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Numerical solution of the system of Volterra integral equations of the first kind

A. Armand * † , Z. Gouyandeh ‡

Abstract

This paper presents a comparison between variational iteration method (VIM) and modified variational iteration method (MVIM) for approximate solution a system of Volterra integral equation of the first kind. We convert a system of Volterra integral equations to a system of Volterra integro-differential equations that use VIM and MVIM to approximate solution of this system and hence obtain an approximation for system of Volterra integral equations. Some examples are given to show the pertinent features of this methods.

Keywords : Volterra integral equation of the first kind; Variational iteration method; Modified variational iteration method.

1 Introduction

THe variational iteration method established in 1999 by He [16]-[21] as a modification of a general Lagrange multiplier method [23]. Insight into the solution procedure of the VIM shows some disadvantages, namely, repeated computations of unneeded terms, which consumes time and effort [3]. However for linear problems, exact solution can be obtained by the only one iteration step due to the fact that the Lagrange multiplier can be exactly identified [29].
As we know the many natural phenomena have been modeled by linear and nonlinear equations, like ordinary or partial differential equations, integral and integro- differential equations [9] that the exact and numerical solutions of this equations are studied in several papers (see e.g. [1, 2, 10, 25]).
In the one decade, the application of the VIM linear and nonlinear problems has been devoted by scientists and engineers, for example, nonlinear systems of ordinary differential equations [11], boundary value problems [22], delay differential equations [19], high order differential equations[1], integral equation [27] and integro-differential equations [28, 26]. In 2007, Abassy et al. proposed the modified variational iteration method (MVIM) for solution some nonlinear problem [3, 4]. They also applied MVIM with Laplace transforms [8] and the Pad technique for solving nonlinear partial differential equations [7]. Moreover this method is used for solving non-

*Corresponding author. atefeh.armand@ymail.com
†Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
‡Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

linear non-homogeneous and homogeneous differential equations in [5, 6].
In this paper, we aim study the solution of systems of Volterra integral equations of the first kind. Some other authors have studied solutions of systems of Volterra integral equations of the first kind by using various methods, such as Adomian decomposition method [24, 12] and Homotopy perturbation method [13, 14]. Now we propose the variational iteration method and the modified variational iteration method for solving systems of Volterra integral equations of the first kind.
The structure of this paper is organized as follows: In the next Section, the VIM and MVIM are introduced. The VIM and MVIM for solving systems of Volterra integral equations of the first kind are presented in Section 3. In Section 4, some numerical results are given to clarify the details and efficiency of the methods . Section 5 ends this paper with a brief conclusion.

2 Methodology

The main points of variational iteration method and its modification are presented in this section, for more details can be refer to [5, 1].

2.1 Description of VIM

Consider the following general non-linear initial value problem

$$L[u(x)] + R[u(x)] + N[u(x)] = g(x), \qquad (2.1)$$

with initial condition

$$u^{(i)}(0) = \alpha_i \qquad i = 0, 1, ..., s-1.$$

where $L = \frac{\partial^s}{\partial x^s}$, $s = 1, 2, 3, ...$ is the highest order of derivative, R is a linear differential operator of order less than s, N expresses the nonlinear terms and $g(x)$ is a nonzero analytical function.
The basic character of the method is to construct a correction functional for the Eq.(2.1), which reads

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)[L[u_n(t)]] + R[\widetilde{u}_n(t)] + N[\widetilde{u}_n(t)] - g(t)dt, n \geq 0 \qquad (2.2)$$

where λ is a general Lagrange multiplier which can be identified optimally via the variation theory, u_n is the nth approximate solution, and the function $\widetilde{u}_n$ is restricted variation [15] i.e.$\delta\widetilde{u}_n = 0$. Therefore, with λ determined and by using iteration formula (2.2), the successive approximations $u_{n+1}(x), n \geq 0$ of the solution $u(x)$ will be readily obtained upon using the obtained zeroth approximation u_0 may be selected by any function that satisfies at initial conditions. Consequently, the exact solution may be obtained by using

$$u(x) = \lim_{n \to \infty} u_n(x).$$

2.2 Description of MVIM

Let Eq.(2.1), according modified variational iteration method that present in [3], we can construct the following iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)[R(u_n - u_{n-1}) + (G_n - G_{n-1})] - (a_{ns}t^{ns} + a_{ns+1}t^{ns+1} + \cdots + a_{s(n+1)-1}t^{s(n+1)-1}]\mathrm{d}t \qquad (2.3)$$

where λ is a general Lagrange multiplier, which is identified optimally via variational theory, $G_n(t)$ is a polynomial of degree $s(n+1)-1$ in t and is obtained from

$$Nu_n(t) = G_n(t) + O(t^{s(n+1)}),$$

and a_n is obtained by Taylors series expansion of $g(t)$ where $g(t) = \sum_{n=0}^{\infty} a_n t^n$.
For obtain an approximate solution for Eq.(2.1), we can use iteration formula (2.3) by

$$u_{-1} = 0,$$
$$u_0 = \alpha_0 + \alpha_1 t + ... + \frac{\alpha_{s-1}}{(s-1)!}t^{s-1}.$$

3 Main Section

We consider the general system of Volterra integral equation of the first kind as follows[13]:

$$f_i(x) = \int_0^x K_i(x,t)G_i(u_1(t), u_2(t), ..., u_m(t))dt \quad i = 1, 2, ..., m. \qquad (3.4)$$

If $G_i(u_1(t), u_2(t), ..., u_m(t))$ are linear, the system (3.4) could be represented as follows:

$$f_i(x) = \int_0^x \sum_{j=1}^m K_{ij}(x,t)u_j(t)dt \quad i = 1, 2, ..., m. \tag{3.5}$$

where $K_{ij}(x,t)$,$i,j = 1, 2, ..., m$ are kernel of integral equations and $u_j(x), j = 1, 2, ..., m$ are the solution to be determined. We assume that system (3.4) have the unique solution [14]. We change Eq.(2.1) to a system of ordinary integro-differential equation or a system of ordinary differential equation.

First we differentiate twice from both sides of system (3.5), with respect to x:

$$\begin{aligned} f_i''(x) &= \sum_{j=1}^m K_{ij}'(x,x)u_j(x) + \sum_{j=1}^m K_{ij}(x,x) \\ & u_j'(x) + \sum_{j=1}^m \frac{\partial K_{ij}(x,t)}{\partial x}u_j(t)\Big|_{t=x} \\ & + \int_0^x \sum_{j=1}^m \frac{\partial^2 K_{ij}(x,t)}{\partial x^2}u_j(t)dt, \end{aligned}$$

then

$$\begin{aligned} u_i'(x) &= \frac{f_i''(x)}{K_{ii}(x,x)} - \sum_{j=1}^m \frac{K_{ij}'(x,x)}{K_{ii}(x,x)}u_j(x) \\ & - \sum_{\substack{j=1 \\ j\neq i}}^m \frac{K_{ij}(x,x)}{K_{ii}(x,x)}u_j'(x) \\ & - \frac{1}{K_{ii}(x,x)} \sum_{j=1}^m \frac{\partial K_{ij}(x,t)}{\partial x}u_j(t)\Big|_{t=x} \\ & - \frac{1}{K_{ii}(x,x)} \int_0^x \sum_{j=1}^m \frac{\partial^2 K_{ij}(x,t)}{\partial x^2}u_j(t)dt \\ & i = 1, 2, ..., m \end{aligned} \tag{3.6}$$

with initial condition $u_i(0) = \alpha_i, \quad i = 1, 2, ..., m$. So, for solving the system of Volterra integral equation of the first kind (3.5) is sufficient that we obtain the solution of system of Volterra integro-differential equation (3.6).

3.1 Using VIM

According to the VIM, to solve the system of Volterra integro-differential equation (3.6), the correction functional is constructed as follows

$$\begin{aligned} u_i^{(n+1)}(x) &= u_i^{(n)}(x) + \int_0^x \lambda_i(x,t)[u_i^{'(n)}(t) \\ & - \frac{f_i''(t)}{K_{ii}(t,t)} + \sum_{j=1}^m \frac{K_{ij}'(t,t)}{K_{ii}(t,t)}\tilde{u}_j^{(n)}(t) + \\ & \sum_{\substack{j=1 \\ j\neq i}}^m \frac{K_{ij}(t,t)}{K_{ii}(t,t)}\tilde{u}_j^{'(n)}(t) \\ & + \frac{1}{K_{ii}(t,t)} \sum_{j=1}^m \frac{\partial K_{ij}(t,s)}{\partial t}\tilde{u}_j^{(n)}(s)\Big|_{s=t} \\ & + \frac{1}{K_{ii}(t,t)} \int_0^t \sum_{j=1}^m \frac{\partial^2 K_{ij}(t,s)}{\partial t^2}\tilde{u}_j^{(n)}(s)ds]dt \\ & i = 1, 2, ..., m \end{aligned} \tag{3.7}$$

where the symbol (n) is the number of iteration steps. Now making the correction functional stationary ,and noticing that $\delta u_i^{(n)}(0) = 0$,

$$\begin{aligned} & \delta u_i^{(n+1)}(x) = \delta u_i^{(n)}(x) \\ & + \delta \int_0^x \lambda_i(x,t)\Big[u_i^{'(n)}(t) - \frac{f_i''(t)}{K_{ii}(t,t)} \\ & + \sum_{j=1}^m \frac{K_{ij}'(t,t)}{K_{ii}(t,t)}\tilde{u}_j^{(n)}(t) + \sum_{\substack{j=1 \\ j\neq i}}^m \frac{K_{ij}(t,t)}{K_{ii}(t,t)}\tilde{u}_j^{'(n)}(t) \\ & + \frac{1}{K_{ii}(t,t)} \sum_{j=1}^m \frac{\partial K_{ij}(t,s)}{\partial t}\tilde{u}_j^{(n)}(s)\Big|_{s=t} \\ & + \frac{1}{K_{ii}(t,t)} \int_0^t \sum_{j=1}^m \frac{\partial^2 K_{ij}(t,s)}{\partial t^2}\tilde{u}_j^{(n)}(s)ds\Big]dt \\ & = \delta u_i^{(n)}(x) + \lambda_i(x,t)\delta u_i^{(n)}(t)\Big|_{t=x} \\ & - \int_0^x \frac{\partial \lambda_i(x,t)}{\partial t}\delta u_i^{(n)}(t)dt = 0 \\ & i = 1, 2, ..., m \end{aligned}$$

for all variations $\delta u_i,\ i = 1, 2, ..., m$,implying following stationary conditions:

$$
\begin{aligned}
-\frac{\partial \lambda_i(x,t)}{\partial t} &= 0 \qquad i = 1,2,...,m\\
1+\lambda_i(x,t)\Big|_{t=x} &= 0 \qquad i = 1,2,...,m
\end{aligned}
$$

The Lagrange multiplier, therefore can be readily identified $\lambda_i(x,t) = -1,\ i = 1,2,...,m.$ Then by substituting λ in (3.7), we obtain following iteration formula

$$
\begin{aligned}
&u_i^{(n+1)}(x) = u_i^{(n)}(x) - \int_0^x [u_i^{'(n)}(t) -\\
&\frac{f_i''(t)}{K_{ii}(t,t)} + \sum_{j=1}^{m} \frac{K'_{ij}(t,t)}{K_{ii}(t,t)} u_j^{(n)}(t)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{K_{ij}(t,t)}{K_{ii}(t,t)} u_j^{'(n)}(t) +\\
&\frac{1}{K_{ii}(t,t)} \sum_{j=1}^{m} \frac{\partial K_{ij}(t,s)}{\partial t} u_j^{(n)}(s)\Big|_{s=t}\\
&+\frac{1}{K_{ii}(t,t)} \int_0^t \sum_{j=1}^{m} \frac{\partial^2 K_{ij}(t,s)}{\partial t^2} u_j^{(n)}(s)ds]dt\\
&i = 1,2,...,m
\end{aligned}
$$

3.2 Using MVIM

The modified variational iteration method introduces a iteration formula for Eq.(3.6) as follows:

$$
\begin{aligned}
&u_i^{(n+1)}(x) = u_i^{(n)}(x) - \int_0^x [R(u_i^{(n)} - u_i^{(n-1)})\\
&+(G_i^{(n)} - G_i^{(n-1)}) - g_{in}t^n]dt \quad i = 1,2,...,m,
\end{aligned}
$$

such that $Nu_i^{(n)}(t) = G_i^{(n)}(t) = 0, \frac{f_i''(x)}{K_{ii}(x,x)} = \sum_{n=0}^{\infty} g_{in}t^n,\ i = 1,2,...,m,$ and

$$
\begin{aligned}
&Ru_i(x) = \sum_{j=1}^{m} \frac{K'_{ij}(x,x)}{K_{ii}(x,x)} u_j(x)\\
&+\sum_{\substack{j=1\\ j\neq i}} \frac{K_{ij}(x,x)}{K_{ii}(x,x)} u'_j(x) +\\
&\frac{1}{K_{ii}(x,x)} \sum_{j=1}^{m} \frac{\partial K_{ij}(x,t)}{\partial x} u_j(t)\Big|_{t=x}\\
&+\frac{1}{K_{ii}(x,x)} \int_0^x \sum_{j=1}^{m} \frac{\partial^2 K_{ij}(x,t)}{\partial x^2} u_j(t)dt,\\
&i = 1,2,...,m \qquad (3.8)
\end{aligned}
$$

In the first step, by iteration formula (3.8) with initial approximation

$$
\begin{aligned}
&u_i^{(-1)}(x) = 0, \quad u_i^{(0)}(x) = u_i(0) = \alpha_i\\
&i = 1,2,...,m
\end{aligned}
$$

we can approximate solution of Eq.(3.5).

4 Illustrative Examples

To show the efficiency of the two methods are described in the previous parts, we present some examples. This tests are chosen such that there exist analytical solutions for them to give an obvious overview of the methods presented in this paper.

Example 4.1 *Consider system of Volterra integral equations of the first kind as follows [13]:*

$$
\begin{cases}
\int_0^x (u(t) + (x-t)u(t)v(t))dt =\\
-\frac{3}{4} + \frac{1}{2}x + \frac{1}{2}x^2 + \frac{1}{12}x^4 + e^x - \frac{1}{4}e^{2x}\\
\int_0^x (v(t) + (x-t)u(t)v(t))dt =\\
\frac{5}{4} + \frac{1}{2}x + \frac{1}{2}x^2 + \frac{1}{12}x^4 - e^x - \frac{1}{4}e^{2x}
\end{cases} \qquad (4.9)
$$

The exact solutions are $u(x) = x + e^x$, $v(x) = x - e^x$.

Following the above procedure of solving the system of Volterra integral equation by twice differentiation from both sides of system (4.9), we drive

$$\begin{cases} u'(x) = 1 + x^2 + e^x - e^{2x} - u(x)v(x) \\ v'(x) = 1 + x^2 - e^x - e^{2x} - u(x)v(x) \end{cases} \quad (4.10)$$

with initial condition $u(0) = 1 \quad , v(0) = -1$. The VIM and MVIM methods are used to approximate the solutions.

- VIM

According to the variational iteration method, to solve the system (4.10), we can construct the following correction functional:

$$\begin{cases} u_{n+1}(x) = u_n(x) + \int_0^x \lambda_1(x,t)[u'_n(t) + \\ \widetilde{u}_n(t)\widetilde{v}_n(t) - 1 - t^2 - e^t + e^{2t}]dt \\ v_{n+1}(x) = v_n(x) + \int_0^x \lambda_2(x,t)[v'_n(t) + \\ \widetilde{u}_n(t)\widetilde{v}_n(t) - 1 - t^2 + e^t + e^{2t}]dt \end{cases} \quad (4.11)$$

Making the above correction functional stationary, and noticing that $\delta u_n(0) = \delta v_n(0) = 0$, conclude that

$$\begin{cases} \delta u_{n+1}(x) = (1 + \lambda_1(x,t))\delta u_n(t)\Big|_{t=x} - \\ \int_0^x \frac{\partial \lambda_1(x,t)}{\partial t} \delta u_n(t)dt = 0, \\ \delta v_{n+1}(x) = (1 + \lambda_2(x,t))\delta v_n(t)\Big|_{t=x} - \\ \int_0^x \frac{\partial \lambda_2(x,t)}{\partial t} \delta v_n(t)dt = 0 \end{cases} \quad (4.12)$$

For δu_{n+1}, δv_{n+1}, implying following stationary conditions:

$$-\frac{\partial \lambda_i(x,t)}{\partial t} = 0 \qquad i = 1,2$$

$$1 + \lambda_i(x,t)\Big|_{t=x} = 0 \qquad i = 1,2.$$

The Lagrange multiplier, therefore can be readily identified $\lambda_i(x,t) = -1$, $i = 1,2$. Then by substituting λ in (4.11), we obtain following iteration formula

$$\begin{cases} u_{n+1}(x) = u_n(x) - \int_0^x [u'_n(t) + \\ u_n(t)v_n(t) - 1 - t^2 - e^t + e^{2t}]dt \\ v_{n+1}(x) = v_n(x) - \int_0^x [v'_n(t) + \\ u_n(t)v_n(t) - 1 - t^2 + e^t + e^{2t}]dt \end{cases} \quad (4.13)$$

Therefore the approximation to the solutions can be readily obtained by initial function $u_0(x) = 1$, $v_0(x) = -1$ and iteration formula (4.13).

- MVIM

Using MVIM for solving (4.14) leads to: $Ru = 0$,$Rv = 0$ and $Nu(t) = Nv(t) = u(t)v(t)$, $s = 1$ same as VIM obtain $\lambda_i(x,t) = -1$, $i = 1,2$ and $g(t) = 1 + t^2 + e^t - e^{2t}$,$f(t) = 1 + t^2 - e^t - e^{2t}$. So, the modified variational iteration formula is constructed as

$$\begin{cases} u_{n+1}(x) = u_n(x) \\ \quad - \int_0^x [(G_n - G_{n-1}) - a_n t^n]dt \\ v_{n+1}(x) = v_n(x) \\ \quad - \int_0^x [(F_n - F_{n-1}) - b_n t^n]dt \end{cases} \quad (4.14)$$

where $u_{-1}(x) = v_{-1}(x) = 0$, $u_0(x) = 1$, $v_0(x) = -1$ and $G_n(t)$, $F_n(t)$ are polynomials of degree n, which are obtained from the formula

$$u_n(t)v_n(t) = G_n(t) + O(t^{n+1}),$$
$$u_n(t)v_n(t) = F_n(t) + O(t^{n+1}).$$

and a_n,b_n obtained by the Taylors series expansion, i.e.

$$1 + t^2 + e^t - e^{2t} = \sum_{n=0}^{\infty} a_n t^n,$$
$$1 + t^2 - e^t - e^{2t} = \sum_{n=0}^{\infty} b_n t^n.$$

The results corresponding for fifth iteration of VIM and MVIM are presented in Table.(??) and Fig.(??)

Example 4.2 *Consider the following system of Voltrra integral equations of the first kind, with exact solutions,* $u(x) = e^x, \quad v(x) = e^{-x}$.

$$\begin{cases} \int_0^x (u(t) + xu(t)v(t))dt = \\ \qquad e^x + \frac{x^2}{2} - 1, \\ \int_0^x (v(t) + xu(t)v(t))dt = \\ \qquad - e^{-x} + \frac{x^2}{2} + 1. \end{cases} \quad (4.15)$$

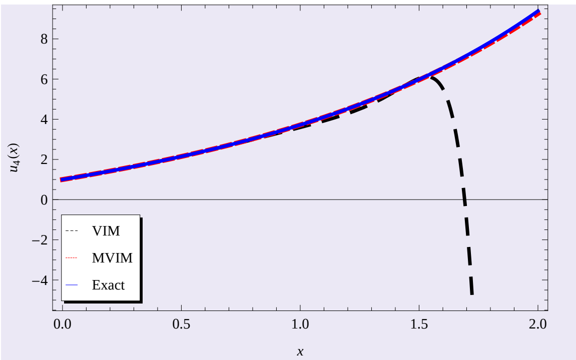
By twice differentiation from both sides of system (4.15), we have

$$\begin{cases} u'(x) + u(x)v(x) + x(u'(x)v(x) \\ +v'(x)u(x)) = 1 + e^x, \\ v'(x) + u(x)v(x) + x(u'(x)v(x) \\ +v'(x)u(x)) = 1 - e^{-x} \end{cases} \quad (4.16)$$

with initial condition $u(0) = 1$,$v(0) = 1$.

	VIM		$MVIM$	
x	$u_5(x)$	$v_5(x)$	$u_5(x)$	$v_5(x)$
0.1	6.3890×10^{-6}	5.8179×10^{-6}	1.40898×10^{-9}	1.40898×10^{-9}
0.2	5.2625×10^{-5}	4.8082×10^{-5}	9.14935×10^{-8}	9.14935×10^{-8}
0.3	5.3892×10^{-6}	1.5973×10^{-5}	1.05758×10^{-6}	1.05758×10^{-6}
0.4	5.7466×10^{-6}	1.2563×10^{-5}	6.03097×10^{-6}	6.03097×10^{-6}
0.5	1.4347×10^{-4}	1.6170×10^{-4}	2.33540×10^{-5}	2.33540×10^{-5}
0.6	6.4121×10^{-4}	6.2985×10^{-4}	7.08004×10^{-5}	7.08004×10^{-5}
0.7	2.1286×10^{-3}	2.0919×10^{-3}	1.81291×10^{-4}	1.81291×10^{-4}
0.8	6.2049×10^{-3}	6.2446×10^{-3}	4.10262×10^{-4}	4.10262×10^{-4}
0.9	1.4992×10^{-2}	1.5062×10^{-2}	8.44861×10^{-4}	8.44861×10^{-4}
1.0	3.2717×10^{-2}	3.2765×10^{-2}	1.61516×10^{-3}	1.61516×10^{-3}

Table 1: Absolute errors of Example 4.1.



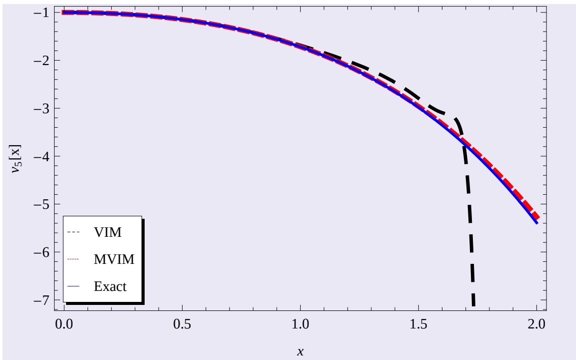


Fig.1. The numerical results and exact solution of Example 4.1.

- VIM

Solving system (4.16) by VIM conclude the following correction functional:

$$\begin{cases} u_{n+1}(x) = u_n(x) - \int_0^x [u_n'(t) + u_n(t)v_n(t) \\ +t(u_n'(t)v_n(t) + v_n'(t)u_n(t)) - 1 - e^t]dt \\ v_{n+1}(x) = v_n(x) - \int_0^x [v_n'(t) + u_n(t)v_n(t) \\ +t(u_n'(t)v_n(t) + v_n'(t)u_n(t)) - 1 + e^{-t}]dt \end{cases} \quad (4.17)$$

Starting with initial approximations $u_0(x) = 1$,$v_0(x) = 1$, by the iteration formula (4.17), we calculate fourth approximation of exact solution. The results is shown in Table.?? and Fig.??.

- MVIM

Solving system (4.16) using MVIM we found that:$Ru(t) = Rv(t) = 0$, $Nu(t) = Nv(t) = u(t)v(t) + t(u'(t)v(t) + v'(t)u(t))$, $g(t) = 1 + e^t$, $f(t) = 1 - e^{-t}$ and $s = 1$ which lead to $\lambda_i(x,t) = -1$, $i = 1, 2$. So,we have the following MVIM formula

$$\begin{cases} u_{n+1}(x) = u_n(x) \\ - \int_0^x [(G_n - G_{n-1}) - a_n t^n]dt \\ v_{n+1}(x) = v_n(x) \\ - \int_0^x [(F_n - F_{n-1}) - b_n t^n]dt \end{cases} \quad (4.18)$$

where $u_{-1}(x) = v_{-1}(x) = 0$, $u_0(x) = 1 = v_0(x) = 1$ and $G_n(t)$, $F_n(t)$ are polynomials of degree n, such that

$$\begin{aligned} u_n(t)v_n(t) + t(u_n'(t)v_n(t) + v_n'(t)u_n(t)) \\ = G_n(t) + O(t^{n+1}), \\ u_n(t)v_n(t) + t(u_n'(t)v_n(t) + v_n'(t)u_n(t)) \\ = F_n(t) + O(t^{n+1}). \end{aligned}$$

and a_n,b_n obtained by the Taylors series expansion of $g(t)$ and $f(t)$ respectively around $t = 0$

$$\begin{aligned} 1 + e^t &= \sum_{n=0}^{\infty} a_n t^n, \\ 1 - e^{-t} &= \sum_{n=0}^{\infty} b_n t^n. \end{aligned}$$

The results corresponding for fourth iteration of MVIM are presented in Table.?? and Fig.??

5 Conclusion

In this paper, the variational iteration method and its modification were successfully employed for solving systems of Volterra integral equations of the first kind. For convenient in explanation of the methods the linear integral equations were considered, but examples were investigated for non-linear system. The results shown that MVIM reduces the size of calculations and gives an accurate power series solution which converges rapidly to the closed form solution in the neighborhood of the initial point.
The computations associated with the examples in this paper were performed using Mathematica 7.

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	VIM		$MVIM$	
x	$u_4(x)$	$v_4(x)$	$u_4(x)$	$v_4(x)$
0.1	9.8790×10^{-3}	1.3734×10^{-1}	8.4742×10^{-8}	8.1964×10^{-8}
0.2	3.8043×10^{-2}	1.3880×10^{-1}	2.7581×10^{-6}	2.5802×10^{-6}
0.3	8.0221×10^{-2}	1.1689×10^{-2}	2.1307×10^{-5}	1.9279×10^{-5}
0.4	1.3008×10^{-1}	3.3147×10^{-1}	9.1364×10^{-5}	7.9953×10^{-5}
0.5	1.8002×10^{-1}	8.4038×10^{-1}	2.8377×10^{-4}	2.4017×10^{-4}
0.6	2.1951×10^{-1}	1.5596	7.1880×10^{-4}	5.8836×10^{-4}
0.7	2.2624×10^{-1}	2.5035	1.5818×10^{-3}	1.2521×10^{-3}
0.8	1.3955×10^{-1}	3.6535	3.1409×10^{-3}	2.4043×10^{-3}
0.9	2.0226×10^{-1}	4.8962	5.7656×10^{-3}	4.2678×10^{-3}
1.0	1.1772	5.9089	9.9484×10^{-3}	7.1205×10^{-3}

Table 2: Absolute errors of Example (4.2)

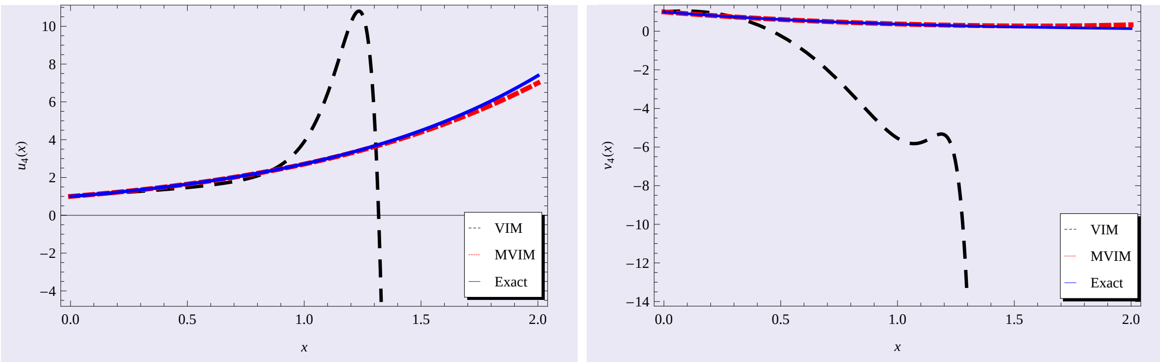

Fig 2. The numerical results and exact solution of Example 4.2.

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Atefeh Armand received B.Sc and M.Sc degrees in Applied Mathematics from Shahr-e-Rey Branch and Science and Research Branch, Islamic Azad University, respeactively. Now she is a PhD student of Applied Mathematics, Numerical analysis field at Science and Research Branch, Islamic Azad University. Her current research interests are in numerical analysis and fuzzy numerical analysis.

Zeinab Gouyandeh is a candidate for PhD at the Department of Mathematics, Science and Research Branch, Islamic Azad University,Tehran, Iran. Now, she is working on numerical analysis and fuzzy numerical analysis.