Available online at http://sanad.iau.ir/journal/ijim/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 16, No. 3, 2024 Article ID IJIM-1629, 15 pages
Research Article

Science and Research Branch (IAU)

A new non-radial DEA-R model to measure the effects of undesirable outputs on the efficiency of production units

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Submission Date: 2023/12/07, Revised Date: 2024/08/08, Date of Acceptance: 2024/08/28

Abstract
The efficiency score of the decision-making units (DMUs) depends on the input and output values. The efficiency score of the DMU in the presence of undesirable outputs is greater or equal to the efficiency score of this DMU in the absence of undesirable outputs. To face this problem, we present a new ratio-based data envelopment analysis (DEA-R) model to measure the effects of undesirable outputs on the efficiency of production units. In this regard, we first introduce the counterpart (hypothetical) units corresponding to the original DMUs. These units use the same amount of input to produce the same desirable outputs as the original DMUs, but produce a small amount of undesirable outputs compared to the original units. In the following, we use non-radial DEA-R models based on slacks corresponding to all the ratios of input components to desirable output and the ratios of undesirable output to desirable output to measure efficiency in the presence of undesirable outputs. Also, let's use the efficiency ratio of the main units to their corresponding counterpart units as a reduction factor to show the impact of undesirable outputs. To show the validity of the proposed approach, we evaluate the performance of thirty paper mills and present the results.

Keywords: Data envelopment analysis, SBM DEA-R, Undesirable output, Weak disposability, Efficiency.

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1. Introduction

Data envelopment analysis (DEA) was originally proposed by Charnes, Cooper and Rhodes [1] to evaluate the performance of a set of homogeneous DMUs that use multiple inputs to produce multiple outputs. Based on DEA models, if the inputs and desirable outputs of a DMUs decrease and increase respectively, then the efficiency score of this DMU is improved. But in the production process, the DMUs may produce other outputs called undesirable outputs in addition to desirable outputs. For example, CO2 and NOX are produced in the combustion of fossil fuel and waste waters are generated in producing pulp and paper. These outputs may not affect the production units, but their production may be dangerous for the environment. Therefore, the production of undesirable outputs should be avoided to protect the environment. But undesirable outputs are produced at the same time as desirable outputs and it cannot completely stop their production. But their amount can be controlled. In most cases, controlling the production of undesirable outputs will be accompanied by high costs. In many cases, a standard level is introduced for these outputs. But controlling the production of undesirable outputs is less than the cost of their production. Therefore, it is important to measure the impact of undesirable outputs on the efficiency of the unit under evaluation (Kao and Hwang, [2,3].

Several methods have been presented to measure the efficiency of the unit under the decision maker's evaluation in the presence of undesirable outputs. Scheel [4] classified them into two categories, direct and indirect methods. Direct methods work on original data while indirect methods work on data transformation. The direct methods are connected to the concept of weak disposability (Färe, Grosskopf, Lovell and Pasurka, [5]). The weak disposability is assumed that undesirable outputs can only be reduced in proportion to desirable outputs. However, the indirect methods are including input inverse, additive inverse, translated inverse, and multiplicative inverse methods.

Zhou, Ang and Poh [6] proposed a similar classification while we used the DEA technique in energy and environmental studies. They described those direct methods work on the original data and can be classified as the slacks-based measure (SBM), the directional distance function (DDF), and hyperbolic models.

Song, An, Zhang, Wang and Wu [7] work on environmental efficiency evaluation in the presence of undesirable outputs and they divided the methods into input reverse, data transformation, and disposability-related methods.

Dakpo, Jeanneaux and Latruffe [8] investigated that papers that proposed on performance benchmarking with undesirable outputs, divided into five approaches: free disposability of the inputs, data transformation, weak disposability of the undesirable outputs, materials balance principles, two sub-technologies and natural and managerial disposability.

You and Yan [9] showed that the SBM method obtained suitable results in terms of distinguishing the performance of the firms and identifying the improvements that can be obtained.

Kao and Hwang [10] used direct models to deal with undesirable outputs. They used the SBM model to show the impact of undesirable outputs on the efficiency score. They showed that the SBM model is a suitable model for investigating the effect of producing excessive amounts of undesirable outputs on the efficiency of DMUs. They proposed a DEA model which allows the production units being evaluated to determine the shadow prices for both the desirable and undesirable outputs by themselves so that the measured efficiency score will achieve the highest possible level.

Kao and Hwang [2] investigated the impact of undesirable outputs on the efficiency score of the unit under evaluation. They used the SBM model based on slack values corresponding to inputs, desirable output, and undesirable outputs. They obtained the lowest undesirable output value corresponding to each of DMUs.

Kao and Hwang [3] investigated the impact of undesirable outputs on the efficiency scores of DMUs. They used the different reduction of the components of undesirable outputs simultaneously with the different increase of desirable outputs based on the principle of weak disposability in DEA.

DEA-R models are one of the efficiency evaluations models in DEA. These models use all ratios of input to output components or vice versa to measure efficiency. These models have a larger space in choosing weights. Because they use weights corresponding to all ratios of input to output components. Therefore, DEA-R models avoid some problems such as under estimation estimating of efficiency, pseudo-inefficiency, compared to traditional DEA models (Wei et al. [11,12]). The studies conducted in DEA in the field of ratio data can be classified into three categories.

In the first category of studies, the ratio data is a decimal number and these decimal numbers are included as a ratio data in the model. These studies put the data in two absolute and ratio forms in the model. The original data corresponding to these decimal numbers may not be available and we will only have their ratio as a decimal number. Among the articles to deal with ratio data in this category, the studies of Olsen et al. [13,14] can be mentioned.

Ratio data in the second class includes ratio data in the form of fractional numbers, where the numerator and denominator of the fractions are available. These data are included as a fraction in the model. The models obtain efficiency based on the numerator and denominator of these fractions. Emrouznejad and amin [15] showed that the assumption of convexity from the set of principles of creating production possibility set (PPS) in DEA is not valid in the presence of ratio data and presented two new solutions to measure efficiency in DEA model. Hatami-Marbini and Toloo [16] showed that the models presented by Emrouznejad and amin [15] have problems. They modified the models presented by Emrouznejad and amin [15] and presented two methods to deal with each of the solutions presented by Emrouznejad and amin [15].

But the last category of DEA models for dealing with ratio data are models that use all ratios of input and output components or vice versa to calculate efficiency. We call these models DEA-R models. Despic et al. [17] investigated DEA-R models and obtained the relationship between the efficiency of DEA and DEA-R models. Subsequently, several studies were conducted in this field as Gerami et al. [18-21].

It can be said that the main contribution of this paper is as follows. We derive DEA-R models to calculate efficiency in the presence of ratio data. We use the slack corresponding to all the ratios of the input components to the desirable output components and the undesirable output components to the components desirable output. In this direction, corresponding to each DMUs, we create new units that called counterpart (hypothetical) units and these units has the same desirable input and output as these units and has less undesirable output compared to the original units. By comparing the efficiency score of the DMUs with the efficiency score of their corresponding counterpart units, we reduce the impact of undesirable outputs on the efficiency calculation, and in this way, the efficiency of the unit under evaluation increases. To investigate the impact of undesirable outputs, we obtain ratio data in a non-radial form to obtain efficiency.

It can be said that the continuation of this paper is organized as follows. The second section presents the DEA-R models in the presence of undesirable outputs. The third section examines the impact of the production of undesirable outputs on efficiency measurement and presents new units corresponding to the original units with less undesirable outputs. The fourth section presents a case study including data related to paper manufacturing companies based on the approach presented in this paper, and at the end we present the results of the research.

2. DEA-R model

Let n DMUs as $DMU_j = (X_j, Y_j),\ j = 1,\dots,n.$ The input and output vectors corresponding to $DMU_j,\ j = 1,\dots,n,$ are $X_j = (x_{1j},\dots,x_{mj})$ and $Y_j = (y_{1j},\dots,y_{sj})$. We suppose that $x_{ij} \ge 0,\ y_{rj} \ge 0,\ i = 1,\dots,m,\ r = 1,\dots,s,\ j = 1,\dots,n.$ Suppose the ratios Suppose the ratios $\frac{X_j}{Y_j},\ j = 1,\dots,n,$ in the input orientation and the ratios $\frac{Y_j}{X_j}, j = 1, \dots, n,$ in the output orientation are defined. Suppose, we consider the components corresponding to this ratio as $\left(\frac{x_{ij}}{y_{rj}}\right),\ i = 1,\dots m,\ r = 1,\dots,s,\ j = 1,\dots,n.$ Based on the idea of Gerami et al. [18], SBM DEA-R model was presented in the input orientation as follows.

$$\sigma_R^* = Min\left(1 - \frac{1}{m \times s}\sum_{r=1}^{s}\sum_{i=1}^{m}\left(\frac{s_{ir}^d}{\left(\frac{x_{io}}{y_{ro}}\right)}\right)\right) \qquad (1)$$

$$\begin{aligned} s.t.\quad & \sum_{j=1}^{n}\mu_j\left(\frac{x_{ij}}{y_{rj}}\right) + s_{ir} = \left(\frac{x_{io}}{y_{ro}}\right), \quad i = 1,\dots m,\ r = 1,\dots,s, \\ & \sum_{k=1}^{n}\mu_j = 1,\ \mu_j \ge 0, \quad j = 1,\dots,n, \\ & s_{ir} \ge 0,\ i = 1,\dots m,\ r = 1,\dots,s. \end{aligned}$$

In the model (1), $s_{ir},\ i = 1,\dots m,\ r = 1,\dots,s,$ are slacks corresponding to all ratios input to output components. $\mu_j,\ j = 1,\dots,n,$ are intensity variable for these ratios.

Definition 1. $DMU_o = (X_o, Y_o)$ is SBM DEA-R efficient if only if $\sigma_R^* = 1.$

Now, based on the model (1), we present SBM DEA-R model in presence of undesirable outputs. Suppose we have n DMUs as $DMU_j=\left(X_j,Y_j,U_j\right),\ j=1,\dots,n$. Each DMUs uses the input vector $X_j=\left(x_{1j},\dots,x_{mj}\right)$ to generate the output vector $Y_j=\left(y_{1j},\dots,y_{sj}\right)$, and $U_j=\left(u_{1j},\dots,u_{hj}\right)$. Then we define the set T_{DEA-R} in presence of undesirable outputs as follows.

$$T_{DEA-R}=\left\{(F,H)\left|\sum_{j=1}^{n}\mu_j\left(\frac{X_j}{Y_j}\right)\le F,\sum_{j=1}^{n}\mu_j\left(\frac{U_j}{Y_j}\right)\le H,\sum_{j=}^{n}\mu_j=1,\mu_j\ge 0,\ j=1,\dots,n\right.\right\}.$$

We define a division data set, which are $m\times s$ and $h\times s$ dimension vectors as follows.

$$\frac{X}{Y}=\left(\frac{x_1}{y_1},\dots,\frac{x_m}{y_1},\frac{x_1}{y_2},\dots,\frac{x_m}{y_2},\dots\dots,\frac{x_1}{y_s},\dots,\frac{x_m}{y_s}\right),X=\left(x_1,\dots,x_m\right),Y=\left(y_1,\dots,y_s\right).$$

$$\frac{U}{Y}=\left(\frac{u_1}{y_1},\dots,\frac{u_h}{y_1},\frac{u_1}{y_2},\dots,\frac{u_h}{y_2},\dots\dots,\frac{u_1}{y_s},\dots,\frac{u_h}{y_s}\right),U=\left(u_1,\dots,u_h\right),Y=\left(y_1,\dots,y_s\right).$$

SBM DEA-R model in presence of undesirable outputs was presented in the input orientation as follows.

$$\tau_R^*=Min\left(1-\frac{1}{s\times(m+h)}\left[\sum_{r=1}^{s}\sum_{i=1}^{m}\left(\frac{s_{ir}^d}{\left(\frac{x_{io}}{y_{ro}}\right)}\right)+\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^u}{\left(\frac{u_{fo}}{y_{ro}}\right)}\right)\right]\right) \tag{2}$$

$$\begin{aligned}
s.t.\quad &\sum_{j=1}^{n}\mu_j\left(\frac{x_{ij}}{y_{rj}}\right)+s_{ir}^d=\left(\frac{x_{io}}{y_{ro}}\right),\quad i=1,\dots m,\ r=1,\dots,s,\\
&\sum_{j=1}^{n}\mu_j\left(\frac{u_{fj}}{y_{rj}}\right)+s_{fr}^u=\left(\frac{u_{fo}}{y_{ro}}\right),\quad f=1,\dots h,\ r=1,\dots,s,\\
&\sum_{k=1}^{n}\mu_j=1,\ \mu_j\ge 0,\quad j=1,\dots,n,\\
&\quad s_{ir}^d\ge 0,\ i=1,\dots m,\ r=1,\dots,s,\\
&\quad s_{fr}^u\ge 0,\ f=1,\dots h,\ r=1,\dots,s.
\end{aligned}$$

In the model (2), $s_{ir}^d, s_{fr}^u, i=1,\dots m,\ f=1,\dots h,\ r=1,\dots,s$, are slacks corresponding to all ratios input components to desirable output components, all ratios undesirable output components to output desirable components.

Definition 2. $DMU_o=\left(X_o,Y_o,U_o\right)$ is SBM DEA-R efficient if only if $\tau_R^*=1$.

3. The impact of undesirable outputs on the efficiency of DMUs

Now we will examine the impact of undesirable outputs on the efficiency of the DMU under evaluation. In this regard, consider a simple numerical example. Consider 5 DMUs according to Table 1.

At first, we obtain the results of model (1). Based on model (1), we do not consider undesirable outputs. As can be seen, the DMU C is efficient and the other DMUs are inefficient. The results are shown in Table 1.

Now we solve model (2) for numerical example data. Based on model (2), we consider undesirable outputs. According to the results in Table 1, the efficiency scores obtained from model (2) are greater or equal to the corresponding scores obtained from model (1). But this result is not logical, because we need to expect that the efficiency scores of the DMUs will decrease in the presence of undesirable outputs, but the efficiency scores of the DMUs have increased in the presence of undesirable outputs.

Therefore, the results of model (2) are not reasonable from the point of view of DEA. Therefore, I have to modify model (2). At first, we consider a situation where we set the undesirable outputs corresponding to each of the DMUs equal to zero and create new units that have undesirable output value equal to zero. We can show these DMUs as follows.

$DMU_j^o=\left(X_j,Y_j,U_j^o=0\right),\ j=1,\dots,n.$ Model (3) considering the new DMUs instead of the original DMUs, then we can calculus the efficiency score of original DMUs based on the new PPs created by new DMUs, therefore, model (3) will be as follows.

$$\tau_R^*=Min\left(1-\frac{1}{s\times(m+h)}\left[\sum_{r=1}^{s}\sum_{i=1}^{m}\left(\frac{s_{ir}^d}{\left(\frac{x_{io}}{y_{ro}}\right)}\right)+\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^u}{\left(\frac{u_{fo}}{y_{ro}}\right)}\right)\right]\right) \tag{3}$$

$$\begin{aligned}
s.t.\quad & \sum_{j=1}^{n}\mu_j\left(\frac{x_{ij}}{y_{rj}}\right)+s_{ir}^d=\left(\frac{x_{io}}{y_{ro}}\right),\quad i=1,\dots m,\ r=1,\dots,s,\\
& \sum_{j=1}^{n}\mu_j\left(\frac{u_{fj}^o}{y_{rj}}\right)+s_{fr}^u=\left(\frac{u_{fo}}{y_{ro}}\right),\quad f=1,\dots h,\ r=1,\dots,s,\\
& \sum_{k=1}^{n}\mu_j=1,\ \mu_j\geq 0,\quad j=1,\dots,n,\\
& s_{ir}^d\geq 0,\ i=1,\dots m,\ r=1,\dots,s,\\
& s_{fr}^u\geq 0,\ f=1,\dots h,\ r=1,\dots,s.
\end{aligned}$$

Table 1. Data for five DMUs as an example and the efficiency score of models (1), (2), and (3).

DMU	**Original DMUs**			**Counterpart DMUS**			**Efficiency score**		
	X	**Y**	**U**	**X**	**Y**	**U**	**Model (1)**	**Model (2)**	**Model (3)**
A	1	0.5	0.5	1	0.5	0.1	0.3333	0.2778	0.1667
B	2	2	0.6	2	2	0.4	0.6667	0.7037	0.3333
C	3	4.5	1	3	4.5	0.9	1	1	0.5
D	4	4	0.8	4	4	0.8	0.6667	1	0.3333
E	5	5	1.5	5	5	1	0.6667	0.7037	0.3333

Considering that each DMUs is defeated by its corresponding new units, therefore the efficiency score of model (3) will not be greater than or equal to the efficiency value of model (2). The results are in the last column of Table 1. According to the results in the last column of Table 1, none of these DMUs are efficient. Therefore, considering that all the DMUs became inefficient in the presence of undesirable outputs, this result shows that we should check the effect of undesirable outputs on the efficiency of the DMUs.

It should be noted that models (2) and (3) were designed based on the assumption of strong disposability, and in this case, we assumed that the production of undesirable outputs in the production process would be prevented. But as we know, this assumption is unrealistic and undesirable outputs along with desirable outputs are produced indefinitely in the production process. In this regard, Färe et al. [5] presented the concept of weak disposability. This assumption claims that undesirable outputs can be reduced in proportion to desirable outputs. On the other hand, undesirable outputs are allowed to be produced freely. Now, we present the PPS in the presence of undesirable outputs and ratio data, taking into account the weak disposability assumption, as follows.

$$T_{DEA-R}^{WD}=\left\{\left(\frac{X}{Y},\frac{U}{Y}\right)\middle|\beta\sum_{j=1}^{n}\mu_j\left(\frac{X_j}{Y_j}\right)\leq\frac{X}{Y},\ \beta\sum_{j=1}^{n}\mu_j\left(\frac{U_j}{Y_j}\right)=\frac{U}{Y},\ \sum_{j=}^{n}\mu_j=1,\right.$$
$$\left.\mu_j\geq 0,\ j=1,...,n,\ 0\prec\beta\leq 1\right\}.$$

Where β is the proportion of the reduction in the desirable and undesirable outputs. Based on this PPS, we proposed SBM DEA-R model for measuring the efficiency of DMU based on the weak disposability assumption as follows.

$$\gamma_R^*=Min\left(1-\frac{1}{s\times m}\left[\sum_{r=1}^{s}\sum_{i=1}^{m}\left(\frac{s_{ir}^d}{\left(\frac{x_{io}}{y_{ro}}\right)}\right)\right]\right) \qquad (4)$$

$$\begin{aligned}
s.t.\quad & \beta\left(\sum_{j=1}^{n}\mu_j\left(\frac{x_{ij}}{y_{rj}}\right)\right)+s_{ir}^d=\left(\frac{x_{io}}{y_{ro}}\right),\quad i=1,...m,\ r=1,...,s,\\
& \beta\left(\sum_{j=1}^{n}\mu_j\left(\frac{u_{fj}}{y_{rj}}\right)\right)=\left(\frac{u_{fo}}{y_{ro}}\right),\quad f=1,...h,\ r=1,...,s,\\
& \sum_{k=1}^{n}\mu_j=1,\ \mu_j\geq 0,\quad j=1,...,n,\\
& 0\prec\beta\leq 1,\ s_{ir}^d\geq 0,\ i=1,...m,\ r=1,...,s.
\end{aligned}$$

Since the ratio of undesirable outputs to desirable output are fixed at the current levels of the assessed DMU, the associated slack variables are always zero, and they do not appear in the objective function. The constraints are nonlinear. By letting $\alpha_j = \beta\mu_j,\ j = 1,...,n,$ model (4) can be converted as follows:

$$\gamma_R^* = Min\left(1 - \frac{1}{s \times m}\left[\sum_{r=1}^{s}\sum_{i=1}^{m}\left(\frac{s_{ir}^d}{\left(\frac{x_{io}}{y_{ro}}\right)}\right)\right]\right) \qquad (5)$$

$$\begin{aligned} s.t. \quad & \sum_{j=1}^{n}\alpha_j\left(\frac{x_{ij}}{y_{rj}}\right) + s_{ir}^d = \left(\frac{x_{io}}{y_{ro}}\right), \quad i = 1,...m,\ r = 1,...,s, \\ & \sum_{j=1}^{n}\alpha_j\left(\frac{u_{fj}}{y_{rj}}\right) = \left(\frac{u_{fo}}{y_{ro}}\right), \quad f = 1,...h,\ r = 1,...,s, \\ & \sum_{k=1}^{n}\alpha_j = \beta,\ \alpha_j \geq 0, \quad j = 1,...,n, \\ & 0 \prec \beta \leq 1,\ s_{ir}^d \geq 0,\ i = 1,...m,\ r = 1,...,s. \end{aligned}$$

where the constraints of model (5) are linear. we can be solved model (5) easily by most nonlinear programming solvers. The results of model (4) are in Table 2. The results show the efficiencies of the five DMUs in the example calculated from this model are different of result of models (2) and (3). The efficiency of all DMUs except for DMU A have decreased compared to model (2). But we still haven't reached a logical conclusion about the presence of undesirable outputs to calculate efficiency, so it seems necessary to present a correct model that presents the effect of the presence of undesirable outputs in calculating efficiency. Therefore, we present a suitable model.

Table 2. The efficiency score of models (4), (6), and (7).

DMUs	**Efficiency score**		**Counterpart DMUS: model (6)**		
	Model (4)	**Model (7)**	**X**	**Y**	**U**
A	1	0.2667	1	0.5	0.1
B	0.6	0.6667	2	2	0.4
C	0.6667	0.9500	3	4.5	0.9
D	0.4	0.8333	4	4	0.8
E	0.6	0.6667	5	5	1

In the case of strong disposability, a set of counterpart DMUs that dominates the original DMUs is established to create the production frontier for measuring efficiencies via model (3). The efficiencies of the original DMUs and the corresponding counterpart DMUs can be used to measure the effect of the undesirable outputs on efficiency. The same idea, yet under weak disposability, can be used here. Now, we obtain the amount of the undesirable output that should be allowed to be generated for different amounts of the desirable output being produced. The undesirable output reduces in proportion to reductions in the desirable output, and this satisfies the assumption of weak disposability. We fix the amount of the desirable output at the current amount of DMUs. We proposed counterpart units have the same input and output levels as the original units, but their undesirable output level is lower than the original units, and to determine the level of undesirable outputs from these DMUs, assume that s_f^{count}, $f = 1,...h$, are slacks corresponding to undesirable output components, we can solve the following model.

$$
\begin{aligned}
& Max \ \sum_{f=1}^{h} s_f^{count} \qquad (6) \\
& s.t. \quad \sum_{j=1}^{n} \mu_j u_{fj} + s_f^{count} = u_{fo}, \quad f = 1,...h, \\
& \sum_{j=1}^{n} \mu_j y_{rj} = y_{ro}, \quad r = 1,...s, \\
& \mu_j \geq 0, \quad j = 1,...,n, \\
& s_f^{count} \geq 0, \ f = 1,...h.
\end{aligned}
$$

Suppose, $\left(\mu^*, s^{count^*}\right)$ that $s^{count^*} = \left(s_1^{count*},...,s_h^{count*}\right)$ is an optimal solution of model (6). In model (6), the amount of desirable outputs remains constant and the minimum level of undesirable outputs that DMU_k is allowed to generate is determined as follows. $u_{fo}^{count} = u_{fo} - s_f^{count^*}, f = 1,...,h.$ Also, we put $U_j^{count} = \left(u_{1j}^{count},...,u_{hj}^{count}\right)$.

The model (6) may have multiple solutions when there is more than one undesirable output. We can consider different weights to factors of different importance to obtain unique solutions. The efficiencies of the original DMUs and their counterpart DMUs are calculated based on the frontier constructed from the counterpart DMUs namely $DMU_j^{count} = \left(X_j, Y_j, U_j^{count}\right), \ j = 1,...,n,$ through the following model.

$$
\varphi_R^{count*} = Min \left(1 - \frac{1}{s \times (m+h)} \left[\sum_{r=1}^{s} \sum_{i=1}^{m} \left(\frac{s_{ir}^d}{\left(\frac{x_{io}}{y_{ro}}\right)} \right) + \sum_{r=1}^{s} \sum_{f=1}^{h} \left(\frac{s_{fr}^u}{\left(\frac{u_{fo}}{y_{ro}}\right)} \right) \right] \right) \qquad (7)
$$

$$
\begin{aligned}
& s.t. \quad \sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right) + s_{ir}^d = \left(\frac{x_{io}}{y_{ro}}\right), \quad i = 1,...m, \ r = 1,...,s, \\
& \sum_{j=1}^{n} \mu_j \left(\frac{u_{fj}^{count}}{y_{rj}}\right) + s_{fr}^u = \left(\frac{u_{fo}}{y_{ro}}\right), \quad f = 1,...h, \ r = 1,...,s, \\
& \sum_{k=1}^{n} \mu_j = 1, \ \mu_j \geq 0, \quad j = 1,...,n, \\
& s_{ir}^d \geq 0, \ i = 1,...m, \ r = 1,...,s, \\
& s_{fr}^u \geq 0, \ f = 1,...h, \ r = 1,...,s.
\end{aligned}
$$

In the model (7), the undesirable outputs are reduced in proportion to the desirable outputs under the assumption of weak disposability. The minimum undesirable outputs U_j^{count} for every DMU can be expressed by their corresponding desirable outputs. The third column of Table 2 shows the efficiencies of the five original DMUs calculated from model (7). Comparing the efficiency calculated under weak disposability with that calculated under strong disposability, it is noted that the results of model (3) is greater than the results of model (7). This is because the model (7) allows the undesirable output to be generated, while the model (3) does not. By comparing the result of model (2) with model (7), we understand that model (7) has the ability to deal with undesirable outputs. The efficiency scores obtained from model (7) are smaller than or equal to the efficiency scores obtained from models (1) and (2) and this shows that model (7) compared to model (2) shows the effect of undesirable outputs in calculating efficiency well. Gives. Therefore, model (7) can be a suitable alternative to the traditional DEA models such as CCR and SBM DEA-R.

Consider the objective function of model (7) as follows.

$$\left(1-\frac{1}{s\times(m+h)}\left[\sum_{r=1}^{s}\sum_{i=1}^{m}\left(\frac{s_{ir}^{d}}{\left(\frac{x_{io}}{y_{ro}}\right)}\right)+\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^{u}}{\left(\frac{u_{fo}}{y_{ro}}\right)}\right)\right]\right)=$$

$$\left[1-\frac{1}{s\times(m+h)}\left[\begin{array}{l}\sum_{r=1}^{s}\sum_{i=1}^{m}\left(\frac{s_{ir}^{d}}{\left(\frac{x_{io}}{y_{ro}}\right)}\right)+\\ \sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^{u}}{\left(\frac{u_{fo}}{y_{ro}}\right)}\right)-\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^{u}}{\left(\frac{u_{fo}^{count}}{y_{ro}}\right)}\right)+\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^{u}}{\left(\frac{u_{fo}^{count}}{y_{ro}}\right)}\right)\end{array}\right]\right]$$

$$=\left(1-\frac{1}{s\times(m+h)}\left[\sum_{r=1}^{s}\sum_{i=1}^{m}\left(\frac{s_{ir}^{d}}{\left(\frac{x_{io}}{y_{ro}}\right)}\right)-\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^{u}}{\left(\frac{u_{fo}^{count}}{y_{ro}}\right)}\right)\right]\right)-$$

$$\left(\frac{1}{(s\times h)}\left[\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^{u}}{\left(\frac{u_{fo}}{y_{ro}}\right)}\right)-\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^{u}}{\left(\frac{u_{fo}^{count}}{y_{ro}}\right)}\right)\right]\right)=\varphi_{R}^{count*}-\gamma. \qquad (8)$$

As can be seen

$$\frac{1}{(s\times h)}\left[\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^{u}}{\left(\frac{u_{fo}}{y_{ro}}\right)}\right)-\sum_{r=1}^{s}\sum_{f=1}^{h}\left(\frac{s_{fr}^{u}}{\left(\frac{u_{fo}^{count}}{y_{ro}}\right)}\right)\right]=\gamma\geq 0,$$ is a reduction factor. Expression (8) shows that the efficiency of a DMU is the different efficiency score of the desirable outputs namely γ_R^* and adjusted by a reduction factor γ that represents the effect of generating excessive amounts of undesirable outputs. Therefore, we have

$$\varphi_R^{count*}=\gamma_R^*-\gamma\ .$$

4. Case study

To show the validity of the proposed approach, we use a practical example provided by You and Yan [20] including data from a dataset of thirty paper mills along the Huai River in Anhui Province, China. These data include two input components, two desirable output components and one undesirable output. The data is given in Table 3. The results of models (1-7) are shown in Table 4.

Table 3. The input and output data for the case study. (You and Yan [20]).

DMU	**Input**		**Desirable output**		**Undesirable output**
	Input1	**Input2**	**Desirable output1**	**Desirable output2**	**Undesirable output1**
DMU1	437	1438	2015	14,667	665
DMU2	884	1061	3452	2822	491
DMU3	1160	9171	2276	2484	417
DMU4	626	10,151	953	16,434	302
DMU5	374	8416	2578	19,715	229
DMU6	597	3038	3003	20,743	1083
DMU7	870	3342	1860	20,494	1053
DMU8	685	9984	3338	17,126	740
DMU9	582	8877	2859	9548	845
DMU10	763	2829	1889	18,683	517
DMU11	689	6057	2583	15,732	664
DMU12	355	1609	1096	13,104	313
DMU13	851	2352	3924	3723	1206
DMU14	926	1222	1107	13,095	377
DMU15	203	9698	2440	15,588	792
DMU16	1109	7141	4366	10,550	524
DMU17	861	4391	2601	5258	307
DMU18	249	7856	1788	15,869	1449

DMU19	652	3173	793	12,383	1131
DMU20	364	3314	3456	18,010	826
DMU21	670	5422	3336	17,568	1357
DMU22	1023	4338	3791	20,560	1089
DMU23	1049	3665	4797	16,524	652
DMU24	1164	8549	2161	3907	999
DMU25	1012	5162	812	10,985	526
DMU26	464	10,504	4403	21,532	218
DMU27	406	9365	1825	21,378	1339
DMU28	1132	9958	2990	14,905	231
DMU29	593	3552	4019	3854	1431
DMU30	262	6211	815	17,440	965

Table 5. The result of models (1), (2), (3), (4), (6), and (7) for case study.

DMU	Efficiency					Model (6)		
	Model (1)	Model (2)	Model (3)	Model (4)	Model (7)	Objective function	$s^{u}_{connter}$	Counterpart DMUs, undesirable output
DMU1	1	1	0.6667	1	0.7511	496.5989	496.5989	168.4011
DMU2	1	1	0.6667	1	1	0	0	491
DMU3	0.1344	0.2179	0.0896	0.1327	0.1491	107.4623	107.4623	309.5377
DMU4	0.2697	0.4078	0.1798	0.1514	0.3024	0	0	302
DMU5	0.6903	1	0.4602	0.1294	0.7178	0	0	229
DMU6	0.8661	0.8519	0.5774	1	0.643	847.9362	847.9362	235.0638
DMU7	0.541	0.6049	0.3607	0.7654	0.4225	738.7824	738.7824	314.2176
DMU8	0.4137	0.5241	0.2758	0.2999	0.3565	563.2908	563.2908	176.7092
DMU9	0.3389	0.4077	0.226	0.3991	0.276	602.5124	602.5124	242.4876
DMU10	0.5976	0.8623	0.3984	0.4329	0.5185	248.5133	248.5133	268.4867
DMU11	0.4358	0.6094	0.2905	0.371	0.3666	491.8996	491.8996	172.1004
DMU12	0.7754	1	0.5169	0.5131	0.6388	103.4195	103.4195	209.5805
DMU13	0.7243	0.6612	0.4829	1	0.5732	659.5925	659.5925	546.4075
DMU14	1	1	0.6667	0.5197	0.8509	168.6554	168.6554	208.3446
DMU15	1	1	0.6667	0.6204	0.7395	619.0609	619.0609	172.9391
DMU16	0.3663	0.6287	0.2442	0.2076	0.3541	61.9265	61.9265	462.0735
DMU17	0.3075	0.5464	0.205	0.1731	0.311	8.3417	8.3417	298.6583
DMU18	0.8086	0.6887	0.5391	1	0.5721	1238.002	1238.002	210.9981
DMU19	0.3477	0.3531	0.2318	1	0.2625	910.4033	910.4033	220.5967
DMU20	1	1	0.6667	0.8764	0.7421	639.0713	639.0713	186.9287
DMU21	0.5601	0.5703	0.3734	0.8342	0.418	1173.947	1173.947	183.0527
DMU22	0.6227	0.7008	0.4151	0.6058	0.4818	872.4832	872.4832	216.5168
DMU23	0.7093	1	0.4729	0.3783	0.6171	256.6097	256.6097	395.3903
DMU24	0.1475	0.2068	0.0983	0.3443	0.1246	740.357	740.357	258.643
DMU25	0.2046	0.331	0.1364	0.2883	0.1911	340.3906	340.3906	185.6094
DMU26	0.6791	1	0.4527	0.1046	1	0	0	218
DMU27	0.5832	0.5707	0.3888	0.7469	0.4297	1000.523	1000.523	338.4766
DMU28	0.2769	0.5505	0.1846	0.0777	0.4129	78.9244	78.9244	152.0756
DMU29	0.5667	0.5232	0.3778	1	0.4122	872.2944	872.2944	558.7056
DMU30	1	1	0.6667	1	1	0	0	965

If the undesirable output is ignored, then the efficiencies of the thirty DMUs can be calculated via model (1). DMU1, DMU2, DMU14, DMU15, DMU20, and DMU30 are efficient and the other DMUs are inefficient. According to model (2), by considering undesirable output, DMU1, DMU2, DMU5, DMU12, DMU14, DMU15, DMU20, DMU23, DMU26, and DMU30 are efficient and the other DMUs are inefficient. Then the efficiency score of model (2) are greater than model (1), this is non logical. Model (3) considering the new DMUs instead of the original DMUs, these new DMUs have undesirable output equal to zero. All DMUs are inefficient by model (3). This result is not suitable. By considering the weak disposability assumption, we can solve model (4), DMU1, DMU2, DMU6, DMU13, DMU18, DMU19, DMU29, and DMU30 are efficient and the other DMUs are inefficient. The results are different of models (2) and (3). According to the results presented in this paper, we solve the model (7) to get the correct results. Based on the model (7), we first solve model (6) and obtain counterpart (hypothetical) units. DMU2, DMU26, and DMU30 are efficient and the other DMUs are inefficient. The efficiency scores of model (7) are less than or equal to the efficiency scores of model (2). This result is correct as we expected. Model (7) shows the correct efficiency scores under the weak disposability assumption.

5. Conclusion

Traditional DEA models do not correctly calculate the relative efficiency score of the DMUs in the presence of undesirable outputs. As we have shown in this paper, the efficiency score obtained from traditional DEA models in the presence of undesirable outputs is greater or equal to the corresponding value obtained from these models in the absence of undesirable outputs, this issue is not logical and correct from the perspective of DEA. To deal with this problem, we presented non-radial DEA-R models based on slacks corresponding to the ratio of input components to desirable output and slacks corresponding to the ratio of components of undesirable outputs to desired outputs. We presented three different models considering three different strategies to evaluate the efficiency in the presence of undesirable outputs. According to the first strategy, we calculated the efficiency of the DMUs in the absence of undesirable outputs. In the second strategy, we obtained the efficiency of the DMUs by setting its undesirable outputs to zero. In the first two strategies, we used strong disposability assumptions. At the end, according to the strategy with the third strategy, at the beginning, we introduced new units corresponding to each of the original DMUs as counterpart (hypothetical) units. These counterpart units have the same desirable input and output as the main units, but have less undesirable output. In the third strategy, we used the weak disposability assumption. In the following, we obtain the efficiency score of each of the main units based on the boundary created by the new units. This efficiency is lower than the efficiency of the original DMUs in the absence of undesirable outputs and is a correct result. Also, by obtaining the relationship between the efficiency values in the presence and absence of undesirable outputs, we showed the effect of producing these undesirable outputs in the production process. We have shown that the presented models can be used to measure performance in the presence of undesirable outputs. The presented models have linear constraints and if the ratio of the data is important, these models are useful in performance evaluation. As future works, we can use the models presented in this paper for the case where the data is imprecise, such as fuzzy data. Also, the models of this paper can be used for other topics of DEA in the presence of undesirable outputs.

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