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Research Article

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Residue Annihilation Method for Solving Ordinary Differential Equations

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Abstract
A new numerical algorithm is proposed for solving ordinary differential equations. The algorithm is names as Residue Annihilation Method (RAM). The method does not require transfer of the equation into a first order system of equations. For a k'th order nonlinear ordinary differential equation, a parametric solution containing k+2 parameters are assumed as an initial step. By imposing the compatibility conditions together with the annihilation of the residue and its first derivative, a nonlinear system with k+2 equations are obtained. Solving the system yields a recursive relation for the parameters. The assumed parameter values therefore vary at each integration step. Evaluating the parametric solution at each integration step yields the discrete numerical solution. A continuous approximate solution valid throughout the whole domain can also be expressed in terms of the Gamma Interval Functions. Sample ordinary differential equations up to third order derivatives are treated with the new method. The method can be applied to initial value problems directly and to boundary value problems when combined with shooting techniques. Depending on the assumed parametric solution, better convergence can be achieved to the real solution. The convergence rate for the algorithm is O(h2), h being the step size. By including higher order derivatives of the residue, convergence rate can be increased.

Keywords: Numerical Analysis, Residue Annihilation, Numerical Iteration Techniques, Convergence, Initial and Boundary Value Problems.

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1. Introduction

Most of the nonlinear ordinary differential equations and even the linear ones with variable coefficients do not possess closed form functional solutions. Lie Group Theory is a systematic way of producing exact closed form solutions for especially nonlinear differential equations [1-3]. The method relies on the symmetry properties of the differential equations with respect to the Lie Group transformations. Although the method is intended to produce purely analytical solutions, the transformed equations may need numerical techniques for retrieving the final solutions. Approximate analytical or numerical solutions are inevitable if the equation does not possess an exact analytical solution. One of the oldest and most common analytical techniques is to solve the equations with a perturbation series which is asymptotic in nature and depends on the small parameter assumption [4]. To overcome the limitations of the small parameter assumption, perturbation-iteration techniques were developed in the last decades [5-7] and successfully applied to many mathematical physics problems [8-15] with a conclusion that the method is effective, straightforward and produced convergent solutions to the real problems.

The alternative to the analytical and semi-analytical techniques is the purely numerical techniques in search of discrete solutions of the differential equations. The Euler, modified Euler and Runge Kutta methods are well established and discussed in standard textbooks [16]. An excellent handbook discussing briefly both analytical and numerical methods for differential equations is due to Zwillinger [17]. A class of solutions which have been treated in the literature extensively is the collocation methods [17-26]. The method can be considered as a semi-analytical method in which the assumed form of the solution is required to satisfy the conditions and the original equation at some discrete points called collocation points [17]. Usually, some base functions are selected with parameters being their coefficients. The coefficients are determined by the requirement that the approximate solution satisfies the equation at some collocation points and convergence is achieved by taking more terms for the base functions. The parameters which are calculated from the linear system of equations are constants and do not vary within the domain of interest. Variations of the method and adaptation to specific equations exist extensively in the literature. Yahaya and Badmus [18] used polynomial type base functions to treat second order differential equations with a self-starting hybrid block method. For the implementation of the method to boundary value problems, see Russell and Shampine [19]. A new Tau method was presented to solve differential equations of polynomial form with respect to the solution function and its derivatives [20]. A block hybrid collocation method was developed for third order ordinary differential equations [21]. Daşcıoğlu & İşler [22] developed a collocation method based on Bernstein polynomials. Chebyshev series were used as base functions in the collocation method [23]. Legendre-Gauss interpolation was incorporated into the collocation methods [24]. Morgan-Voyce polynomials were employed for solving nonlinear differential equations with quadratic and cubic terms [25]. Stiff initial value problems and Fredholm integral equations were solved with a special local collocation method [26].

In this work, a new version of the collocation method is presented and the method is named as "Residue Annihilation Method" (RAM) to distinguish the method from the previous ones. A form of solution is assumed as an initial guess with sufficient number of parameters. If the equation is k'th order, then k+2 parameters are needed in the assumed solution. Imposing the compatibility conditions at each integration step together with annihilation of the residue and its first derivative, a nonlinear algebraic system of equations for parameters are derived and the solutions lead to recursive expressions for the parameters. Since parameters alter at each

integration step, a simpler assumed solution is sufficient to determine the approximate solution. The differences between the standard collocation method (CM) and the new method are: 1) Parameters are coefficients of base functions in CM whereas one does not require such a condition in RAM. The parameters may be coefficients or implicit arguments of the functions. 2) Base functions are selected and convergence is achieved by increasing the number of base functions whereas in RAM relatively simpler functions can be selected and convergence is achieved locally through variation of the parameters, not the base functions. 3) Collocation methods are more suitable for boundary value problems whereas RAM is a solver of initial value problems. 4) Parameters are solved for given collocation points once and they are not altered in the solution whereas in RAM, piecewise continuous solutions are obtained with parameters changing at each integration step.
The Local Annihilation Method [26] (LAM) proposed by Abdollahi and Babolian is closer to our work in the sense that they also used piecewise continuous functions and allowed variation of the parameters. However, the parameters are coefficients of base functions whereas one does not require such condition in RAM. Depending on the assumed form of solution, nonlinear algebraic equations appear for the parameters in RAM whereas in LAM, linear systems of equations are retrieved. One of the advantages of the method is that by defining a Gamma Interval Function (GIF), the solutions can be expressed as continuous solutions not as discrete solutions as in the case of standard numerical techniques.

2. Residue Annihilation Method

The essential steps of the algorithm are outlined in this section. Consider a k'th order nonlinear ordinary differential equation

$$F\left(x, y, y', y'', \dots, y^{(k)}\right) = 0 \qquad (2.1)$$

having initial conditions

$$y^{(j)}(0) = \alpha_j \quad , \qquad j = 0,1,2 \dots k-1 \qquad (2.2)$$

with $y^{(0)} = y$ and $y^{(j)} = \frac{d^j y}{dx^j}$. The algorithm is:
A1) Assume a solution of the form

$$y = u\left(x, a_j\right), j = 0,1,2, \dots, k+1 \qquad (2.3)$$

possessing totally k+2 parameters. The parameters need not be coefficients of some base functions rather, they may appear as arguments inside the functions.
A2) At each integration step i, the compatibility conditions, annihilation of the residue and its first derivative yields

$$u_i^{(j)}(x_i) = u_{i-1}^{(j)}(x_i) \quad , \qquad j = 0,1,2 \dots k-1; \quad i = 0,1,2, \dots n \qquad (2.4)$$
$$R_i^{(j)}(x_i) = 0 \quad , \qquad j = 0,1; \quad i = 0,1,2, \dots n \qquad (2.5)$$

where $x_i = ih$ for a constant step size h. Therefore, for each integration step i, one has k+2 nonlinear algebraic equations to be solved in terms of the parameters.
A3) Solve the system (2.4) and (2.5) as a recursive relation

$$a_{ij} = g(a_{i-1m}), \; i = 1,2, \dots n; \; j = 0,1,2 \dots k+1; \; m = 0,1,2 \dots k+1 \qquad (2.6)$$

A4) Determine a_{0j} from the initial conditions, Residue equation and its first derivative.
A5) The approximate discrete solution at each point x_i is

$$u_i(x_i, a_{ij}),\ i = 0,1,2, \ldots n;\ \ j = 0,2, \ldots, k+1 \tag{2.7}$$

A6) The continuous solution over the whole domain of interest in terms of the Gamma Interval Function is

$$u = \sum_{i=0}^{n} u_i(x)\gamma[x_i, x_{i+1})(x) \tag{2.8}$$

where GIF is defined as

$$\gamma[a,b)(x) = \begin{cases} 1 & a \le x < b \\ 0 & x < a, x \ge b \end{cases} \tag{2.9}$$

If one needs to increase the rate of convergence, then higher order derivatives of the residue may be included with each inclusion increasing the number of parameters in the assumed solution by one for consistency. The rate of convergence is more specifically discussed in the subsequent sections.

3. Differential Equations up to Second Order

Three sample problems will be treated numerically by the method. All problems possess exact solutions for the intent of testing the algorithm.

Sample Problem 1

Consider the first order differential equation

$$y' - 2xy = 0\ , \quad y(0) = 1 \tag{3.1}$$

which has an exact solution for the given initial condition

$$y_e(x) = e^{x^2} \tag{3.2}$$

To apply the algorithm, the assumed form of solution requires three parameters

$$u_i(x) = a_i e^{b_i x} + c_i \quad , \quad i = 0,1,2, \ldots n \tag{3.3}$$

Equations (2.4) and (2.5) lead to

$$a_i e^{b_i x_i} + c_i = a_{i-1} e^{b_{i-1} x_i} + c_{i-1} \tag{3.4}$$

$$a_i e^{b_i x_i}(b_i - 2x_i) - 2x_i c_i = 0 \tag{3.5}$$

$$a_i e^{b_i x_i}(b_i^2 - 2 - 2x_i b_i) - 2c_i = 0 \tag{3.6}$$

where $x_i = ih$ for a constant step size h. Solving the nonlinear algebraic equations by elimination method, the recursive relations are

$$b_i = \frac{1+2x_i^2}{x_i} \tag{3.7}$$

$$a_i = \frac{2x_i}{b_i} e^{-b_i x_i}(a_{i-1} e^{b_{i-1} x_i} + c_{i-1}) \tag{3.8}$$

$$c_i = -a_i e^{b_i x_i} + a_{i-1} e^{b_{i-1} x_i} + c_{i-1} \tag{3.9}$$

The initial condition, the Residue equation and its first derivative require

$$a_0 = 0, \quad b_0 = 0, \quad c_0 = 1 \tag{3.10}$$

Note that (3.7), (3.9) must be calculated in order. The discrete solution at each station x_i is then

$$u_i(x_i) = a_i e^{b_i x_i} + c_i, \quad i = 0,1,2,\ldots n \tag{3.11}$$

The continuous approximate solution valid throughout the whole domain of interest is

$$u(x) = \sum_{i=0}^{n}(a_i e^{b_i x} + c_i)\gamma[x_i, x_{i+1})(x) \tag{3.12}$$

To identify the differences between the exact and approximate solution visually, a fairly large step size is chosen, i.e., $h = 0.1$, in Figure 1.

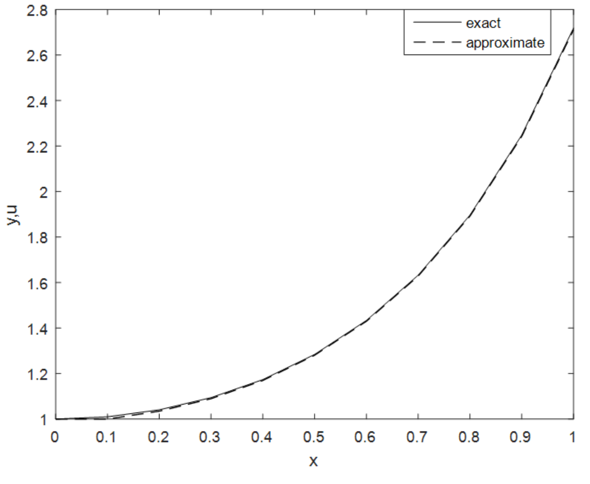

Figure 1. Comparison of approximate and exact results for sample problem 1 ($h = 0.1$)

The maximum error is defined as

$$e_m = \max_{0\le i\le n}|y_e(x_i) - u_i(x_i)| \tag{3.13}$$

For various step sizes, the maximum errors are listed in Table 1.

Table 1. Maximum Errors for Sample Problem 1

h	e_m
0.1	0.0101
0.01	$1.4947\ 10^{-4}$
0.001	$3.5929\ 10^{-6}$

The maximum errors are approximately proportional to $O(h^2)$.

Sample Problem 2

Consider the nonlinear first order differential equation

$$y' + y^2 = 0, \qquad y(0) = 1 \tag{3.14}$$

with an exact solution

$$y_e(x) = \frac{1}{1+x} \tag{3.15}$$

The three-parameter solution

$$u_i(x) = a_i e^{b_i x} + c_i \quad , \quad i = 0,1,2, \ldots n \tag{3.16}$$

is selected as our approximate solution. Equations (2.4) and (2.5) lead to

$$a_i e^{b_i x_i} + c_i = a_{i-1} e^{b_{i-1} x_i} + c_{i-1} \tag{3.17}$$

$$a_i b_i e^{b_i x_i} + (a_i e^{b_i x_i} + c_i)^2 = 0 \tag{3.18}$$

$$b_i + 2a_i e^{b_i x_i} + 2c_i = 0 \tag{3.19}$$

where $x_i = ih$ for a constant step size h. Solving the nonlinear algebraic equations by elimination method, the recursive relations are

$$b_i = -2(a_{i-1} e^{b_{i-1} x_i} + c_{i-1}) \tag{3.20}$$

$$c_i = -\frac{b_i}{4} \tag{3.21}$$

$$a_i = -\left(c_i + \frac{b_i}{2}\right) e^{-b_i x_i} \tag{3.22}$$

The initial condition, the Residue equation and its first derivative require

$$a_0 = \frac{1}{2}, \quad b_0 = -2, \quad c_0 = \frac{1}{2} \tag{3.23}$$

The discrete solution is then

$$u_i(x_i) = a_i e^{b_i x_i} + c_i \quad , \quad i = 0,1,2, \ldots n \tag{3.24}$$

with the corresponding continuous approximate solution being

$$u(x) = \sum_{i=0}^{n} (a_i e^{b_i x} + c_i) \gamma[x_i, x_{i+1})(x) \tag{3.25}$$

To distinguish the curves from each other, the step size is chosen as $h = 0.5$ in Figure 2.

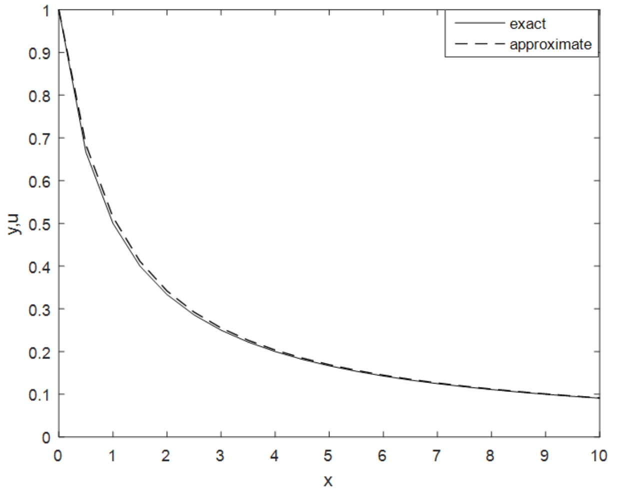

Figure 2. Comparison of approximate and exact results for sample problem 2 ($h = 0.5$)

For various step sizes, the maximum errors are listed in Table 2.

Table 2. Maximum Errors for Sample Problem 2

h	e_m
0.5	0.0173
0.1	5.3590 10^{-4}
0.01	4.9795 10^{-6}

The maximum errors are approximately proportional to $O(h^2)$.

Sample Problem 3

Consider the second order differential equation

$$(1+x)y'' + y' = 0\,, \quad y(0) = 0, \quad y'(0) = 1 \tag{3.26}$$

with an exact solution

$$y_e(x) = ln\,(1+x) \tag{3.27}$$

Since the equation is second order, one needs four parameters in the assumed solution

$$u_i(x) = a_i + b_i x + c_i e^{d_i x} \quad , \quad i = 0,1,2,\ldots n \tag{3.28}$$

The algebraic equations to be solved are

$$a_i + b_i x_i + c_i e^{d_i x_i} = a_{i-1} + b_{i-1} x_i + c_{i-1} e^{d_{i-1} x_i} \tag{3.29}$$

$$b_i + c_i d_i e^{d_i x_i} = b_{i-1} + c_{i-1} d_{i-1} e^{d_{i-1} x_i} \tag{3.30}$$

$$b_i + \big((1+x_i)d_i + 1\big) c_i d_i e^{d_i x_i} = 0 \tag{3.31}$$

$$(1+x_i)d_i + 2 = 0 \tag{3.32}$$

The solutions are

$$d_i = -\frac{2}{1+x_i} \tag{3.33}$$

$$c_i = \frac{e^{-d_i x_i}}{2d_i}\left(b_{i-1} + c_{i-1} d_{i-1} e^{d_{i-1} x_i}\right) \tag{3.34}$$

$$b_i = -c_i d_i e^{d_i x_i} + b_{i-1} + c_{i-1} d_{i-1} e^{d_{i-1} x_i} \tag{3.35}$$

$$a_i = -b_i x_i - c_i e^{d_i x_i} + a_{i-1} + b_{i-1} x_i + c_{i-1} e^{d_{i-1} x_i} \tag{3.36}$$

The initial values for the parameters are

$$a_0 = \frac{1}{4}, \quad b_0 = \frac{1}{2}, \quad c_0 = -\frac{1}{4}\,, \quad d_0 = -2 \tag{3.37}$$

The discrete solution is then

$$u_i(x_i) = a_i + b_i x_i + c_i e^{d_i x_i} \;, \quad i = 0,1,2,\ldots n \tag{3.38}$$

with the corresponding continuous solution being

$$u(x) = \sum_{i=0}^{n} (a_i + b_i x + c_i e^{d_i x})\gamma[x_i, x_{i+1})(x) \tag{3.39}$$

To distinguish the curves from each other, the step size is chosen as $h = 0.2$ in Figure 3.

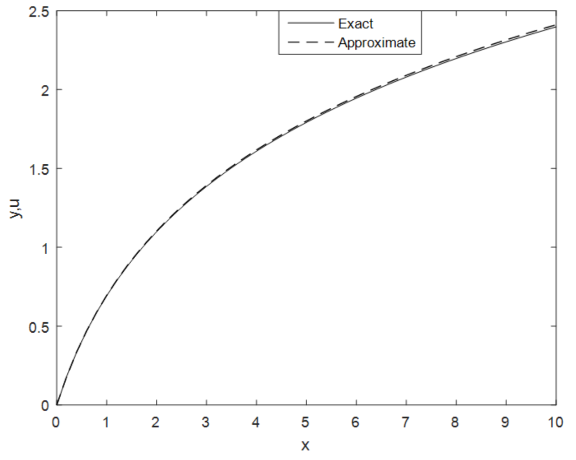

Figure 3. Comparison of approximate and exact results for sample problem 3 ($h = 0.2$)

For various step sizes, the maximum errors are listed in Table 3.

Table 3. Maximum Errors for Sample Problem 3

h	e_m
0.2	0.0132
0.02	$1.2749\ 10^{-4}$
0.002	$1.2687\ 10^{-6}$

The maximum errors are proportional to $O(h^2)$. Note that the functional value of any intermediate point not calculated numerically can be evaluated with the aid of the continuous function (3.39).

4. Blasius Equation

Blasius equation is one of the most famous equations in fluid dynamics derived from the boundary layer equations of a fluid flowing over a plate. The equation is third order and the problem is essentially a boundary value problem, not an initial value problem. The equation will be solved with the new numerical technique

$$f''' + \frac{1}{2} f f'' = 0 \, , f(0) = 0, \quad f'(0) = 0, f'(\infty) = 1 \tag{4.1}$$

The problem can be converted into an initial value problem by replacing

$$f''(0) = \alpha \tag{4.2}$$

with the last condition at infinity. α is determined by trial and error (Shooting technique) so that the last condition is satisfied at a fairly large value. The method requires an initial solution with five parameters. One of the simplest choices is to try a polynomial function

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3 + e_i x^4 \tag{4.3}$$

The coefficients are found as

$$e_i = \frac{f_i''}{96}(f_i^2 - 2f_i') \tag{4.4}$$

$$d_i = -4e_i x_i - \frac{1}{12} f_i f_i'' \tag{4.5}$$

$$c_i = -3d_i x_i - 6e_i x_i^2 + \frac{1}{2} f_i'' \tag{4.6}$$

$$b_i = -2c_i x_i - 3d_i x_i^2 - 4e_i x_i^3 + f_i' \tag{4.7}$$

$$a_i = -b_i x_i - c_i x_i^2 - d_i x_i^3 - e_i x_i^4 + f_i \tag{4.8}$$

where f_i, f_i' and f_i'' are all known functions defined by

$$f_i = a_{i-1} + b_{i-1} x_i + c_{i-1} x_i^2 + d_{i-1} x_i^3 + e_{i-1} x_i^4 \tag{4.9}$$

$$f_i' = b_{i-1} + 2c_{i-1} x_i + 3d_{i-1} x_i^2 + 4e_{i-1} x_i^3 \tag{4.10}$$

$$f_i'' = 2c_{i-1} + 6d_{i-1} x_i + 12e_{i-1} x_i^2 \tag{4.11}$$

The initial parameter values are

$$a_0 = 0, \quad b_0 = 0, \quad c_0 = \frac{\alpha}{2}, \quad d_0 = 0, \quad e_0 = 0 \tag{4.12}$$

The discrete solution is then

$$f_i(x_i) = a_i + b_i x_i + c_i x_i^2 + d_i x_i^3 + e_i x_i^4 \ , i = 0,1,2, \ldots n \tag{4.13}$$

with the corresponding continuous solution being

$$u(x) = \sum_{i=0}^{n} (a_i + b_i x + c_i x^2 + d_i x^3 + e_i x^4) \gamma[x_i, x_{i+1})(x) \tag{4.14}$$

The shooting technique dictates an initial value for the second order derivative as

$$\alpha = 0.3320573 \tag{4.15}$$

Results of an adaptive step size Runge-Kutta method with the constant step size Residue Annihilation method are contrasted in Figures 4 (*f*), and 5 (f'). Within the domain of interest, both solutions agree well for a step size of $h = 0.1$ for RAM.

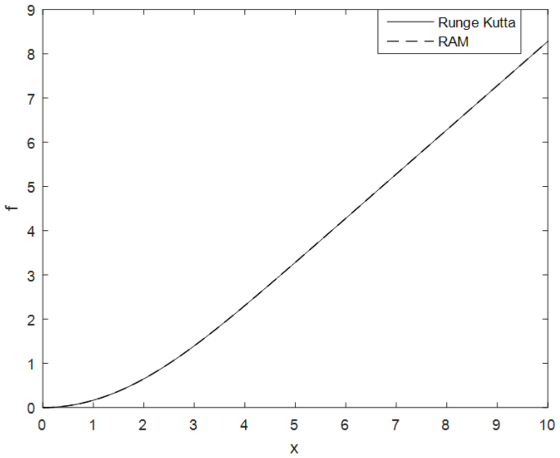

Figure 4. Comparison of RAM and Runge Kutta solutions (*f*) for the Blasius equation ($h = 0.1$)

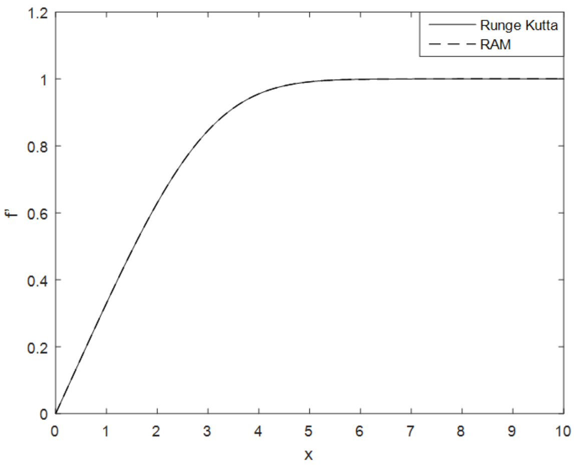

Figure 5. Comparison of RAM and Runge Kutta solutions (f') for the Blasius equation ($h = 0.1$)

Since both solutions are approximate, there is no point to calculate relative errors since the exact analytical solution is not available.

5. Improvement in the Convergence Rates

An improvement in the rate of convergence can be achieved by including higher order derivatives of the residues. For this task, reconsider sample problem 1 again.

$$y' - 2xy = 0\,, \quad y(0) = 1 \tag{5.1}$$

and include second order derivatives of the Residue also. The compatibility equation and the residue equations are

$$u_i(x_i) = u_{i-1}(x_i),\, R_i(x_i) = 0,\, R_i'(x_i) = 0,\, R_i''(x_i) = 0,\, i = 0,1,2, \dots n \tag{5.2}$$

For simplicity, one may assume a polynomial with four parameters since the equations to be solved are four in number,

$$u_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i \;\; , i = 0,1,2, \dots n \tag{5.3}$$

Substituting (5.3) into (5.2), the recursive relations for the coefficients are

$$a_i = \frac{u_i'''}{6} \tag{5.4}$$

$$b_i = \frac{u_i''}{2} - 3a_i x_i \tag{5.5}$$

$$c_i = u_i' - 3a_i x_i^2 - 2b_i x_i \tag{5.6}$$

$$d_i = u_i - a_i x_i^3 - b_i x_i^2 - c_i x_i \tag{5.7}$$

Where

$$u_i = a_{i-1}x_i^3 + b_{i-1}x_i^2 + c_{i-1}x_i + d_{i-1} \quad (5.8)$$

$$u_i' = 2x_i u_i \quad (5.9)$$

$$u_i'' = 2u_i + 2x_i u_i' \quad (5.10)$$

$$u_i''' = 4u_i' + 2x_i u_i'' \quad (5.11)$$

The discrete solution is then

$$u_i(x_i) = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i \quad , \; i = 0,1,2, \dots n \quad (5.12)$$

with the corresponding continuous solution being

$$u(x) = \sum_{i=0}^{n} (a_i x^3 + b_i x^2 + c_i x + d_i)\gamma[x_i, x_{i+1})(x) \quad (5.13)$$

Figure 6 shows the comparisons of the approximate and exact solutions for $h = 0.1$.

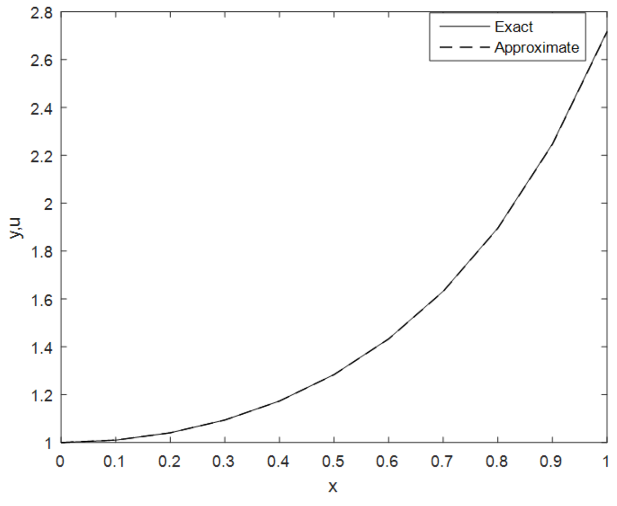

Figure 6. Comparison of improved approximate and exact results for sample problem 1 ($h = 0.1$)

The match is excellent. For various step sizes, the maximum errors are listed in Table 4.

Table 4. Maximum Errors for Improved Solution of Sample Problem 1

h	e_m
0.1	0.0030
0.01	$3.4751\ 10^{-6}$
0.001	$3.5279\ 10^{-9}$

The maximum errors are proportional to $O(h^3)$. From the numerical solutions and the character of the equations, for a k'th order ordinary differential equation and inclusion of the m'th derivatives of the residues, the conclusions are

- The solution must contain $k + m + 1$ parameters
- The maximum errors are proportional to $O(h^{m+1})$.

- If the recursive relations yield constant solutions for the parameters that do not vary depending on the integration steps, then the assumed solution is the exact solution.

6. Concluding Remarks

A new numerical technique is proposed to solve ordinary differential equations. The method is more suitable for initial value problems albeit it can be applied to boundary value problems if combined with shooting techniques. The method can directly be implemented to the equation without reduction to system of first order equations as is done in standard numerical techniques for initial value problems. An assumed form of solution is proposed which contains 2 more parameters than the order of the equation. The compatibility conditions, the residue equation and its derivative lead to a system of algebraic nonlinear equations which results in recursive relations for the parameters. The discrete solution can also be expressed as a continuous solution in terms of the Gamma Interval function. It is found that the exact and numerical solutions match with each other for fairly large step sizes. The numerical simulations reveal that the maximum errors are proportional to h^2, h being the step size. The third order boundary value problem of Blasius equation is also solved with the method displaying a good match with the adaptive step size Runge Kutta method.

If the number of parameters is increased with an inclusion of residue annihilations of higher order derivatives, the convergence rate can be increased. For inclusions up to m'th order derivatives of the residue, the order of convergence would be proportional to h^{m+1} with a disadvantage of increasing the computational algebra due to increased number of parameters. The implementation of the method to many physical problems may appear in the future.

7. Declarations

Ethical Approval- Not applicable
Availability of Supporting Data- Additional data available upon request from the author.
Competing Interests- Author declares no competing interests
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Author Contribution- M. Pakdemirli is solely responsible from all stages of the manuscript.
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