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Research Article

A random walk with exponential travel times

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Abstract

Consider the random walk among N places with $N(N-1)/2$ transports. We attach an exponential random variable X_{ij} to each transport between places P_i and P_j and take these random variables mutually independent. If transports are possible or impossible independently with probability p and $1-p$, respectively, then we give a lower bound for the distribution function of the smallest path at point $\log N$ as Np is large.

Keywords : Smallest path; Random walk; Pure birth process; Random recursive tree.

1 Introduction

A random walk is a mathematical formalization of a path that consists of a succession of random steps. For example, the path traced by a molecule as it travels in a liquid or a gas, the search path of a foraging animal, the price of a fluctuating stock and the financial status of a gambler can all be modeled as random walks, although they may not be truly random in reality. The term random walk was first introduced by Karl Pearson [10] in 1905. Random walks have been used in many fields: ecology, economics, psychology, computer science, physics, chemistry, and biology. Random walks explain the observed behaviors of processes in these fields, and thus serve as a fundamental model for the recorded stochastic activity. Although random walk problem is introduced as the one-dimensional motion of particles, but it is not restricted to one dimension nor is the applications limited to the wanderings of inebriates. However, the motivation of studying the random walks and related fields is multifold and there are some nice articles in this subject and its applications [1, 2, 4, 5, 3, 6, 11, 13, 14]. In all random walks in the mentioned sciences, we label two arbitrary places with 1 and N and assess the random walk between these places.

2 Modelling

For two arbitrary places which we label 1 and N, we attach an exponential random variable X_{ij} with mean 1 to each transport between places P_i and P_j and take these random variables mutually independent. i.e., $X_{ij} \sim f(x) = e^{-x} I_{(0,\infty)}(x)$. A possible path can be shown as $R_N : 1 \longrightarrow P_{i_1} \longrightarrow P_{i_2} \longrightarrow \cdots \longrightarrow P_{i_j} \longrightarrow N$ where $(i_1, ..., i_j)$ is any permutation of $(2, ..., N-2)$. Suppose $X_N = X_{1i_1} + X_{i_2 i_3} + \cdots + X_{i_j N}$. In fact, we define X_N as

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the weight of R_N. Thus if S_N is the weight of the smallest path, then we define $S_N = \min_{R_N} X_N$ where we minimize over all possible paths R_N from 1 to N. Now let T_N be the smallest path of a particle, then T_N is the number of transports in a path with weight S_N.

Now, let $Y_N = (Y_N(t);\ t \geq 0)$ be a continuous time Markov chain which is a pure birth process with state space $\{1, 2, ..., N\}$. The random variable $Y_N(t)$ represents the number of places that can be reached from place 1 in a travel time less than or equal to t. Then the birth rate of this process is $n(N-n)$. For this, we describe that the following bijection:
Bijection. The process $Y_N = (Y_N(t);\ t \geq 0)$ starts at time 0 with one particle and will eventually be absorbed in place N. Let $Z_N = (Z_N(t);\ t \geq 0)$ denotes the number of distinct places (including place 1) that can be reached over the exponential steps starting from place 1 within time t. Thus, the two processes are identical in distribution (This follows from the memoryless property of the exponential distribution). When n places are reached, each of these n places can be connected to the set of $N-n$ remaining places over $N-n$ different step. This bijection shows that the rate of Y_N is $\lambda_n = n(N-n)$.

A tree with n vertices labeled $1, 2, ..., n$ is a recursive tree if the node labeled with 1 is distinguished as the root, and for each k such that $2 \leq k \leq n$, the labels of the vertices in the unique path from the root to the node labeled with k form an increasing sequence. The usual model of randomness on the space of n node recursive trees is to assume that all $(n-1)!$ trees are equally likely. It is easy to see that given a random tree T_{n-1} on $n-1$ nodes, we obtain a random tree on n nodes by choosing a node (a parent) of T_{n-1} uniformly at random and joining a node labeled n (a child) to it (see Javanian and Vahidi-Asl [8]).

Definition 2.1 *[9] The depth of a node in a tree is the number of edges from the root node to that node .*

The evolution of the above discussed process can be visualized by a random recursive tree of N nodes. Also each birth in the pure birth process corresponds to connecting an edge of unit length randomly to one of the existing nodes in the associated tree.

3 The Main Results

As our first result we prove the following Lemma.

Lemma 3.1 *Let $B(\alpha, \beta)$ be the Beta function and $\phi_{T_N}(t)$ be the probability generating function of T_N. Then*

$$\begin{aligned} \mathbf{E}[T_N] &\sim \log N + \gamma - 1, \\ \mathbf{V}ar[T_N] &\sim \log N + \gamma - \frac{\pi^2}{6}, \\ \phi_{T_N}(t) &= \frac{N\left(\frac{(N+t+2)(N+t-1)}{B(N+1,t+1)} - \frac{1}{N}\right)}{N-1}, \end{aligned} \tag{3.1}$$

where γ is Euler's constant.

Proof. By bijection introduced in Section 2, the random variable T_N is equal to the depth D_N of particle N in the recursive tree. Thus proof is completed (see Smythe and Mahmoud [12] for details).□

Lemma 3.2 *Let $\phi_{S_N}(t)$ be the probability generating function of S_N. Then*

$$\phi_{S_N}(t) = \frac{1}{N-1} \sum_{j=1}^{N-1} \prod_{i=1}^{j} \frac{\lambda_i}{\lambda_i - t}. \tag{3.2}$$

Proof. Suppose that $Y_N(t) = n$ and that the associated tree has n nodes. After an exponential time with rate $n(N-n)$ the pure birth process gives birth to a new node, which is born with equal probability out of any of the n nodes. In the tree we connect the new node to one of the n existing nodes by a unit edge with equal probability. Since the tree of size n is uniform, it follows that the final tree with N nodes is uniform. Then each of the $N-1$ possibilities of positions for node N is equally likely and the generating function of the $\sum_{i=1}^{j} E_i$, where $E_i \sim Exp(i(N-i))$ equals: $\phi_j(t) = \prod_{i=1}^{j} \frac{\lambda_i}{\lambda_i - t}$. Thus

$$\phi_{S_N}(t) = \frac{1}{N-1} \sum_{j=1}^{N-1} \phi_j(t)$$

and proof is completed.□

Now, let transports are possible or impossible independently with probability p and $1-p$, respectively. The above discussion was based on the fact that from each node in a cluster of size n there are a constant number $N-n$ of outgoing links. Now each node in the cluster of the root when this cluster has size n, the number of outgoing links is binomial with parameters $N-n$ and p. As indicated in Lemma 3.1, we expect that the probability that T_N exceeds a large multiple of $\log N$ is small. If T_N is bounded by a multiple of $\log N$, then the exponential weights over the shortest path are likely to be bounded by another multiple of $\log N$ times the typical weight over each edge of the shortest path. These typical weights are of order $(Np)^{-1}$. The size of a typical weight of an edge belonging to the shortest path follows, because each node has on the average Np edges and the minimum of Np independent exponentials each with weight 1 has expectation $(Np)^{-1}$.

Theorem 3.1 *(Cramér's theorem) Take a sequence of IID copies of the random variable $X_1, ..., X_n$ and consider the sum S_n of the first copies. The cumulative generating function of S_n is $nK_X(t)$ where $K_X(t) = \log \mathbf{E}[e^{tX}]$. Then*

$$\log P\left(\frac{S_n}{n} \geq \epsilon\right) \leq \inf_{t\downarrow 0} -nt\epsilon + nK_X(t).$$

Theorem 3.2 *There exists constant $\epsilon > 0$ such that for Np sufficiently large,*

$$F_{S_N}\Big(\frac{\log N}{Np}\Big) \geq 1 - \frac{1}{N^\epsilon}. \tag{3.3}$$

Proof. If W_n is the sum of n independent exponentials with mean 1, then $W_n \sim Gamma(n, 1)$. Thus (Hofstad and et. al [7]),

$$S_N \leq \frac{2W_{4j+1}}{Np},$$

where $j = \lceil \log \sqrt{N}/\log 2 \rceil$ and $\lceil y \rceil$ is the smallest integer larger than y. Apply Cramér's theorem to W_{4j+1},

$$\begin{aligned} F_{S_N}\Big(\frac{\log N}{Np}\Big) &= P\Big(S_N \leq \frac{\log N}{Np}\Big) \\ &= P\Big(\frac{2W_{4j+1}}{Np} \geq \frac{\log N}{Np}\Big) \\ &= P\Big(W_{4j+1} \geq \frac{\log N}{2}\Big) \\ &\geq 1 - \frac{1}{N^\epsilon}. \end{aligned}$$

□

In the following theorem , we give a lower bound for the distribution function of the smallest path at point $\log N$ as Np is large.

Theorem 3.3 *There exists constant $\epsilon > 0$ such that for Np sufficiently large,*

$$F_{T_N}(\log N) \geq 1 - \frac{2}{N^\epsilon}.$$

Proof. By definition and applying Cramér's theorem, we have

$$\begin{aligned} & F_{T_N}(\log N) \\ = \; & P(T_N \leq \log N) \\ = \; & 1 - P(T_N > \log N) \\ = \; & 1 - P\Big(T_N > \log N, S_N > \frac{\log N}{Np}\Big) \\ - \; & P(T_N > \log N, S_N \leq \frac{\log N}{Np}) \\ \geq \; & 1 - \frac{1}{N^\epsilon} - P(W_{[\log N]} \leq \log N) \\ \geq \; & 1 - \frac{1}{N^\epsilon} - \frac{1}{N^\epsilon} = 1 - \frac{2}{N^\epsilon}. \end{aligned}$$

□

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