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Research Article

A new method to determine a well-dispersed subsets of non-dominated vectors for MOMILP problem

G. Tohidi *†, SH. Razavyan ‡

Abstract

Multi-objective optimization is the simultaneous consideration of two or more objective functions that are completely or partially in conflict with each other. The optimality of such optimizations is largely defined through the Pareto optimality. Multiple objective integer linear programs (MOILP) are special cases of multiple criteria decision making problems. Numerous algorithms have been designed to solve MOILP and multiple objective mixed integer linear programs. However, MOILP have not received the algorithmic attention that continuous problems have. This paper uses the data envelopment analysis (DEA) technique to find a well-dispersed non-dominated vectors of multiple objective mixed integer linear programming (MOMILP) problem with bounded or unbounded feasible region, while the previous methods consider only problems with bounded feasible region. To this end, it uses the L_1-norm and the modified slack-based measure (MSBM) model. The proposed method does not need the filtering procedures and it ranks the elements of well-dispersed non-dominated vectors of MOMILP problem. The proposed algorithm is illustrated by using two numerical examples.

Keywords : Well-dispersed non-dominated vectors; DEA; L_1-norm; MOMILP; Non-dominated vectors.

1 Introduction

Since the most real-life problems include conflicting objectives, multiple objective optimization provides a means for obtaining more realistic models [1, 2, 8]. Multiple objective mixed integer linear programming (MOMILP) problem is an important research area as many practical situations require discrete representations by integer variables and many decision makers have to deal with several objectives [16]. Some noteworthy practical environments where the MOILP problems find their applications are supply chain design, logistics planning, scheduling and financial planning.

Numerous algorithms have been designed to solve MOILP [5, 10, 11, 13, 14, 16] and MOMILPs [9, 13]. Sylva and Crema's [13] proposed an algorithm to find well-dispersed subsets of non-dominated vectors for MOMILP with bounded feasible region. But, in some cases feasible region of a MOMILP problem is unbounded. Therefore, a MOMILP problem can have infinite objective values [15]. These cases have not been considered in [11] and [13].

Data envelopment analysis (DEA), provides a nonparametric methodology for evaluating the efficiency of each of a set of comparable Decision Making Units (DMUs), relative to one another. Charnes et al. [4], CCR model, proposed the DEA technique, which allows any DMU to select

*Corresponding author. ghatohidi@yahoo.com
†Department of Mathematics, Islamic Azad University, Central Tehran Branch, Tehran, Iran.
‡Department of Mathematics, Islamic Azad University, South Tehran Branch, Tehran, Iran.

their most favorable weights while requiring the resulting ratios of the sum of weighted outputs to the sum of weighted inputs of all DMUs to be less than or equal to a constant value. After introducing the first model in DEA, the CCR model by Charnes et al. [4], Banker et al. [3] developed the DEA technique by providing the BCC model. Nowadays DEA has allocated a wide variety of research in Operations Research to itself. For instance, Jahanshahlo et al. [7] used DEA technique to find efficient solutions of a 0-1 multi objective programming problem.

In this paper, we use the modified slack-based measure (MSBM) model [12] as a DEA technique and L_1-norm to propose an algorithm to find a well-dispersed non-dominated vectors of MOMILP problem with bounded and unbounded feasible regions. The density of the well-dispersed of non-dominated vector can be determined by using decision maker opinions.

The paper is organized as follows. Section 2 presents a brief background about MOMILP problem. Section 3 introduces the proposed method to find the well-dispersed subsets of non-dominated vectors MOMILP problem. Illustration with two numerical examples are given in Section 4. Finally, the concluding results are presented.

2 Preliminaries

2.1 MOMILP problem

An MOMILP problem is a special case of multi objective programming program and can be defined as follows:

$$\begin{aligned} &\{C_1W,\ldots,C_sW\}\\ \text{s.t.}\quad &A_iW\le b_i,\quad i=1,\ldots,m \qquad (2.1)\\ &W\ge 0, w_j\in Z^+, j\in J \end{aligned}$$

where $C_r=(c_{1r},\ldots,c_{nr})$ $(r=1,\ldots,s)$, $A_i=(a_{i1},\ldots,a_{in})$ $(i=1,2,\ldots,m)$, $J\subseteq\{1,\ldots,n\}$, $Z^+=\{0,1,2,\ldots\}$ and $W=(w_1,\ldots,w_n)^T$. The set X, which is defined as follows:

$$X=\Big\{W \mid A_iW\le b_i, i=1,\ldots,m, \quad W\ge 0, w_j\in Z^+, j\in J\Big\} \qquad (2.2)$$

is called the set of feasible solutions of problem (2.1). Corresponding to each $W\in X$ the vector Y is defined as follows [6]:

$$Y=(y_1,\cdots,y_s)^T=(C_1W,\ldots,C_sW)^T. \qquad (2.3)$$

Definition 2.1 *The vector $Y=(y_1,\ldots,y_s)^T$ dominates the vector $Y^o=(y_1^o,\ldots,y_s^o)^T$ if for each $r(r=1,\ldots,s)$, $y_r\ge y_r^o$ and there is at least one l such that $y_l>y_l^o$.*

Definition 2.2 *Let $F=\{Y \mid Y=(C_1W,\ldots,C_sW)^T, W\in X\}$. F is called the values space of objective functions in problem (2.1).*

Let $g_r=C_rW_r^*(r=1,\ldots,s)$, where W_r^* is the optimal solution of the following single objective mixed integer programming problem:

$$\begin{aligned} g_r=\max\quad &C_rW\\ \text{s.t.}\quad &W\in X. \end{aligned} \qquad (2.4)$$

Let X be bounded and $g=(g_1,\ldots,g_s)^T=(C_1W_1^*,\ldots,C_sW_s^*)^T$. g is called the ideal vector of model (2.1) [6]. As can be seen, for each $W\in X$ as a feasible solution of problem (2.1), the vector g dominates the vector $Y=(C_1W,\ldots,C_sW)^T\neq g$.

2.2 MSBM model

The objective values of an MOMILP problem are physical quantities and they have true zero points. Hence, we can use the MSBM model with natural negative/positive data to find the efficient solutions of an MOMILP problem. Consider n DMUs ($\text{DMU}_j, j=1,\ldots,n$) where DMU_j consumes the inputs $\mathbf{x}_j=(x_{1j}\ldots,x_{mj})^T$ to produce the outputs $\mathbf{y}_j=(y_{1j}\ldots,y_{sj})^T$. Sharp et al. [12] defined the MSBM model for the case of variable returns to scale (VRS) technology as follows:

$$\begin{aligned} \max \quad & \rho_o^* = \frac{1+\sum_{r=1}^{s} v_r s_r^+/p_{ro}^+}{1-\sum_{i=1}^{m} w_i s_i^-/p_{io}^-} \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \ i = 1,\dots,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \ r = 1,\dots,s \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j, s_i^-, s_r^+ \geq 0, j = 1,\dots,n, \\ & i = 1,\dots,m, r = 1,\dots,s \end{aligned} \tag{2.5}$$

where $\sum_{r=1}^{s} v_r = 1, \sum_{i=1}^{m} w_i = 1$, $p_{ro}^+ = \max_j\{y_{rj}\} - y_{ro}, r = 1,\dots,s$, and $p_{io}^- = x_{io} - \min_j\{x_{ij}\}, i = 1,\dots,m$.

Corresponding to each feasible solution of model (2.1), say W_j, the vector $\mathbf{y}_j$ is defined as $\mathbf{y}_j = (y_{1j},\dots,y_{sj})^T$ where, $y_{rj} = C_r W_j = \sum_{k=1}^{n} c_{rk} w_k$, $r = 1,\dots,s$. We need the DEA techniques to determine a well-dispersed subsets of non-dominated vectors of problem (2.1). To this end, corresponding to W_j as a feasible solution of model (2.1) we consider a DMU, say DMU$_j$, with s outputs $\mathbf{y}_j$ and 1 input. If $m = 1$ and $x_{1j} = 1$ for $j = 1,\dots,n$, then $s_1^- = 0$ and model (2.5) is converted the following model:

$$\begin{aligned} & \rho^* = \max \rho = 1 + \sum_{r=1}^{s} v_r s_r^+/p_{ro}^+ \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \ r = 1,\dots,s \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j, s_r^+ \geq 0, j = 1,\dots,n, r = 1,\dots,s. \end{aligned} \tag{2.6}$$

Model (2.6) is feasible and bounded and it will be used to obtain the relative efficiency of constructed DMUs. In fact, the efficiency of a DMU, say DMU$_o$, by model (2.6) is as $1/\rho_o^* = 1/(1 + \sum_{r=1}^{s} v_r s_r^{+*}/p_{ro}^+)$. When $s_r^{+*} = 0, (r = 1,\dots,s)$, DMU$_o$ is efficient by model (2.6) and vice versa. Hence, we have the following definition.

Definition 2.3 *DMU$_o$ is efficient under model (2.6) (i.e., MSBM-efficient) if and only if $1/\rho_o^* = 1$, i.e. $s_r^{+*} = 0, r = 1,\dots,s$.*

The following Theorem states a relationship between MSBM-efficiency and non-dominated vector of MOMILP problem.

Theorem 2.1 *Let DMU$_o$ be efficient by using model (2.6), then W_o as the corresponding feasible solution of model (2.1) is an efficient solution of model (2.1).*

Proof: Let $1/\rho_o^* = 1$ and by contradiction suppose that W_o is not an efficient solution of (2.1). Therefore, there exists $\overline{W}$ such that

$$C_r(\overline{W}) \geq C_r(W_o), \quad r = 1,\dots,s$$

and the inequality holds strictly for at least one index. That is, there exists $p \in \{1,\dots,s\}$ such that $C_p(\overline{W}) > C_p(W_o)$, i.e., $\overline{y}_{op} > y_{op}$. Hence, there is a feasible solution of model (2.6), say $(\lambda_o = 1, \lambda_j = 0, j \neq o, s_r^+ = 0, r \neq p, s_p^+ > 0)$ such that $1/\rho_o = 1/(1 + s_p^+) < 1$. This is a contradiction.□

We need the dual of model (2.6) as follows to determine the supporting hyperplane of production possibility set (PPS) which has been created by the constructed DMUs.

$$\begin{aligned} \max \quad & g_p = \sum_{r=1}^{s} u_r y_{ro} + u_o \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} + u_o \leq 0, j = 1,\dots,n \\ & u_o \text{ free}, u_r \geq v_r/p_{ro}^+, r = 1,\dots,s. \end{aligned} \tag{2.7}$$

Let $(u^*, u_o^*) = (u_1^*,\dots,u_s^*,u_o^*)$ be an optimal solution of model (2.7). When DMU$_o$ is efficient in model (2.7) $u^*Y + u_o^* = 0$ is the supporting hyperplane on the PPS constructed by efficient DMUs. By using p DMUs (i.e., $Y_1,\dots,Y_p$) the PPS is defined as follows [7]:

$$PPS = \{Y \mid Y \leq \sum_{j=1}^{p} \lambda_j Y_j, \ \sum_{j=1}^{p} \lambda_j = 1, \ \lambda_j \geq 0, j = 1,\dots,p\}.$$

By using the supporting hyperplanes of the above PPS the set PPS^c is defined as PPS^c = R^s-PPS.

3 Well-dispersed subsets of non-dominated vectors for MOMILP problem

This paper uses the DEA technique to find a well-dispersed non-dominated vectors of MOMILP problem with bounded or unbounded feasible region, while the previous methods [11] consider only problems with bounded feasible region. To obtain a well-dispersed subsets of non-dominated vectors of problem (2.1), a feasible solution, say $W \in X$, is specified such that $g - Y = (g_1 - C_1W, \ldots, g_s - C_sW)^T$ is minimized. To this end, the following MOMILP problem is solved:

$$\begin{array}{ll} \min & \{g_1 - C_1W, \ldots, g_s - C_sW\} \\ \text{s.t.} & W \in X. \end{array} \tag{3.8}$$

According to $g_r \geq C_rW$ $(r = 1, \ldots, s, W \in X)$ and by using the L_1-norm we have:

$$\begin{aligned} &\min_{W\in X} \sum_{r=1}^{s} |g_r - C_rW| = \min_{W\in X} \sum_{r=1}^{s} (g_r - C_rW) \\ &= \sum_{r=1}^{s} g_r + \min_{W\in X} \sum_{r=1}^{s} (-C_rW) \\ &= \sum_{r=1}^{s} g_r - \max_{W\in X} \sum_{j=1}^{n} \sum_{r=1}^{s} c_{rj} w_j. \end{aligned}$$

Hence, to find some efficient solutions of the MOMILP problem the following mixed linear integer programming problem is solved:

$$\begin{array}{ll} \theta_o^* = \max & \sum_{r=1}^{s} C_rW \\ \text{s.t.} & W \in X. \end{array} \tag{3.9}$$

Theorem 3.1 *The optimal solutions of problem (3.9) are efficient solutions of model (2.1).*

Proof: The proof is similar to that of Theorem 2.3 in [6] and is omitted. □

Let $D_o = \{W_1^*, \ldots, W_p^*\}$ be the set of optimal solutions of problem (3.9) and $Y_d = (C_1W_d^*, \ldots, C_sW_d^*)^T, (d = 1, \ldots, p)$. These solutions are used to construct the PPS and the constructed PPS is used to find the other members of the well-dispersed subset of the non-dominated vectors for MOMILP problem.

To find another member of the well-dispersed subset of the non-dominated vectors for MOMILP problem a point, say $Y_h = (C_1W_h, \ldots, C_sW_h)^T \in \text{PPS}^c$ such that it has the minimal distance from the ideal point. In other words, there is a supporting hyperplane of PPS such that

$$u^*Y_h + u_h^* > 0. \tag{3.10}$$

By using Y_h inequality (3.10) is converted to $\sum_{r=1}^{s} C_rWu_{rh}^* + u_h^* > 0$. On the other hand, we need to add the constraint $\sum_{r=1}^{s} C_rW \leq \theta_o^* - \phi$ to obtain an efficient solution, which its distance from the ideal point is more than the distance of previous founded efficient solution/solutions, where ϕ is a small positive constant and is determined to obtain a well-dispersed subset of non-dominated vectors for MOMILP problem. It is evident, using different ϕ's we can obtain different well-dispersed subsets of non-dominated vectors of MOMILP problem.

According to the above discussion the following model is considered:

$$\begin{array}{ll} \theta_1^* & = \max \sum_{r=1}^{s} C_rW \\ \text{s.t.} & W \in X \\ & \sum_{r=1}^{s} C_rW \leq \theta_o^* - \phi \\ & \sum_{r=1}^{s} C_rWu_{rd}^* + u_d^* > -Mt_{rd}, \\ & r = 1, \ldots, s, d = 1, \ldots, h \\ & \sum_{r=1}^{s} t_{rd} \leq p - 1, d = 1, \ldots, h \\ & t_{rd} \in \{0, 1\}, r = 1, \ldots, s, d = 1, \ldots, h. \end{array} \tag{3.11}$$

When $t_d = 1$ the constraint $\sum_{r=1}^{s} C_rWu_{rd}^* + u_d^* > -Mt_d$ is redundant and the constraint $\sum_{r=1}^{p} t_d \leq p - 1$ implies that at least one of the constraints $\sum_{r=1}^{s} C_rWu_{rd}^* + u_d^* > -Mt_d$ is not redundant [7].

Let $A_1 = \{W_{p+1}^*, \ldots, W_k^*\}$ be the set of optimal solutions of problem (3.9), $D_1 = D_0 \cup A_1$ and $Y_d = (C_1W_d^*, \ldots, C_sW_d^*)^T$ for $d = p + 1, \ldots, k$ the corresponding output vectors. Similarly, after q iterations we set $D_q = D_{q-1} \cup A_q = \{W_1^*, \ldots, W_k^*\}$. Therefore, in general at the $(q+1)^{th}$ iteration the following problem is solved:

$$\begin{aligned} &\theta^*_{q+1} = \max \sum_{r=1}^{s} C_r W \\ \text{s.t.}\quad & W \in X \\ & \sum_{r=1}^{s} C_r W \le \theta^*_q - \phi \\ & \sum_{r=1}^{s} C_r W u^*_{rd} + u^*_d > -M t_{rd}, \\ & r = 1,\dots,s, d = 1,\dots,k \\ & \sum_{r=1}^{s} t_{rd} \le s-1, d = 1,\dots,k \\ & t_{rd} \in \{0,1\}, r = 1,\dots,s, d = 1,\dots,k. \end{aligned} \tag{3.12}$$

Theorem 3.2 *The optimal solutions of problem (3.12) are efficient solutions of model (2.1).*

Proof: Let $\widehat{W}^*$ be an optimal solution of model (3.12) and by contradiction assume that it is not an efficient solution of model (2.1). Therefore, there is a feasible solution of model (2.1), say W', such that

$$C_r W' \ge C_r \widehat{W}^*, r = 1,\dots,s,\ \exists\, l \in \{1,\dots,s\}, \quad C_l W' > C_l \widehat{W}^*. \tag{3.13}$$

By multiplying u^*_{rp} in $C_r W^o \ge C_r W^*_d$ $(r = 1,2,\cdots,s)$ and summing them, we will have [7]:

$$\begin{aligned} & \sum_{r=1}^{s} u^*_{rp} C_r W^o \ge \sum_{r=1}^{s} u^*_{rp} C_r W^*_d, p = 1,\dots,l \\ & \sum_{j=1}^{n}\sum_{r=1}^{s} u^*_{rp} C_{rj} w^o_j \ge \sum_{j=1}^{n}\sum_{r=1}^{s} u^*_{rp} C_{rj} w^*_{jd}, \\ & p = 1,\dots,l \\ & \sum_{j=1}^{n}\sum_{r=1}^{s} u^*_{rp} C_{rj} w^o_j > -u^*_{op} - t_{rp} M, \\ & p = 1,\dots,l, r = 1,\dots,s. \end{aligned}$$

Also, W^o holds in the inequalities $\sum_{j=1}^{n} a_{ij} w_j \le b_i$, $i = 1,2,\dots,m$. Therefore, W^o is a feasible solution of the problem (3.12). From (3.13), we have $\sum_{r=1}^{s} C_r W^o > \sum_{r=1}^{s} C_r W^*_d (Z_{W_o} > Z_{W^*_d})$ which is a contradiction. □

Let $D_{p-1} = \{W^*_1,\dots,W^*_{p-1}\}$ be the subset of the well-dispersed efficient solution of problem until $(p-1)^{th}$ iteration and W^*_p be an optimal solution of problem (3.11) and $A = \{W^*_{i_{k+1}}\}$. To find the other well-dispersed efficient solution of problem (2.1), corresponding to W^*_{k+1}, we add the following constraints to problem (3.11):

$$\begin{aligned} & C_r W > C_r W^*_{k+1} - M t_{rk+1}, r = 1,\dots,s \\ & \sum_{r=1}^{s} t_{rq} \le s-1, q = 1,\dots,k \\ & \delta_{rq} \ge \varepsilon, r = 1,\dots,s, q = 1,\dots,k \end{aligned}$$

where ε is a very small positive real number. Therefore, the pth iteration's problem is as follows:

$$\begin{aligned} \max\quad & \sum_{r=1}^{s} C_r W \\ \text{s.t.}\quad & W \in X \\ & \sum_{r=1}^{s} C_r W_r \le \theta^*_{p-1} - \phi \\ & C_r W \ge \delta_{rq} + C_r W^*_q - M t_{rq}, \\ & r = 1,\dots,s, q = 1,\dots,k,k+1 \\ & \sum_{r=1}^{s} t_{rq} \le s-1, q = 1,\dots,k,k+1 \\ & t_{rq} \in \{0,1\}, r = 1,\dots,s, \\ & q = 1,\dots,k,k+1 \\ & \delta_{rq} \ge \varepsilon, r = 1,\dots,s, \\ & q = 1,\dots,k,k+1. \end{aligned} \tag{3.14}$$

When X is bounded, this process is continued until problem (3.14) becomes infeasible.

Theorem 3.3 *Each optimal solution of problem (3.14) is an efficient solution for MOMILP problem.*

Proof: The proof is similar to that of Theorem 2.4 in [6] and is omitted. □

Using the above discussions, in the following cases an MOMILP problem has well-dispersed subset of efficient solutions.

1. When X is nonempty and bounded.
2. When X is unbounded and there is no $d \ne 0$ such that $A_i d \le 0$, $i = 1,\dots,m, C_r d \ge 0, r = 1,\dots,s$ with at least one $p(p \in \{1,\dots,s\})$ such that $C_r d > 0$ and $d_j \in Z^+$ for $j \in J$, [15].

Therefore, to find a well-dispersed subsets of non-dominated vectors for MOMILP problem with bounded and unbounded feasible regions we consider the following Algorithm.

3.1 The proposed Algorithm

Stage 0: Solve the system $A_i d \leq 0$, $i = 1, \ldots, m, C_r d \geq 0, r = 1, \ldots, s$, with at least one $p (p \in \{1, 2, \ldots, s\})$ such that $C_p d > 0$ and $d_j \in Z^+$ for $j \in J$. If this system has solution, then there is no efficient solution for problem (2.1) and go to stage 3. Otherwise go stag 1,

Stage 1:

Step 1-1: Let $k = 0$ and solve problems (2.4) and specify $G_o = \{W_1^*, \ldots, W_h^*\}$. If G_o is empty go to step 1-2, otherwise let $k = h$ and go to step 1-3,

Step 1-2: Determine the optimal solutions of problem (3.9) and let $G_o = \{W_1^*, \ldots, W_\beta^*\}$ as optimal solutions set of model (3.9) and let $\beta = k$,

Step 1-3: Determine an optimal solution of problem (3.11) and let $A = \{W_{k+1}^*\}$,

Step 1-4: If A is not empty, let $G_1 = G_o \cup A$ and go to stage 2. Otherwise, stop, G_o is a well-dispersed subset of efficient solutions of (2.1),

Stage 2:

Step 2-1: Determine an optimal solution of problem (3.14), say W_{k+1}^* and let $B = \{W_{k+1}^*\}$,

Step 2-2: If B is not empty, let $G_{k+1} = G_k \cup B$ and go to stage 2. Otherwise, stop, the set G_k is the well-dispersed subset of efficient solutions of (2.1),

Stage 3: End.

If W_p^* and W_{p+1}^* are the two well-dispersed non-dominated vectors of MOMILP which have been obtained in p^{th} and $(p+1)^{th}$ iterations, respectively, then the distance of Y_p is less than Y_{p+1} from g by using L_1 norm and so the rank of W_p^* is higher than W_{p+1}^*. Hence, the elements of the well-dispersed subsets of non-dominated vectors of MOMILP problem are ranked by using the proposed algorithm.

4 Examples

The proposed algorithm is illustrated for MOMILP problems with bounded and unbounded feasible regions.

Example 4.1 *Consider the following MOMILP problem:*

$$\begin{array}{ll} \max & w_1 + w_2 \\ \max & 4w_1 + 3w_2 \\ \text{s.t.} & -3w_1 + 2w_2 \leq 6 \\ & -6w_1 + 10w_2 \leq 60 \\ & w_1,\ w_2 \in\ Z^+. \end{array}$$

It can be seen, there is $d = \binom{d_1}{d_2}$ such that $A_i d \leq 0$, $i = 1, 2, C_r d > 0, r = 1, 2, d_1 \leq 0$ and $d_2 \in Z^+$, where $A_1 = (-3, 2), A_2 = (-6, 10), C_1 = (1, 1)$ and $C_2 = (4, 3)$. That is, the feasible region is unbounded and the objective functions can become infinite together. Therefore, there is not any efficient solution for this problem.

Example 4.2 *Consider the following MOMILP problem:*

$$\begin{array}{ll} \max & -2w_1 + w_2 \\ \max & w_1 - 3w_2 \\ \text{s.t.} & -4w_1 + w_2 \leq 4 \\ & -9w_1 + 5w_2 \leq 45 \\ & w_1 \geq 0,\ w_2 \in\ Z^+. \end{array} \tag{4.15}$$

There is d, say $d = \binom{d_1}{d_2} = \binom{1}{1}$, such that $A_i d \leq 0$, $i = 1, 2, d_1 \geq 0, d_2 \in Z^+$ where $A_1 = (-4, 1)$ and $A_2 = (-9, 5)$. That is, feasible region of this problem is unbounded. But, there is no recession direction such that, $C_r d \geq 0, r = 1, 2, \exists p \in \{1, 2\}, C_p d > 0, d_1 \geq 0, d_2 \in Z^+$, where $A_1 = (-1, 1), A_2 = (-4, 6), C_1 = (-2, 1)$ and $C_2 = (1, -3)$. Therefore, this problem has efficient solution.

Stage1, Step 1-1: Consider the following single objective integer programming problems:

$$\begin{array}{ll} \max & -2w_1 + w_2 \\ \text{s.t.} & -4w_1 + w_2 \leq 4 \\ & -9w_1 + 5w_2 \leq 45 \\ & w_1 \geq 0,\ w_2 \in\ Z^+ \end{array} \tag{4.16}$$

and

$$\begin{array}{ll} \max & w_1 - 3w_2 \\ \text{s.t.} & -4w_1 + w_2 \leq 4 \\ & -9w_1 + 5w_2 \leq 45 \\ & w_1 \geq 0,\ w_2 \in\ Z^+. \end{array} \tag{4.17}$$

$W_1^* = (2.25, 13)^T$ is optimal solution of the problem (4.16) and $Y^1 = (8.5, -36.75)^T$ is its objectives values vector. But, optimal value of the problem (4.17) is infinite and this problem doesn't has optimal solution. Therefore, $G_o = \{(2.25, 13)^T\}$.

Step 1-2: The corresponding problem of G_o is as follows:

$$\begin{array}{ll} \max & -w_1 - 2w_2 \\ \text{s.t.} & -4w_1 + w_2 \leq 4 \\ & -9w_1 + 5w_2 \leq 45 \\ & -2w_1 + w_2 - \delta_1 + 100t_{11} \geq 8.5 \\ & w_1 - 3w_2 - \delta_2 + 100t_{12} \geq -36.75 \\ & t_{11} + t_{12} \leq 1 \\ & t_{11},\ t_{12} \in \{0,1\}, w_1 \geq 0,\ w_2 \in\ Z^+ \\ & \delta_1, \delta_2 \geq \varepsilon. \end{array} \tag{4.18}$$

$W_2^* = (0,0)^T$ is optimal solution of the problem (4.18) and $Y^2 = (0,0)^T$ is the corresponding objectives values vector. Therefore, $A = \{(0,0)^T\}$.
Step 1-3: $G_1 = A \cup G_o = \{(2.25, 13)^T, (0,0)^T\}$

Stage 2, Iteration 1

Step 2-1: To obtain a set of efficient solutions with a appropriate density we let $\phi = 0.5$ (the value of ϕ can be obtained by using decision maker opinions) and to obtain a member of well-dispersed subset of efficient solutions the corresponding problem is as follows:

$$\begin{array}{ll} \max & -w_1 - 2w_2 \\ \text{s.t.} & -4w_1 + w_2 \leq 4 \\ & -9w_1 + 5w_2 \leq 45 \\ & -w_1 - 2w_2 \leq -0.5 \\ & -2w_1 + w_2 - \delta_1 + 100t_{11} \geq 8.5 \\ & w_1 - 3w_2 - \delta_2 + 100t_{12} \geq -36.75 \\ & t_{11} + t_{21} \leq 1 \\ & -2w_1 + w_2 - \delta_3 + 100t_{21} \geq 0 \\ & w_1 - 3w_2 - \delta_4 + 100t_{22} \geq 0 \\ & t_{21} + t_{22} \leq 1 \\ & t_{11},\ t_{12}, t_{21}, t_{22}, \in \{0,1\} \\ & w_1 \geq 0,\ w_2 \in\ Z^+, \delta_1, \delta_2, \delta_3, \delta_4 \geq \varepsilon. \end{array} \tag{4.19}$$

$W_3^* = (0.5,0)^T$ is optimal solution of the problem (4.19) and $Y^3 = (-1, 0.5)^T$ is its objectives values vector. Hence, $B = \{(0.5,0)^T\}$.
Iteration 2

Step 2-1: The corresponding problem of G_1 is as follows:

$$\begin{array}{ll} \max & -w_1 - 2w_2 \\ \text{s.t.} & -4w_1 + w_2 \leq 4 \\ & -9w_1 + 5w_2 \leq 45 \\ & -w_1 - 2w_2 \leq -1 \\ & -2w_1 + w_2 - \delta_1 + 100t_{11} \geq 8.5 \\ & w_1 - 3w_2 - \delta_2 + 100t_{12} \geq -36.75 \\ & t_{11} + t_{21} \leq 1 \\ & -2w_1 + w_2 - \delta_3 + 100t_{21} \geq 0 \\ & w_1 - 3w_2 - \delta_4 + 100t_{22} \geq 0 \\ & t_{12} + t_{22} \leq 1 \\ & -2w_1 + w_2 - \delta_5 + 100t_{31} \geq -1 \\ & w_1 - 3w_2 - \delta_6 + 100t_{32} \geq 0.5 \\ & t_{13} + t_{23} \leq 1 \\ & t_{11},\ t_{12}, t_{21}, t_{22}, t_{31}, t_{32} \in \{0,1\} \\ & w_1 \geq 0, w_2 \in\ Z^+ \\ & \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \geq \varepsilon. \end{array} \tag{4.20}$$

$W_4^* = (0,1)^T$ is an optimal solution of problem (4.20) and $Y^4 = (1, -3)^T$ is its objectives values vector. Therefor, $B = \{(0,1)^T\}$. Using the other single objective integer problems we find that for each $n \in Z^+, W^* = (0,n)^T$ is an efficient solution of the problem (4.15). Hence, the number of efficient solution of this problem is infinite and the proposed approach finds a well-dispersed subset of efficient solutions as $\{(2.25,13)^T, (0,0)^T, (0.5,0)^T, (0,1)^T, (0,2)^T\}$.

5 Conclusion

This paper used the DEA technique and proposed an algorithm to find a well-dispersed subsets of non-dominated vectors of MOMILP problems with bounded and unbounded feasible regions, while the previous methods [11] consider only problems with bounded feasible region. In each iteration of the proposed algorithm, at least one well-dispersed efficient solution of MOMILP problem is found.

The elements of the well-dispersed subsets of non-dominated vectors of MOMILP problem are ranked by using the proposed algorithm, so it does not need filtering procedures. The proposed method illustrated by two numerical examples.

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Ghasem Tohidi is Associate Professor in Mathematics department at the Islamic Azad University Tehran central Branch in Iran. His teaching has been in operation research and data envelopment analysis (DEA). His major research interests are multi objective programming and DEA. His publications have been appeared in several journals, including Journal of the Operational Research Society, Applied Mathematics and Computation, Applied Mathematical Modelling, and Journal of the Operational Research Society of Japan, among others.

Shabnam Razavyan is Assistant Professor in Mathematics department at the Islamic Azad University Tehran south Branch in Iran. Her teaching has been in operation research and DEA. Her major research interests are multi objective programming and DEA. Her publications have been appeared in several journals, including Journal of the Operational Research Society, Applied Mathematics and Computation, Applied Mathematical Modelling, and Journal of the Operational Research Society of Japan, among others.