representing the behavior of parallel structures in which the internal processes are connected in a parallel mode. Models proposed by Tone and Tsutsui, in 2009 [11], and Lozano, in 2011 [13], are envelopment form of the parallel models and the model proposed by Kao in 2009 [12] is in multiplier form of the parallel models. Later Kao extended (2010) its model to general network having a combination of the series and the parallel models. In these studies the efficiency of the DMUs has been discussed. However, common set of the weights in network structure is not being studied considerably.Kao and Hungs in 2005 [7] and Yang and Liu in 2012 [14] have studied common set of the weights only for special cases.
This paper suggests a model for the general network structure so that each input/output and intermediate indicator has the same weight for evaluating efficiency of all involved processes. A new model is presented in order to obtain a common set of weights in such a way that all DMUs simultaneously are achieved the highest possible efficiency rating while the efficiency measures of divisions do not violates one.
The content of this study is organized as follows. Section 2 has a literature of the multiplier model with network structure and common set of weights model. In section 3, a new MOP method is suggested and the common set of weights in the network structure using goal programming is obtained and reported. Also a solution for a multiple optimal solutions problem is presented. Some numerical examples of obtaining the common set of weights in the general network structure are provided in the Section 4. Finally, section 5 analyzes obtained results and make a conclusion.

2 Review of the literature

In this section the common weight model based on Kaos multiple network model approach [12] and Hosseinzadeh et al. [6] is presented.

2.1 obtaining common weight using MOP

Hosseinzadeh et al.(2013), [6] evaluated efficiency of the DMUs by common weights and using MOP. MOP method is an optimization process for two or more possibly conflicting optimization processes and subjected to certain restrictions.

Suppose that J number of the DMUs consume m input DMU to produce s output. The following MOF problem gives the maximum simultaneous efficiency of the all DMUs:

$$\begin{aligned} Max \quad & \left\{\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i y_{ij}} \middle| j = 1, 2, \cdots, J\right\} \\ s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i y_{ij}} \leq 1; \quad \forall j \\ & u_r \geq \varepsilon; \quad \forall r \\ & v_i \geq \varepsilon; \quad \forall i \end{aligned} \tag{2.1}$$

There are several approach for solving problem (2.1). Goal programming is one of the main methods of the MOP [15]. In goal programming approach, decision maker consider all ideal levels for objective functions. Hence, the sum of the deviations from ideal levels, as the objective function of goal programming problem will be minimized. Accordingly, if $A_j, j = 1, 2, \cdots, J$, presents the goal of the jth objective function and φ_j^+, φ_j^- are negative deviation (under-achievement) and positive deviation (over-achievement)of the jth goal, respectively, Model (2.1) can be written as follows:

$$\begin{aligned} min \quad & \sum_{j=1}^{J} \varphi_j^- + \varphi_j^+ \\ s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} + \varphi_j^- - \varphi_j^+ = A_j; \ \forall j \quad (a) \\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1; \quad \forall j \quad (b) \\ & u_r \geq \varepsilon; \quad \forall r \\ & v_i \geq \varepsilon; \quad \forall i \end{aligned} \tag{2.2}$$

On the other hand, according to constraint (2.2 b), the positive deviation (φ_j^+) lacks a positive value and as a result, $\varphi_j^+ = 0$. Thus, this Constraint (2.2b) is redundant and the constraint (2.2a), considering $A_j = 1$, can be written as follows:

$$\sum_{r=1}^{s} u_r y_{rj} + \varphi_j^- \sum_{i=1}^{m} v_i x_{ij} = \sum_{i=1}^{m} v_i x_{ij} \quad \forall j \tag{2.2-1}$$

Thus, the non-linear model (2.2-1) cannot be transformed into a linear form. Therefore, Hosseinzadeh et al., for linearization of Model (2.2) using the concept of goal programming and substituting $A_j = 1$ in Model (2.2), presented the following model to obtain the common set of weights:

$$\begin{aligned} min & \sum_{j=1}^{J} \varphi_j \\ s.t. & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i y_{ij} + \varphi_j = 0 \qquad \forall j \\ & u_r, v_i \geq \varepsilon, \qquad \forall r, i \\ & \varphi_j \geq 0, \qquad \forall j \end{aligned} \tag{2.2-2}$$

in which φ_j is the deviation from goal (unity value for the efficiency score).

Using this model, the common set of weights for a set of DMUs under evaluation is obtained and the units utilizing obtained common set of the weights are evaluated. However, when this model has multiple optimal solutions, ranking units may vary. This is considered a drawback for Hosseinzadeh et al. study and this paper suggests strategies to avoid this issue.

2.2 NDEA multiplier models

Many researchers suggested multiplier model for network structures and a few of them were introduced in the introduction section. Kao[12](2009)introduced a multiplier model of DEA for network structure for series, parallel and general states. He provided a methodology to convert the system with an overall network structure to a series network structure while each structure has parallel structure (s).

Kazemi Matin and Azizi [16] in 2015 introduced the integrated NDEA model to measure the production systems performance and showed that the model presented by Kao (2009), is a special case of the presented model. Kao introduced a general network structure using an example in which each unit had the third divisions (Figure 1).

In Kao example, system main inputs and outputs are X_1 and X_2 and Y_1, Y_2 and Y_3, respectively. Division 1 may consumes only some of X_1 and X_2 values for producing Y_1 and a part of Y_1 may remains for division 3. Division 2 consumes a specific value of X_1 and X_2 producing $Y2$ similar to division 1 and a part of Y_2 for division 3. Division 3 consumes the rest of X_1 and X_2 alongside with the parts produced Y_1 and Y_2 resulting from divisions 1 and 2 for producing Y_3.

Assume that $X_{ij}^{(k)}$ indicates the ith input of division $k(k = 1, 2, 3)$ from DMU_j. Particularly, sum of all inputs of three divisions $(X_{ij}^{(1)} + X_{ij}^{(2)} + X_{ij}^{(3)})$ for system input are $X_{ij}(j = 1, \cdots, J, i = 1, 2)$. It means that

$$(X_{ij}^{(1)} + X_{ij}^{(2)} + X_{ij}^{(3)}) = X_{ij}; \quad i = 1, 2, \quad j = 1, \cdots, J.$$

The output of division 1 is separated as $Y_1^{(I)}, Y_1^{(O)}$, where, $Y_1^{(O)}$ is the system final output and $Y_1^{(I)}$ is a value consumed by the division 3 as an input. Similarly, output of division 2 is $Y_2^{(I)}, Y_2^{(O)}$, where $Y_2^{(o)}$ is the system final output and $Y_2^{(I)}$ is a value consumed by division 3 as an input. Accordingly,

$$Y_{rj}^{(I)} + Y_{rj}^{(O)} = Y_{rj}; \quad r = 1, 2; \quad j = 1, \cdots, J.$$

Multiplier model of general network structure of figure 1 is as follows:

$$\begin{array}{lll}
E_o: & for\ o = 1, \cdots, J & \\
Max & u_1 y_{1o}^{(o)} + u_2 y_{2o}^{(o)} + u_3 y_{3o} & \\
s.t. & v_1 x_{1o} + v_2 1 x_{2o} = 1 & \\
 & (u_1 y_{1j}^{(o)} + u_2 y_{2j}^{(o)} + u_3 y_{3j}) & \\
 & -(v_1 x_{1j} + v_2 x_{2j}) \leq 0 & \forall j \\
 & u_1 y_{1j} - (v_1 x_{1j}^{(1)} + v_2 x_{2j}^{(1)}) \leq 0 & \forall j \\
 & u_2 y_{2j} - (v_1 x_{1j}^{(2)} + v_2 x_{2j}^{(2)}) \leq 0 & \forall j \\
 & u_1 y_{3j} - (v_1 x_{1j}^{(3)} + v_2 x_{2j}^{(3)} & \\
 & + u_1 y_{1j}^{(I)} + u_2 y_{2j}^{(I)}) \leq 0 & \forall j \\
 & u_1, u_2, u_3, v_1, v_2, \geq \varepsilon &
\end{array} \quad (2.3)$$

Where, u_r indicates the allocated weight to rth output $(r = 1, 2, 3)$ and v_i is the allocated weight to the ith input $(i = 1, 2)$ used for measuring system efficiency DMU_o of the each process. As observed in model 2.3, X_1 input weight is always v_1 no matter to be used by division 1 for $x_{1j}^{(1)}$ input, division 2 as $x_{1j}^{(2)}$ or division 3 as $x_{1j}^{(3)}$; or that y_1 output weight is always u_1 no matter to be used by division 3 as input or to be the final output of the system. Other indices complies in a similar condition.

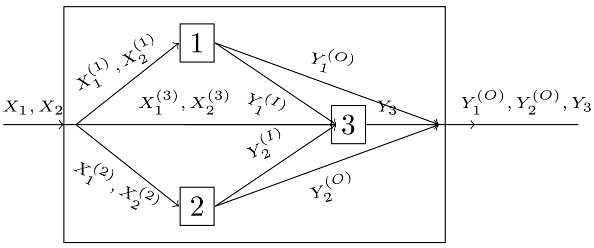

Figure 1: A network system with three division.

Kao also showed that every general network structure could be converted into a two-stage network structure through introducing dummy divisions, where each stage has a parallel structure. If dummy divisions 4 and 5 are added in Figure 1 as an example, Figure 2 with two-stage parallel structure will be created.

The process for calculating the efficiency score of the divisions in Figure 1 is as fol-

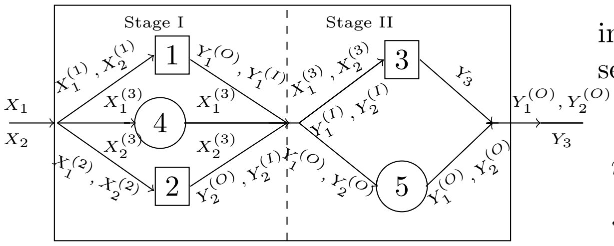

Figure 2: An equivalent tandem system where each stage has a parallel structure.

lowing.

$$\begin{aligned} E_o^{(1)} &= \frac{u_1^* y_{1o}}{v_1^* x_{1o}^{(1)} + v_2^* x_{2o}^{(1)}} \\ E_o^{(2)} &= \frac{u_2^* y_{2o}}{v_1^* x_{1o}^{(2)} + v_2^* x_{2o}^{(2)}} \\ E_o^{(3)} &= \frac{u_3^* y_{3o}}{v_1^* x_{1o}^{(3)} + v_2^* x_{2o}^{(3)} + u_1^* y_{1o}^{(I)} + u_2^* y_{2o}^{(I)}} \end{aligned} \tag{2.4}$$

The efficiency score of the two stages shown in Figure 2, can be obtained using following relation.

$$\begin{aligned} E_o^I &= \frac{u_1^* y_{1o} + (v_1^* x_{1o}^{(3)} + v_2^* x_{2o}^{(3)}) + u_2^* y_{2o}}{v_1^* x_{1o} + v_2^* x_{2o}} \\ E_o^{II} &= \frac{u_1^* y_{1o}^{(o)} + u_2^* y_{2o}^{(o)} + u_3^* y_{3o}}{u_1^* y_{1o} + (v_1^* x_{1o}^{(3)} + v_2^* x_{2o}^{(3)}) + u_2^* y_{2o}} \end{aligned} \tag{2.5}$$

So that according to 2.5, the overall efficiency is equal to the product efficiency of the two stages, in other words: $E_o = E_o^I \times E_o^{II}$

3 Common weights in network structures

3.1 Common weights considering units efficiency deviation

Suppose J is the number of the network structures and each structure consist of K divisions (K=3 for the Kao example). Each division can receive input(s) from the outside or from other division(s), consuming them in production process to generate the main output of the system. The generated output also can be received by the another division of the system.

For Kao's network structure a model is introduced to obtain the common weights set.

$$\begin{aligned} min \quad & \sum_{j=1}^{J} \varphi_j \\ s.t. \quad & (u_1 y_{1j}^{(o)} + u_2 y_{2j}^{(o)} + u_3 y_{3j}) \\ & -(v_1 x_{1j} + v_2 x_{2j}) + \varphi_j = 0 \quad ;\forall j \\ & u_1 y_{1j} - (v_1 x_{1j}^{(1)} + v_2 x_{2j}^{(1)}) \le 0 \quad ;\forall j \\ & u_2 y_{2j} - (v_1 x_{1j}^{(2)} + v_2 x_{2j}^{(2)}) \le 0 \quad ;\forall j \\ & u_3 y_{3j} - (v_1 x_{1j}^{(3)} + v_2 x_{2j}^{(3)} \\ & + u_1 y_{1j}^{(I)} + u_2 y_{2j}^{(I)}) \le 0 \quad ;\forall j \\ & u_1, u_2, u_3, v_1, v_2 \ge \varepsilon \quad \forall j \\ & \varphi_j \ge 0 \quad \forall j \end{aligned} \tag{3.6}$$

Where φ_j is the efficiency deviation of DMU_j and u_1, u_2, u_3, v_1, v_2 are the common weight indicators. In this model the goal is to minimize the sum of φ_j, subject to maximizing the efficiency scores of the network structure and divisions.

If (v_r^*, u_i^*) is the optimal solution, the efficiency score for DMU_j is calculated as follows:

$$E_j^* = \frac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} u_i^* x_{ij}} \tag{3.7}$$

The units can be ranked using obtained efficiency scores from equation (3.7).

When a problem has multiple optimal solutions, the objection of "the ranking is not stable" is appeared. Thus, in order to resolve the objection, the problem is converted into a two-phase problem, in which the first phase is solving the Model (3.6), and the second is solving following model:

$$\begin{aligned} Max\,min \quad & \{\frac{(u_1 y_{1j}^{(O)} + u_2 y_{2j}^{(O)} + u_3 y_{3j})}{(v_1 x_{1j} + v_2 x_{2j})} | \forall j\} \\ s.t. \quad & (u_1 y_{1j}^{(O)} + u_2 y_{2j}^{(O)} + u_3 y_{3j}) \\ & -(v_1 x_{1j} + v_2 x_{2j}) + \varphi_j^* = 0 \quad \forall j \\ & u_1 y_{1j} - (v_1 x_{1j}^{(1)} + v_2 x_{2j}^{(1)}) \le 0 \quad \forall j \\ & u_2 y_{2j} - (v_1 x_{1j}^{(2)} + v_2 x_{2j}^{(2)}) \le 0 \quad \forall j \\ & u_3 y_{3j} - (v_1 x_{1j}^{(3)} + v_2 x_{2j}^{(3)} \\ & + u_1 y_{1j}^{(I)} + u_2 y_{2j}^{(I)}) \le 0 \quad \forall j \\ & u_1, u_2, u_3, v_1, v_2 \ge \varepsilon \end{aligned} \tag{3.8}$$

Variable

$\lambda = min\{\frac{u_1y_{1j}^{(O)}+u_2y_{2j}^{(O)}+u_3y_{3j}}{v_1x_{1j}+v_2x_{2j}}|\forall j\}$

is defined to change the multi-objective model (3.8) into the nonlinear model as follows:

$$\begin{array}{lll} Max & \lambda & \\ s.t. & \frac{u_1y_{1j}^{(O)}+u_2y_{2j}^{(O)}+u_3y_{3j}}{v_1x_{1j}+v_2x_{2j}} \geq \lambda & \\ & (u_1y_{1j}^{(O)} + u_2y_{2j}^{(O)} + u_3y_{3j}) & \\ & -(v_1x_{1j} + v_2x_{2j}) + \varphi_j^* = 0 & \forall j \\ & u_1y_{1j} - (v_1x_{1j}^{(1)} + v_2x_{2j}^{(1)}) \leq 0 & \forall j \\ & u_2y_{2j} - (v_1x_{1j}^{(2)} + v_2x_{2j}^{(2)}) \leq 0 & \forall j \\ & u_3y_{3j} - (v_1x_{1j}^{(3)} + v_2x_{2j}^{(3)} & \\ & +u_1y_{1j}^{(I)} + u_2y_{2j}^{(I)}) \leq 0 & \forall j \\ & u_1, u_2, u_3, v_1, v_2, \lambda \geq \varepsilon & \end{array} \tag{3.9}$$

By solving the nonlinear model (3.9) and obtaining the weights, (v_r^*, u_i^*), the efficiency score of DMU_j is calculated as (3.7).

In Model (5),

$$(v_1, v_2, u_1, u_2, u_3, \varphi) = (0, 0, 0, 0, 0, 0)$$

satisfies in the first four constrains, and considering the fifth constrains and the objective function in the optimum solution the weights reaches into near zero.

Because of the computer limited memory, answers are strongly depend on the value of epsilon; and sometimes irrational results on obtaining error propagation. Therefore, to resolve with this problem, the epsilon has to be obtained from [17] and the optimum value is used in Models (3.6) to (3.12).

$$\begin{array}{lll} Max & \varepsilon & \\ s.t. & v_1x_{1j} + v_2x_{2j} \leq 1 & \forall j \\ & (u_1y_{1j}^{(o)} + u_2y_{2j}^{(o)} + u_3y_{3j}) & \\ & -(v_1x_{1j} + v_2x_{2j}) \leq 0 & \forall j \\ & u_1y_{1j} - (v_1x_{1j}^{(1)} + v_2x_{2j}^{(1)}) \leq 0 & \forall j \\ & u_2y_{2j} - (v_1x_{1j}^{(2)} + v_2x_{2j}^{(2)}) \leq 0 & \forall j \\ & u_3y_{3j} - (v_1x_{1j}^{(3)} + v_2x_{2j}^{(3)} & \\ & +u_1y_{1j}^{(I)} + u_2y_{2j}^{(I)}) \leq 0 & \forall j \\ & u_1 - \varepsilon \geq 0 & \\ & u_2 - \varepsilon \geq 0 & \\ & u_3 - \varepsilon \geq 0 & \\ & v_1 - \varepsilon \geq 0 & \\ & v_2 - \varepsilon \geq 0 & \end{array} \tag{3.10}$$

For example, if the optimum value for the Model (3.10) is ε^*, the Model (3.6) constraint Type 5 is as follows:
$u_1, u_2, u_3, v_1, v_2 \geq \varepsilon^*, \varphi_j \geq 0 \quad \forall j$
Using ε^* in other models is alike.
Model (3.10) is used for finding ε^* . ε^* is used for finding, divisions, stage, and overall efficiency of the system. It should be noted that the obtained values for the efficiency using ε^* may vary slightly with the values obtained using Kaoa's model.

3.2 Common weights of network structures with efficiency deviation of the units and divisions

The model presented in the previous section is obtained based on the the efficiency deviation of the entire system. In this section this model is expanded using the efficiency deviation of both the divisions, and the entire system; and maintaining the maximum efficiency of the system, and the intermediate divisions. Considering these conditions the model is as follows:

$$\begin{array}{ll} min & \sum_{j=1}^{J}(\varphi_j + \sum_{k=1}^{3}\varphi_{kj}) \\ s.t. & (u_1 y_{1j}^{(o)} + u_2 y_{2j}^{(o)} + u_3 y_{3j}) \\ & -(v_1 x_{1j} + v_2 x_{2j}) + \varphi_j = 0 \\ & u_1 y_{1j} - (v_1 x_{1j}^{(1)} + v_2 x_{2j}^{(1)}) + \varphi_{1j} = 0 \\ & u_2 y_{2j} - (v_1 x_{1j}^{(2)} + v_2 x_{2j}^{(2)}) + \varphi_{2j} = 0 \\ & u_3 y_{3j} - (v_1 x_{1j}^{(3)} + v_2 x_{2j}^{(3)} \\ & +u_1 y_{1j}^{(I)} + u_2 y_{2j}^{(I)}) + \varphi_{3j} = 0 \\ & j = 1, 2, \cdots, J \\ & \varphi_j, \varphi_{kj} \geq 0; \quad \forall k, j \\ & u_1, u_2, u_3, v_1, v_2 \geq \varepsilon^* \end{array} \tag{3.11}$$

By solving this model, the common weights set of the system units under evaluation is achieved using the maximum efficiency of units and divisions.

Suppose $(v_r^*, u_i^*, \varphi_j^*, \varphi_{kj}^*)$ is the optimal solution of model (3.11), the efficiency score of DMU_j is obtained using the relationship as (3.7). Hence, the DMUs can be ranked based on the obtained efficiency scores.

However, when the problem has multiple optimal solutions, the ranking of the DMUs will be unstable. To resolve this objection, a two-phase model has to be solved. The first phase of model is (3.11) and the second phase of the model is as follows:

$$\begin{array}{ll} Max\ min & \{\frac{u_1 y_{1j}^{(O)} + u_2 y_{2j}^{(O)} + u_3 y_{3j}}{v_1 x_{1j} + v_2 x_{2j}} | \forall j\} \\ s.t. & (u_1 y_{1j}^{(O)} + u_2 y_{2j}^{(O)} + u_3 y_{3j}) \\ & -(v_1 x_{1j} + v_2 x_{2j}) + \varphi_j^* = 0 \\ & u_1 y_{1j} - (v_1 x_{1j}^{(1)} + v_2 x_{2j}^{(1)}) + \varphi_{1j}^* = 0 \\ & u_2 y_{2j} - (v_1 x_{1j}^{(2)} + v_2 x_{2j}^{(2)}) + \varphi_{2j}^* = 0 \\ & u_3 y_{3j} - (v_1 x_{1j}^{(3)} + v_2 x_{2j}^{(3)} \\ & +u_1 y_{1j}^{(I)} + u_2 y_{2j}^{(I)}) + \varphi_{3j}^* = 0 \\ & j = 1, 2, \cdots, J \\ & u_1, u_2, u_3, v_1, v_2 \geq \varepsilon^* \end{array} \tag{3.12}$$

The multi-objective model (3.12) is transferred to the model (3.13) by introducing the below variable

$$\lambda = min\{\frac{u_1 y_{1j}^{(O)} + u_2 y_{2j}^{(O)} + u_3 y_{3j}}{v_1 x_{1j} + v_2 x_{2j}} | \forall j\}$$

$$\begin{array}{lll} Max & \lambda & \\ s.t. & \frac{u_1 y_{1j}^{(O)} + u_2 y_{2j}^{(O)} + u_3 y_{3j}}{v_1 x_{1j} + v_2 x_{2j}} \geq \lambda & \\ & (u_1 y_{1j}^{(O)} + u_2 y_{2j}^{(O)} + u_3 y_{3j}) & \\ & -(v_1 x_{1j} + v_2 x_{2j}) + \varphi_j^* = 0 & \forall j \\ & u_1 y_{1j} - (v_1 x_{1j}^{(1)} + v_2 x_{2j}^{(1)})\varphi_{1j}^* = 0 & \forall j \\ & u_2 y_{2j} - (v_1 x_{1j}^{(2)} + v_2 x_{2j}^{(2)})\varphi_{2j}^* = 0 & \forall j \\ & u_3 y_{3j} - (v_1 x_{1j}^{(3)} + v_2 x_{2j}^{(3)} & \\ & +u_1 y_{1j}^{(I)} + u_2 y_{2j}^{(I)})\varphi_{3j}^* = 0 & \forall j \\ & u_1, u_2, u_3, v_1, v_2, \lambda \geq \varepsilon^* & \end{array} \tag{3.13}$$

The efficiency scores is obtained by substituting the presented model results in equation (3.7). Using obtained scores DMUs are ranked, and the challenge of zero weights is eliminated.

4 Numerical example

To demonstrate performance of the proposed models, the models are investigated considering two examples of Kao in [9] and [12]. The first is a simple example including five decision-making units A, B, C, D, E, with three intermediate divisions as its structure is shown in the Figure 1. The second example is a case study introduced by Kao about Twenty four Non-life insurance companies in Taiwan. He considered these companies as decision making units (system), each consisting two intermediate divisions.

Example 4.1. *Consider five decision-making units A, B, C, D, E, each consisting three Divisions. The structure of each decision-making unit is shown in Figure 1. The inputs/outputs of all the systems are listed in Table 1.*

According to the data shown in Table 1, implementing Toloo's model, the overall assurance value will be equal to $\varepsilon^{**} = 0.0344828$.

Table 1: Input/output data of Kao example in the Year 2009.

DMU	P	X_1	X_2	$y_1^{(o)}$	$y_1^{(I)}$	$y_2^{(o)}$	$y_2^{(I)}$	y_3
A		11	14	2	-	2	-	1
	1	3	5	2	2	-	-	-
	2	4	3	-	-	2	1	-
	3	4	6	-	2	-	1	1
B		7	7	1	-	1	-	1
	1	2	3	1	1	-	-	-
	2	2	1	-	-	1	1	-
	3	3	3	-	1	-	1	1
C		11	14	1	-	1	-	2
	1	3	4	1	1	-	-	-
	2	5	3	-	-	1	1	-
	3	3	7	-	1	-	1	2
D		14	14	2	-	3	-	1
	1	4	6	2	1	-	-	-
	2	5	5	-	-	3	1	-
	3	5	3	-	1	-	1	1
E		14	15	3	-	2	-	3
	1	5	6	3	1	-	-	-
	2	5	4	-	-	2	2	-
	3	4	5	-	1	-	2	3

Thus, the overall assurance interval is (0,0.0344828]. Table 2 shows the values for the traditional CCR model, and Kao model (3.9) in two modes, without value and with overall assurance value, and CSW model.

Table 2: Comparing 5 DMU system performances independently calculated through ordinary model CCR, Kao model and the model presented here.

DMU	E-CCR	E-CCRε	EN-CCR	EN ε	EN-CSW
A	1.0000	0.9266	0.5227	0.4744	04667
B	0.8980	0.8832	0.5952	0.5895	0.5833
C	0.8485	0.7377	0.5682	0.5209	0.5133
D	1.0000	1.0000	0.4821	0.4706	0.4702
E	1.0000	1.0000	0.8000	0.7931	0.7931

As it is shown in the table 2, applying a common set of the weights, DMU_A and DMU_D rankings are replaced comparing with the rank obtained from Kao network structure efficiency scores . And this replacement is due to applying the common weights for evaluating the units. Furthermore, considering calculation of epsilon using Toloos model, values are obtained for CCR efficiencies and network model are slightly different from Kaos solution [?]. For instance, in Kao CCR model, DMU_A efficiency value is 1 regarded as an efficient unit. But, with using the epsilon obtained equal to 0.9266, it is considered as an inefficient unit.

As Table 2 shows, using ε assurance value in CCR model, Unit A is converted from efficient state to inefficient state, and the efficiency scores of units C,and B are dropped. In addition, using the assurance value ε in NDEA-CCR model, the efficiency scores of all the units are reduced; though, ranking of the units are still constant.
The Weights obtained from Kao's model and CSW proposed model are given in Table 3.

Table 3: The weights of the five DMUs calculated independently via Kaos model, and CSW proposed model

DMU	v_1	v_2	w_1	w_2	u_3
A	0.0470	0.0345	0.0784	0.0643	0.1891
B	0.0345	0.1085	0.1613	0.0888	0.3395
C	0.0470	0.0345	0.0784	0.0643	0.1891
D	0.0369	0.0345	0.0708	0.0542	0.1665
E	0.0345	0.0345	0.0690	0.0517	0.1609
CSW	0.0345	0.0345	0.0690	0.0517	0.1609

In the above table, Rows 2 to 6 are the weights obtained from Kao network structure model and the last row is the common set of the weights obtained using set of common weight model. The efficiency of stages and divisions of 5 evaluation units considering Kao models and common weight set are presented in the following table.

Table 4: Efficiency scores processes and stages calculated from the Kaos network model

DMU	P.eff.Kao			S.eff.Kao	
	E_1	E_2	E_3	E_{S1}	E_{S2}
A	1.0000	0.6613	0.3070	0.9013	0.5264
B	0.8188	1.0000	0.5003	0.9286	0.6348
C	0.5618	0.3796	0.7204	0.6677	0.7801
D	0.5990	0.6069	0.4029	0.7174	0.6561
E	0.7273	0.6667	1.0000	0.7931	1.0000

Now, the models applied for the data of 23 insurance companies in Taiwan. The data are related to Kao 2008 [9].

Table 5: Efficiency scores processes and stages calculated from the CSW proposed model

DMU	P.eff.CSW			S.eff.CSW	
	E_1	E_2	E_3	E_{S1}	E_{S2}
A	1.0000	0.6429	0.3011	0.9000	0.5185
B	0.8000	1.0000	0.4912	0.9286	0.6282
C	0.5714	0.3750	0.6914	0.6800	0.7549
D	0.6000	0.6000	0.4058	0.7143	0.6583
E	0.7273	0.6667	1.0000	0.7931	1.0000

Example 4.2. *Consider the example of 24 Taiwanese insurance companies extracted from Kao and Hwang paper 2.4 in which the structure of each is similar to Figure 3. Inputs/outputs of the insurance companies are listed in Table 3.*

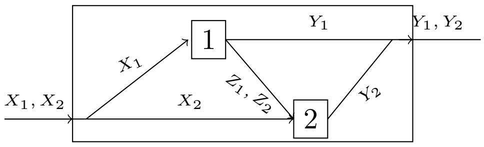

Figure 3: Network structure of the insurance operation system.

Table 6: Input/output table of Kao, case study: Taiwanese insurance companies in 2008.

DMU	X_1	X_2	Z_1	Z_2	Y_1	Y_2
DMU_1	1178744	673512	7451757	856735	984143	681687
DMU_2	1381822	1352755	10020274	1812894	1228502	834754
DMU_3	1177994	592790	4776548	560244	293613	658428
DMU_4	601320	594259	3174851	371863	248709	177331
DMU_5	6699063	3531614	37392862	1753794	7851229	3925272
DMU_6	2627707	668363	9747908	952326	1713598	415058
DMU_7	1942833	1443100	10685457	643412	2239593	439039
DMU_8	3789001	1873530	17267266	1134600	3899530	622868
DMU_9	1567746	950432	11473162	546337	1043778	264098
DMU_{10}	1303249	1298470	8210389	504528	1697941	554806
DMU_{11}	1962448	672414	7222378	643178	1486014	18259
DMU_{12}	2592790	650952	9434406	1118489	1574191	909295
DMU_{13}	2609941	1368802	13921464	811343	3609236	223047
DMU_{14}	1396002	988888	7396396	465509	1401200	332283
DMU_{15}	2184944	651063	10422297	749893	3355197	555482
DMU_{16}	1211716	415071	5606013	402881	854054	197947
DMU_{17}	1453797	1085019	7695461	342489	3144484	371984
DMU_{18}	757515	547997	3631484	995620	692731	163927
DMU_{19}	159422	182338	1141950	483291	519121	46857
DMU_{20}	145442	53518	316829	131920	355624	26537
DMU_{21}	84171	26224	225888	40542	51950	6491
DMU_{22}	15993	10502	52063	14574	82141	4181
DMU_{23}	54693	28408	245910	49864	0.1	18980
DMU_{24}	163297	235094	476419	644816	142370	16976

Implementing Toloo model using the data in Table 3 results on the overall assurance interval $\varepsilon^{**} = 1.04573e-8$. Thus, the overall assurance interval is $(0, 1.04573e-8]$. Kao implementation results are listed in Tables 4and 5 in the following two states:
1. Without any value;
2. With overall assurance value

In this table, the second, third, fourth and sixth columns are the results of implementing traditional $CCR-\varepsilon$ models without overall assurance value, traditional $CCR-\varepsilon$ with overall assurance value, network without an overall assurance value, and network with the overall assurance value obtained from model NDEA-PZ, respectively. The fifth and seventh columns are units rankings in network models without overall assurance value and with overall assurance value respectively.

Table 7: Comparing the efficiencies of 24 insurance companies in Taiwan independently calculated through ordinary CCR model and Kao model.

DMU	E-CCR	E-CCR-ε	EN-Kao	R-Kao
DMU_1	0.984	0.978	0.996	4
DMU_2	1.000	1.000	1.000	1.5
DMU_3	0.988	0.970	0.936	5
DMU_4	0.488	0.488	0.488	11
DMU_5	1.000	1.000	0.979	3
DMU_6	0.594	0.588	0.390	15
DMU_7	0.470	0.467	0.374	17
DMU_8	0.415	0.415	0.295	20
DMU_9	0.327	0.327	0.280	22
DMU_{10}	0.781	0.772	0.705	9
DMU_{11}	0.283	0.277	0.283	21
DMU_{12}	1.000	1.000	0.714	8
DMU_{13}	0.353	0.351	0.337	18
DMU_{14}	0.470	0.468	0.394	14
DMU_{15}	0.979	0.972	0.737	7
DMU_{16}	0.472	0.472	0.321	19
DMU_{17}	0.635	0.633	0.427	13
DMU_{18}	0.427	0.426	0.385	16
DMU_{19}	0.822	0.821	0.487	12
DMU_{20}	0.935	0.934	0.850	6
DMU_{21}	0.333	0.333	0.268	23
DMU_{22}	1.000	1.000	1.000	1.5
DMU_{23}	0.599	0.598	0.580	10
DMU_{24}	0.257	0.256	0.172	24

According to results of the tables 7 and 8, it is observed that DMU_2, DMU_5,DMU_{12} and DMU_{22} have the efficiency equal to one in CCR basic model. however in Kao network model, just DMU_{22} has the efficiency equal to one.

In addition, in CSW model, there is no unit with efficiency value equal to one. The rank of DMU_{15} in Model CSW was promoted significantly compared to the Kao two network models and CCR. DMU_{24} in

Table 8: Comparing the efficiencies of 24 insurance companies in Taiwan independently calculated through the model presented here.

DMU	EN-ε	EN-CSW-ε	R-EN-CSW-ε
DMU_1	0.913	0.477	5
DMU_2	0.805	0.301	9
DMU_3	0.894	0.473	6
DMU_4	0.450	0.145	22
DMU_5	0.599	0.546	2
DMU_6	0.403	0.332	8
DMU_7	0.325	0.193	17
DMU_8	0.293	0.221	15
DMU_9	0.262	0.161	21
DMU_{10}	0.582	0.237	13
DMU_{11}	0.266	0.098	23
DMU_{12}	0.711	0.618	1
DMU_{13}	0.320	0.175	19
DMU_{14}	0.362	0.200	16
DMU_{15}	0.729	0.530	3
DMU_{16}	0.320	0.269	11
DMU_{17}	0.420	0.266	12
DMU_{18}	0.345	0.178	18
DMU_{19}	0.480	0.232	14
DMU_{20}	0.848	0.461	7
DMU_{21}	0.268	0.174	20
DMU_{22}	1.000	0.493	4
DMU_{23}	0.579	0.273	10
DMU_{24}	0.167	0.057	24

basic CCR, network Models, and CSW presented Model have the lowest efficiency and ranking.
The common weights obtained by solving CSW Model are as follows:

Table 9: The weights of the 24 DMUs calculated via CSW proposed model.

v_1	v_2	w_1	w_2	u_1	u_2
2.3334e-7	1.046e-8	1.046e-8	1.046e-8	1.046e-8	1.0346e-7

The efficiency of divisions and stages of Example 24 in life insurance Company in Taiwan are investigated in Tables 10 and 11:

According to 2.4, the total efficiency is equal to multiplication of efficiencies of the two stages.
$E_O = E_O^I \times E_O^{II}$
However, the above relationship is not true for Kaos results. For example, in Table 6, Kao [12] relationship is not applied for DMU_6. Because the total efficiency is equal to 0.390, while, the product of efficiency of stages is $0.736 \times 0.324 = 0.238$.

Table 10: Efficiency scores processes and stages of the 24DMUs calculated from the Kaos network model .

DMU	Process Eff. Of Kao		Stage Eff. Of Kao	
	1	2	I	II
DMU_1	0.618	0.971	0.940	0.972
DMU_2	0.433	0.981	0.821	0.981
DMU_3	0.426	0.971	0.921	0.971
DMU_4	0.306	0.501	0.896	0.503
DMU_5	0.596	0.883	0.667	0.898
DMU_6	0.736	0.322	0.743	0.543
DMU_7	0.421	0.386	0.805	0.404
DMU_8	0.479	0.275	0.500	0.586
DMU_9	0.616	0.278	0.915	0.287
DMU_{10}	0.359	0.716	0.806	0.722
DMU_{11}	0.354	0.018	0.367	0.725
DMU_{12}	1.000	0.694	1.000	0.711
DMU_{13}	0.464	0.127	0.479	0.669
DMU_{14}	0.420	0.408	0.866	0.418
DMU_{15}	1.000	0.394	1.000	0.729
DMU_{16}	0.621	0.287	0.626	0.512
DMU_{17}	0.503	0.440	0.786	0.535
DMU_{18}	0.481	0.366	0.931	0.371
DMU_{19}	0.538	0.485	0.966	0.497
DMU_{20}	0.851	0.442	0.851	0.996
DMU_{21}	0.451	0.208	0.451	0.593
DMU_{22}	1.000	0.501	1.000	1.000
DMU_{23}	0.800	0.585	0.991	0.585
DMU_{24}	0.241	0.173	0.958	0.174

Table 11: Efficiency scores processes and stages of the 24DMUs calculated from the CSW presented model here.

DMU	Process Eff. Of CSW		Stage Eff. Of CSW	
	1	2	I	II
DMU_1	0.618	0.711	0.646	0.738
DMU_2	0.433	0.625	0.458	0.657
DMU_3	0.426	1.000	0.473	1.000
DMU_4	0.286	0.423	0.317	0.456
DMU_5	0.596	0.847	0.628	0.869
DMU_6	0.832	0.308	0.858	0.387
DMU_7	0.421	0.327	0.454	0.424
DMU_8	0.533	0.278	0.572	0.386
DMU_9	0.616	0.192	0.642	0.250
DMU_{10}	0.359	0.548	0.387	0.613
DMU_{11}	0.623	0.018	0.667	0.147
DMU_{12}	0.835	0.684	0.860	0.718
DMU_{13}	0.601	0.127	0.632	0.278
DMU_{14}	0.420	0.355	0.454	0.440
DMU_{15}	1.000	0.411	1.000	0.530
DMU_{16}	0.741	0.271	0.771	0.348
DMU_{17}	0.462	0.388	0.492	0.540
DMU_{18}	0.435	0.301	0.468	0.381
DMU_{19}	0.527	0.260	0.545	0.427
DMU_{20}	0.674	0.442	0.709	0.651
DMU_{21}	0.544	0.183	0.601	0.289
DMU_{22}	0.635	0.501	0.658	0.750
DMU_{23}	0.467	0.536	0.509	0.536
DMU_{24}	0.241	0.131	0.264	0.217

This is considered as an objection to Kaos results.

5 Conclusion

it is observed that there is no need for DMU to have the full efficiency in network structure. Most methods are applied only for obtaining the efficient units which cannot be used for ranking. Hence, the suggested modle of the common weights can be used for ranking of the units with network structure. The obtained efficiency scores in the common weight model is reduced compared to Kao model; and DMUs ranking may changed. Further research, ranking of the units under evaluation with network structures when the efficiency scores are equal, is suggested.

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