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The Semi-Additive Production Technology in Presence Flexible Measures in DEA

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Abstract

Data envelopment analysis (DEA) obtains the relative efficiency of decision-making units (DMUs) based on their inputs and outputs. In many situations, some variables can play the role of input or output for a DMU. It is important to provide a suitable model that can maximize the efficiency of the unit under evaluation by correctly choosing the role of these variables. One of the production technologies in DEA is semi-additive production technology. In addition to the observed DMUs, this technology also considers the set of all aggregations corresponding to these DMUs in the evaluation of efficiency. In this paper, we presented the semi-additive production technology in DEA in the presence of flexible measures. This technology is created based on the observed DMUs and their corresponding aggregations DMUs. We have shown that we can provide semi-additive production technology in the presence of flexible measures based only on the observed DMUs by removing the region with decreasing returns to scale from the production possibility set (PPS). In the following, we present two different approaches for measuring the efficiency of DMUs in the presence of flexible measures in semi-additive production technology. These two approaches allocate flexible measure as input or output in such a way that the efficiency of the unit under evaluation is maximized. Also, we use the proposed approach to evaluate the efficiency of data sets related to academic units.

Keywords : DEA; Semi-additive production technology; Efficiency; Classification; Flexible measures.

1 Introduction

DEA is a mathematical programming methodology used to evaluate the relative efficiency of a set of DMUs which was introduced by Charnes et al. [9]. The evaluation of the efficiency of each of the DMUs depends on the input and output variables, and the selected production technology. One of the important issues in evaluating the efficiency of DMUs is the selection of input and output variables. In many applications of the real world, certain variables may play the role of input or output for DMUs, and it is important for the decision-maker (DM) to choose these variables as input or output, and the appropriate choice can determine the efficiency of the DMU correctly. Such data are called flexible measures. Some of factors in DEA can play both input and output roles. These factors called flexible measures. As can be seen, efficiency score is the ratio of outputs over inputs in DEA, also, the output is

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a variable and when it increases it causes increase efficiency score. Also, input is a variable whose reduction increases efficiency score. Research and development can be considered as both input and output factors in the supplier selection problem. Therefore, if we reduce the Research and development then it increases the expenditure of financial resources, in this case, Research and development can be considered as an input factor. If we consider it as an input, it imposes tremendous costs on the supplier, and if we consider it as an output, it acts as an agent of innovation that leads to better products, services and processes. Beasley [6] considered the research funding factor as a flexible measure in evaluating US universities. The research funding factor is a proxy for the quality of the research program, which leads to acquiring the income for universities on the one hand, and supports different outputs, such as graduate students, on the other hand.

Cook and Zhu [12] developed traditional DEA models to deal with flexible data in constant returns to scale (CRS) technology. The model presented by them was a fractional programming model in the presence of binary variables. Their model determined the best choice for these data as input or output components by maximizing the efficiency according to each DMU. They presented the necessary transformation in order to transform their model into a mixed integer programming model. In the following, other approaches were presented to deal with flexible measures in DEA.

Cook and Green [11] evaluate power plant efficiency in flexible measures. Toloo [32] presented a mixed integer linear programming to deal with flexible measures in DEA. He showed that if we consider alternative optimal solutions in the model proposed by Cook and Zhu [12], in this case, I, we cannot obtain more correct results for choosing flexible measures as input or output. His model solved the problem of considering alternative optimal solutions.

Toloo [23] reviewed the models presented by Cook and Zhu [12]. He showed that in the model presented by Cook and Zhu [12] by choosing large values for M, the efficiency score may not be obtained correctly. He modified the model presented by Cook and Zhu [12] and solved the computational problems of this model. Farzipoor Saen [16] proposed a novel model for determining third-party reverse logistics providers in the presence of multiple dual-role factors. Their model can be easily solved by computer and used as a decision tool to help decision makers.

Azadi and Farzipoor Saen [3] proposed a new chance-constrained DEA approach to assist the DMs to determine the most appropriate third-party reverse logistics providers in the presence of both dual-role factors and stochastic data. Amirteimoori and Emrouznejad [1] investigated the effect of the presence of flexible measures on the PPS and investigated the effect of the presence of these measures on the efficiency score corresponding to each DMU. They showed that their model is more appropriate in terms of input and output compared to the model proposed by Cook and Zhu [12]. Their model obtains the efficiency score truly. Because the model presented by Cook and Zhu [12] was a very optimistic model, while their model was always overestimating the efficiency. Amirteimoori et al. [2] proposed a flexible slack-based measure of efficiency to maximize the performance. They developed a slacks-based classification DEA model to deal with the presence of flexible measures in DEA. Their model maximizes the efficiency score corresponding to the DMU and considered the flexible measures corresponding to some units as input and for others as output. They showed that when the operational unit is efficient, the flexible measure for this unit can have both input and output roles.

Kordrostami et al. [26] proposed DEA models for classifying inputs and outputs by integer-valued data envelopment analysis. Kordrostami and Noveiri [27] developed a new model to evaluate the efficiency of DMUs where flexible and negative data exist. Azizi and Amirteimoori [4] investigated the efficiency evaluation of DMUs in the presence of imprecise data and presented efficiency evaluation models in the simultaneous presence of imprecise and flexible data. These imprecise data were in the form of intervals. Tohidi and Matroud [31] presented a non-oriented model to deal with flexible measure. Their model provided in CRS and variable returns to scale (VRS) technologies. Their model was also able to deal with negative data. Toloo and Barat [35] proposed models to handle to dual-role factor in supplier selection problem. They show that the dual-role factor can consider as nondiscretionary in the returns to scale concepts. They present a mixed integer linear programming approach for dealing with the dual-role factor. Joulaei et al. [23] presented a suitable model to deal with the simultaneous presence of imprecise and flexible data through fuzzy concepts. Toloo et al. [35] presented a non-radial directional distance-based DEA model for the classification problem of inputs and outputs. They proposed two approaches, pessimistic and optimistic, from both individual and summative points of view. They applied their approach to the banking industry.

Toloo et al. [36] proposed a pair of interval DEA models based on the pessimistic and optimistic standpoints with the aim of handling with interval dual-role factors. Their model determines a unique status of each imprecise dual-role factor. Also, they provided an aggregate model for reallocating the dual-role factors for all DMUs. They applied their models for a data set of 20 banks. Kiyadeh et al. [24] presented a slacks-based classification DEA model to evaluate the efficiency of the DMUs in the presence of flexible measures. They developed DEA models to determine the type of flexible measures as input or output variables. They showed that their model has more adaptability to achieve the desired goals in DEA compared to previous approaches.

Sharifi et al. [29] developed a new stochastic model for classifying flexible measures in DEA. They showed that the orientation of flexible data is not pre-determined, and flexible data can be treated as random variable in terms of both input and output selection. The reason is that the number of DMUs is fixed and all the DMUs are independent. Hosseini Monfared et al. [21] proposed new radial models for classifying flexible measures in two-stage network DEA. Their models can select the status of each flexible measure as an input or output and target setting to inefficient units. Hosseini Monfared et al. [22] Classified flexible measures in two-stage network DEA. They provide an additive efficiency decomposition approach so that the overall efficiency is as a weighted sum of the efficiency of the individual stages. They developed their approach in the both CRS and VRS technologies. Amirteimoori et al. [2] proposes an alternative model that distinguishes the same efficient and inefficient units, however, their model accomplishes different projections for inefficient units obtaining then approach can obtain different classifications of flexible measures. In this way, Boa [7] proposed a comment on the paper of Amirteimoori et al. [2] to solve the problem of the presented model.

Ebrahimi and Hajizadeh [14] proposed a novel DEA model for measuring performance DMUs with flexible measures. They applied their model to Tehran Stock Exchange. They proposed a novel mixed binary linear DEA model for classify the flexible measures and select the best DMU. Also, they developed an algorithm for finding a suitable epsilon score, they rank all efficient units. Tavana et al. [30] proposed a new non-radial directional distance DEA model for dealing with negative and flexible measures. They developed a new extended non-radial directional distance model, that is a variant of the weighted additive model, to handle with negative data. They extend their model in the presence flexible measures, that play the role of both inputs and outputs and have unknown status. They proposed a case study in the automotive industry to show the efficacy of the proposed approach. Eydi and Rastgar [15] developed a DEA model with dual-role factors and fuzzy data for selecting third-party reverse logistics provider. They applied their model in Hospital waste collection. Toloo et al. [37] developed DEA models for classifying flexible measures. They used role of non-Archimedean epsilon. They show that proposed previous models to deal with flexible measures ignore the role of non-Archimedean epsilon in the input-output classification process. They show that these epsilon-free models may ignore some flexible measures in the performance evaluation process and then the status of flexible measures can be randomly and inappropriately identified. They proposed a pair of epsilon-based multiplier and envelopment classifier models. They provided an approach in order to find a suitable epsilon score for their developed classifier models.

Ghiyasia and Cook [19] presented performance evaluation models in DEA in the presence of flexible measures or dual role variables. They showed that if we apply the model proposed by Cook and Zhu [12] in variable scale efficiency technology for the aggregate unit obtained of all DMUs, the model may have an unbounded optimal solution. To deal with the above result, they modified the model proposed by Cook and Zhu [12] and presented a new model that has a bounded optimal solution in the evaluation of the aggregate unit.

The production technology or PPS is a significant part in the efficiency analysis by the DEA. The first production technology by Charnes et al. [9]. It was presented that this technology had the property of CRS. In continuation Banker et al. [5]. They presented a different technology that has VRS property. Koopmans [25] presented another technology that had the property of increasing VRS. Deprins et al. [13] presented Free Disposal Hall (FDH) technology by removing the convexity axiom from the set of axioms in creating PPS. Green and Cook [11] proposed a model in non-convex production technology that obtains the efficiency of each DMU in terms of the observed DMUs and their corresponding aggregate DMUs. As discussed, some production technologies in data coverage analysis, in addition to the observed decision-making units, also consider aggregations units corresponding to them in the creation of production technology. (Ghiyasi and Cook, [20]). Inclusion of aggregations units in production technologies in DEA refers to the additive assumption. This assumption states that if the two DMUs A and B have the possibility of production and belong to production technology, then the aggregations unit corresponding to them in the form of A+B will have the possibility of production and will belong to this technology. (Ghiyasi and Cook, [20]).

Ghiyasi [18] investigated different technologies in DEA and introduced a new assumption called semi-additive assumption by modifying the additive assumption. He introduced semi-additive production technology in DEA. This assumption states that if two different DMUs, A and B have the possibility of production and belong to this set, then the aggregation unit corresponding to them in the form of A+B will also have the possibility of production and will belong to this set. The difference between the additive and semi-additive assumption is that according to the semi-additive assumption, two units must be different in the formation of the aggregation unit. They presented the semi-additive production technology in terms of the power set corresponding to the set of all observed DMUs.

Ghiyasi and Cook [20] investigated the semi-additive production technology and presented the axioms of the construction of this technology. They showed that the relative efficiency evaluation model in semi-additive production technology has a large number of constraints and variables and is not suitable from the computational point of view. In this regard, they presented the PPS related to semi-additive production technology by removing the decreasing DRS area of PPS, and proposed semi-additive production technology only in terms of the observed DMUs. Their new model had fewer variables than their original model and significantly reduced the amount of computation.

The semi-additive assumption states that if two different observed DMUs have the possibility of production, aggregates of the observed DMUs also has the possibility of production. The semi-additive assumption is also called relaxed additivity. The difference between additivity and semi-additive assumptions is that, according to the additivity assumption, the sum of two identical DMUs can also be produced in this technology, while according to the semi-additive assumption, these two DMUs must necessarily be different. In other words, if A is an observed DMU, then according to the additivity assumption, new aggregate DMUs in the form of 2A or 3A are also possible to produce, while according to the semi-additive assumption, new aggregated DMUs in the form of 2A or 3A are not possible to produce. According to the semi-additive postulate, if A and B are two different observed DMUs, then according to the additive postulate, a new aggregated DMU can be produced in the form of A+B. Classic DEA model calculus the efficiency of DMUs based on this issue that type of input and output variables is already known. These models classified input and output variable as either input or output. However, in some real world there are variables that can play both input or output role. They are known as flexible measures. Most of the previous suggested approaches for determining the status of flexible measures are oriented. Therefore, input and output model may obtain the different efficiency scores. As we know, the efficiency score of a DMUs depends on the technology in which we evaluate this DMU. So, in different technologies, we may face the result that a flexible measure is considered as an input in the evaluation of one technology, while it is considered as an output in another technology. Also, the different technologies may produce different efficiency scores for under evaluation DMU. Then, we can be expected a flexible measure is selected as an input variable in one model but an output variable in the other model. In addition, in all of the previous studies did not point to in semi-additive production technology, but the semi-additive assumption is prevailed on many real applications. The resulting VRS model can be unbounded in VRS technology, to alleviate this circumstance, we develop the first approach which is feasible and bounded. To deal with these issues, this study proposes two different approaches for measuring the efficiency of DMUs in the presence of flexible measures in semi-additive production technology.

The semi-additive production technology considers the aggregation of production units in the process of efficiency evaluation in DEA. This technology must consider not only the full set of DMUs, however as well the entire power set, namely the set of all aggregations of the set of DMUs. Therefore, this paper obtains the efficiency of DMUs in the presence of flexible measures in semi-additive production technology in the simultaneous presence of observed DMUs and their corresponding aggregation DMUs. The current paper develops two new model that significantly decreases the computational complexity of models based on the semi-additive production technology in the presence of flexible measures. These two approaches allocate flexible measure as input or output in such a way that the efficiency of the unit under evaluation is maximized in the semi-additive production technology. In this paper, we presented performance evaluation models in semi-additive production technology in the presence of flexible measures. The first approach proposes a model in the envelopment form, this model allows the number of variables to be decreases. The second approach is non-oriented model that not only selects the status of each flexible measure as an input or output but also determines it in presence of aggregations DMUs. Performance evaluation models of DMUs in the presence of flexible measures have not been presented in semi-additive production technology so far. We also compared the results obtained from the previous approaches in VRS and semi-additive production technology, considering the importance of these two technologies in the real world.

It can be said that the main contribution of this paper is as follows. In this paper, we present the semi-additive production technology in the presence of flexible measures based on the observed DMUs and their corresponding aggregation DMUs. With regard to semi-additive production technology, we present two new approaches for calculating the efficiency of DMUs in the presence of flexible measures. Both presented models are always feasible and have a bounded optimal solution. The presented models obtain the efficiency of the unit under evaluation only based on the observed DMUs. These models allocate the flexible measures as input or output to the DMUs in such a way that the efficiency of the unit under evaluation is maximized.

It can be said that the remainder of this paper is organized as follows. In the second section, we will introduce semi-additive production technology in DEA in the presence of flexible measures and present two different approaches to measure the efficiency of DMUs in semi-additive production technology. In the third section, we illustrate the proposed approaches with two different numerical examples. In the fourth section, we propose an application of the proposed approaches in this paper, we use the data set related to universities and compare the results in the semi-additive production technology and VRS technology. In the sixth section, we present the results of the research.

2 The semi-additive production technology with flexible measures

In this section, at first, we illustrate semi-additive technology with simple a numerical example geometrically. In the following, we proposed different two approach to deal to presence of flexible measures in semi-additive technology.

2.1 A geometric explanation of semi-additive production technology

In order to illustrate PPS of semi-additive technology geometrically, let we have three DMUs as follows. $A = (2, 0.5)$ $B = (3, 2.5)$ and $C = (5, 3)$. In this way, we can create aggregated DMUs as follows. $D = A+B = (5, 3)$, $F = A+C = (7, 3.5)$, $G = B+C = (8, 5.5)$, $E = A+B+C = (10, 6)$. To show PPS, we consider horizontal axis as input-axis and vertical axis as output-axis. DMUs A , B, and C are efficient. We consider DMUs in three different technologies including constant returns to scale (CRS), variable returns to scale (VRS), and semi-additive technology geometrically. The CRS technology is the biggest technology that includes all the other technologies. The PPS under CRS property is included the region restricted by the input-axis and the right-hand side of the line starting from the origin and passing the B that show as the dashdot line. The PPS under VRS property is included the bounded region by the input-axis starting from x_A and the segment A-B-C and the horizontal extension from B. The PPS in the semi-additive technology is bounded by the input-axis starting from x_A passing the segment of A-B-G-E and horizontal extension from E. The PPS under semi-additive assumption is bigger than the PPS of the BCC model. Consider a DMU C that is efficient DMU in VRS technology but an inefficient DMU in semi-additive technology. In order to evaluate the efficiency of this DMU, we can project it on efficient frontier of semi-additive technology. We depicts DMU C at point C_1 on the efficiency frontier of the PPS corresponding to semi-additive technology radially. As shown in Fig. 1, the efficiency score is calculated as the ratio $|OC_1^X/OC^X| = 0.6$. C^X and C_1^X, represent the image of the points C and C_1 on the input-axis, respectively.

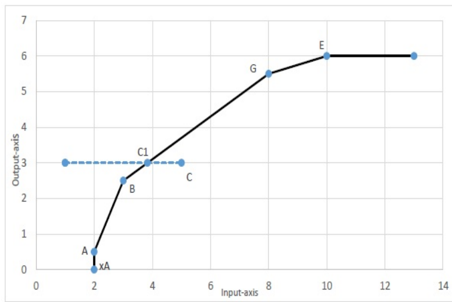

Figure 1: PPS of semi-additive technology.

2.2 Modelling the semi-additive production technology in the presence flexible measures

In this section, we will introduce the semi-additive production technology in the presence of flexible measures and present the efficiency evaluation models based on this production technology. Suppose that there are n DMUs observed as $DMU_j = (X_j, Z_j, Y_j)$, $j = 1, ..., n$, where the input vector $X_j = (x_{1j}, ..., x_{mj}) \in R^m$ are used to produce the output vector $Y_j = (y_{1j}, ..., y_{sj}) \in R^s$. Also assume that each of these DMUs has a flexible measure as $Z_j = (z_{1j}, ..., z_{Dj}) \in R^D$. This measure is considered as an input for some DMUs and as an output for others. Also, this measure can play the role of input or output for the unit under evaluation. Now we introduce the general production technology as follows.

$T = \{$(X,Z,Y)|X *Can produce* (Z,Y)
or (X,Z) *Can produce* Y$\}$.

To introduce semi-additive production technology in the presence of flexible measures, we first introduce the following axioms.

A1. Feasibility of observations.

This axiom implies that all observed DMUs belong to the production technology, i.e. $(X_j, Z_j, Y_j) \in T$, $j = 1, ..., n$.

A2. Free (strong) disposability.

This axiom states that if $(X_1, Z_1, Y_1) \in T$ and if a point (X_2, Z_2, Y_2) is such that, $X_2 \geq X_1$, $Z_2 \geq Z_1$ or $Z_2 \leq Z_1$, $Y_2 \leq Y_1$, then $(X_2, Z_2, Y_2) \in T$.

A3. Convexity.

This axiom states that if $(X_1, Z_1, Y_1) \in T$, $(X_2, Z_2, Y_2) \in T$, then $\mu(X_1, Z_1, Y_1) + (1 - \mu)(X_2, Z_2, Y_2) \in T$ for all $\mu \in (0, 1)$.

A4. Radial rescaling.

This axiom states that if $(X, Z, Y) \in T$, then $\mu(X, Z, Y) \in T$, for all $\mu \geq 0$.

A5. Semi-additive.

This axiom implies that $(X_i, Z_i, Y_i) \in T$, $(X_j, Z_j, Y_j) \in T$, then $((X_i, Z_i, Y_i) + (X_j, Z_j, Y_j)) \in T$, that $i \neq j$.

The semi-additive axiom states that if we consider two DMUs, then their corresponding aggregation DMU also belongs to the production technology.

A6. Minimum extrapolation.

This axiom states that the set T is the smallest set that holds in the above axioms. In other words, the T set is the subscription of all sets of production technologies that have the above properties.

As seen in the numerical example in section 2.1., the PPS of VRS technology is created only based on DMUs, A, B, C as observed DMUs. This PPS is formed only based on observed DMUs, i.e. $J = \{1, ..., n\}$.

Now we consider the power set corresponding to the set $J = \{1, ..., n\}$ as the set $P(J)$. This set includes all subsets of set J. We obtain new aggregations units based on the observed DMUs according to the semi-additive axiom. In fact, we can consider semi-additive production technology as an aggregate form of the VRS technology. It should be noted that the set J' is the power set corresponding to the set J, which includes all the subsets of the set J except the null set, that is, this production technology does not include the origin. Therefore, we put $J' = P(J)/\emptyset$ where $P(J)$ represents the power set of J set and $\emptyset$ show empty set. It should be noted that semi-additive production technology accepts the axioms of including feasibility of observations, free (strong) disposability, convexity, radial rescaling, semi-additive, and minimum extrapolation.

To create the PPS in semi-additive production technology, we must create the power set $P(J)$ corresponding to all the DMUs. We remove the origin from this set. All the aggregated DMUs in the power set belong to semi-additive production technology. Therefore, the semi-additive production technology is created based on all the observed DMUs and the aggregated DMUs corresponding to them in the power set, while the VRS production technology is created only based on the existing observed DMUs. In order to build the PPS in semi-additive production technology, we accept assumptions $A_1 - A_3$, A_5 and A_6. Therefore, semi-additive production technology can be considered as an aggregated form of BCC model. The semi-additive production technology has the property of VRS. The semi-additive production technology allows for an IRS-like technology of the BCC model.

Therefore, we can present the set T_{SF} based on the above axioms as follows.

$$\begin{aligned} T_{SF} = \{(X, Z, Y) \mid & \sum_{j \in J'} \lambda_j X_j \leq X, \\ (\sum_{j \in J'} \lambda_j Z_j \leq Z \text{ or } & \sum_{j \in J'} \lambda_j Z_j \geq Z), \\ & \sum_{j \in J'} \lambda_j Y_j \geq Y, \\ \sum_{j \in J'} \lambda_j = 1, & \lambda_j \geq 0, j \in J'\}. \end{aligned} \tag{2.1}$$

Theorem 2.1. *The set T_{SF} is the smallest set that satisfy in the axioms $A_1 - A_6$. Proof: It is clear that the set T_{SF} satisfy in the axioms $A_1 - A_6$. To show that the set T_{SF} is the smallest set that satisfy in the axioms $A_1 - A_6$. Assume that the set $\hat{T}$ is another set that satisfy in the axioms $A_1 - A_6$. We show that $T_{SF} \subseteq \hat{T}$. Assume that the set $\hat{T}$ applies to the above* *axioms. Suppose that the vector (X, Z, Y) satisfy in the following relation.*

$$\sum_{j\in J'} \lambda_j X_j \leq X,$$
$$(\sum_{j\in J'} \lambda_j Z_j \leq Z \; or \sum_{j\in J'} \lambda_j Z_j \geq Z),$$
$$\sum_{j\in J'} \lambda_j Y_j \geq Y,$$
$$\sum_{j\in J'} \lambda_j = 1, \lambda_j \geq 0, j \in J'\}.$$

Let, vector corresponding to $\lambda = (\lambda_1, ..., \lambda_n)$ in the set $\hat{T}$ is as $(\bar{X}, \bar{Z}, \bar{Y}) = (\sum_{j\in J'} \lambda_j X_j, \sum_{j\in J'} \lambda_j Z_j, \sum_{j\in J'} \lambda_j Y_j)$ where $\sum_{j\in J'} \lambda_j = 1, \lambda_j \geq 0, j \in J'$.

Considering that the set $\hat{T}$ applies to the assumptions of including observations, feasibility, convexity, ray infinity, quasi-collectivity, therefore $(\bar{X}, \bar{Z}, \bar{Y}) \in \hat{T}$. Therefore, the vector $(\bar{X}, \bar{Z}, \bar{Y})$ dominates the vector (X, Z, Y). According to the assumption of feasibility, we conclude that $(X, Z, Y) \in \hat{T}$ and the proof is complete.□

If we design the efficiency evaluation model based on set T_{SF}, then these model has $1+2^n$ decision variables. If the number of observed DMUs is large. Solving the above model is not appropriate from a computational point of view. But in Theorem (2) it was shown that semi-additive production technology can be presented only based on the observed DMUs. In this case, we define a new set as T_{SF}^{new} that it only based on the observed DMUs.

Theorem 2.2. *The set of semi-additive production technology, T_{SF}, is equal to the set below.*

$$T_{SF}^{new} = \{(X, Z, Y) \mid \sum_{j=1}^{n} \lambda_j X_j \leq X,$$
$$(\sum_{j=1}^{n} \lambda_j Z_j \leq Z \; or \sum_{j=1}^{n} \lambda_j Z_j \geq Z),$$
$$\sum_{j=1}^{n} \lambda_j Y_j \geq Y, \tag{2.2}$$
$$\sum_{j=1}^{n} \lambda_j \geq 1, \lambda_j \leq 1, \lambda_j \geq 0, j = 1, ..., n\}.$$

Proof: To show that $T_{SF} = T_{SF}^{new}$. We show that $T_{SF}^{new} \subseteq T_{SF}$ and $T_{SF} \subseteq T_{SF}^{new}$. First, we show that $T_{SF} \subseteq T_{SF}^{new}$. Suppose that $(X, Z, Y) \in T_{SF}$, according to the definition of the set T_{SF}, there exists a vector $\lambda = (\lambda_1, ..., \lambda_n)$ such that

$$\sum_{j\in J'} \lambda_j X_j \leq X,$$
$$(\sum_{j\in J'} \lambda_j Z_j \leq Z \; or \sum_{j\in J'} \lambda_j Z_j \geq Z),$$
$$\sum_{j\in J'} \lambda_j Y_j \geq Y, \tag{2.3}$$
$$\sum_{j\in J'} \lambda_j = 1, \lambda_j \geq 0, j \in J'.$$

Suppose we divide the indexes set relating to DMUs in the set J' into two groups. The first set includes the index corresponding to the observed DMUs, namely J, and aggregation DMUs that belong to the $J' - J$ set. We consider two different cases. In the first case, suppose that we have $\lambda_j > 0$ only for the observed DMUs, where $j \in J$ and for $j \in J' - J$ units we have $\lambda_j = 0$. Therefore, according to relation (3), we will have

$$\sum_{j=1}^{n} \lambda_j X_j \leq X,$$
$$(\sum_{j=1}^{n} \lambda_j Z_j \leq Z \; or \sum_{j=1}^{n} \lambda_j Z_j \geq Z),$$
$$\sum_{j=1}^{n} \lambda_j Y_j \geq Y,$$
$$\sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., n.$$

According to the constraint $\sum_{j=1}^{n} \lambda_j = 1$, $\lambda_j \geq 0$,

$j = 1, ..., n$, we will have $\sum_{j=1}^{n} \lambda_j \geq 1,\ 0 \leq \lambda_j \leq 1$, $j = 1, ..., n$. *So we conclude that* $(X, Z, Y) \in T_{SF}^{new}$.

In the second case, suppose that we have an index $J' - J$, *such that* $\lambda_j > 0$. *Suppose that we define the set* $\hat{J} \subseteq J$ *to contain the indices of the observed DMUs that produce the unit* (X, Z, Y), *i.e.* $X = \sum_{j \in \hat{J}} \lambda_j X_j,\ Z = \sum_{j \in \hat{J}} \lambda_j Z_j,\ Y = \sum_{j \in \hat{J}} \lambda_j Y_j$.

Put in this case $\bar{\lambda}_j = 1,\ j \in \hat{J},\ \bar{\lambda}_j = 0,\ j \in J - \hat{J}$.

So we will have $\sum_{j=1}^{n} \bar{\lambda}_j \geq 1,\ 0 \leq \bar{\lambda}_j \leq 1,\ j = 1, ..., n$, *and*

$$\sum_{j \in J} \bar{\lambda}_j X_j \leq X,$$
$$(\sum_{j \in J} \bar{\lambda}_j Z_j \leq Z \ \ or \sum_{j \in J} \bar{\lambda}_j Z_j \geq Z),$$
$$\sum_{j \in J} \bar{\lambda}_j Y_j \geq Y.$$

namely $(X, Z, Y) \in T_{SF}^{new}$, *then* $T_{SF} \subseteq T_{SF}^{new}$.

To show that $T_{SF}^{new} \subseteq T_{SF}$, *suppose that* $(X, Z, Y) \in T_{SF}^{new}$ *according to the definition of the set* T_{SF}^{new}, *there is a vector* $\lambda = (\lambda_1, ..., \lambda_n)$ *such that* $\sum_{j=1}^{n} \lambda_j X_j \leq X$,

$$(\sum_{j=1}^{n} \lambda_j Z_j \leq Z \ or \sum_{j=1}^{n} \lambda_j Z_j \geq Z),$$
$$\sum_{j=1}^{n} \lambda_j Y_j \geq Y,$$
$$\sum_{j=1}^{n} \lambda_j \geq 1, \lambda_j \leq 1, \lambda_j \geq 0, j = 1, ..., n.$$

We consider two different cases. First, assume that $\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0, j = 1, ..., n$. *Put in this case* $\tilde{\lambda}_j = \lambda_j,\ j \in J,\ \tilde{\lambda}_j = 0,\ j \in J' - J$.

So we will have $\sum_{j \in J'} \tilde{\lambda}_j X_j \leq X$,

$$(\sum_{j \in J'} \tilde{\lambda}_j Z_j \leq Z \ or \sum_{j \in J'} \tilde{\lambda}_j Z_j \geq Z),$$
$$\sum_{j \in J'} \tilde{\lambda}_j Y_j \geq Y,$$
$$\sum_{j \in J'} \tilde{\lambda}_j = 1, \tilde{\lambda}_j \geq 0, j \in J',$$

then $(X, Z, Y) \in T_{SF}$.

In the second case, assume that $\sum_{j=1}^{n} \lambda_j > 1,\ \lambda_j \geq 0$. *Put in this case*

$$\tilde{\lambda}_j = \lambda_j / \sum_{j=1}^{n} \lambda_j,\ j \in J,\ \tilde{\lambda}_j = 0,\ j \in J' - J.$$

So we will have $\sum_{j \in J'} \tilde{\lambda}_j X_j \leq X$,

$$(\sum_{j \in J'} \tilde{\lambda}_j Z_j \leq Z \ or \sum_{j \in J'} \tilde{\lambda}_j Z_j \geq Z),$$
$$\sum_{j \in J'} \tilde{\lambda}_j Y_j \geq Y,$$
$$\sum_{j \in J'} \tilde{\lambda}_j = 1, \tilde{\lambda}_j \geq 0, j \in J',$$

then $(X, Z, Y) \in T_{SF}^{new}$.

Considering that we have shown that $T_{SF}^{new} \subseteq T_{SF}$, *and* $T_{SF} \subseteq T_{SF}^{new}$, *therefore the proof is complete.* □

Therefore, we created the set T_{SF}^{new} based on all the observed DMUs and the aggregated DMUs corresponding to them. In semi-additive production technology T_{SF}^{new}, the DRS region is removed from the VRS technology. For this purpose, the expression $\sum_{j=1}^{n} \lambda_j \geq 1$ is used. This constraint removes the DRS region from the VRS technology and ensures that we have the IRS region in this technology and this constraint do not allow for any scaling up. Also, the constraint $0 \leq \lambda_j \leq 1$, $j = 1, ..., n$, allows us to have all observed DMUs and the aggregated DMUs corresponding to them in the power set in the semi-additive production technology.

Based on the definition of the T_{SF}^{new} set, we present the evaluation model of the efficiency of the unit under evaluation, i.e. $DMU_o = (X_o, Z_o, Y_o)$ in the input oriented in the presence of flexible measures based on semi-additive production technology as follows.

$$\min \ \{\theta^{SF} \mid (\theta^{SF} X_o, (\theta^{SF} Z_o \ \ or \ \ Z_o), Y_o) \in T_{SF}^{new}\}. \tag{2.4}$$

According to the definition of the set T_{SF}^{new}, therefore, model (2.4) will be as follows.

$$\begin{aligned}
&\min \ \theta^{SF} \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta^{SF} x_{io}, \quad i=1,\cdots,m, \\
&\quad (\sum_{j=1}^{n} \lambda_j z_{fj} \leq \theta^{SF} z_{fo}, \\
&\quad or \sum_{j=1}^{n} \lambda_j z_{fj} \geq z_{fo}), \qquad f=1,\cdots,D, \\
&\quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \qquad r=1,\cdots,s, \\
&\quad \sum_{j=1}^{n} \lambda_j \geq 1, \ 0 \leq \lambda_j \leq 1, \ j=1,\cdots,n.
\end{aligned} \tag{2.5}$$

We easily convert model (2.5) into a mixed integer programming model. Suppose that M is a large number. Due to the fact that the relations in second constraint of model (2.5) cannot exist simultaneously, we use two binary variables to include this constraint in model (2.5). We present the model corresponding to model (2.5) as follows.

$$\begin{aligned}
&\theta^{SF*} = \min \ \theta^{SF} \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta^{SF} x_{io}, \quad i=1,\cdots,m, \\
&\sum_{j=1}^{n} \lambda_j z_{fj} \leq \theta^{SF} z_{fo} + M\alpha_f, \ f=1,\cdots,D, \\
&\quad \sum_{j=1}^{n} \lambda_j z_{fj} \geq z_{fo} - M\beta_f, \ f=1,\cdots,D, \\
&\quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \qquad r=1,\cdots,s, \\
&\quad \sum_{j=1}^{n} \lambda_j \geq 1, \ 0 \leq \lambda_j \leq 1, \ j=1,\cdots,n, \\
&\qquad \alpha_f + \beta_f = 1, \qquad f=1,\cdots,D, \\
&\qquad \alpha_f, \beta_f \in \{0,1\}, \qquad f=1,\cdots,D.
\end{aligned} \tag{2.6}$$

In model (2.6), by choosing $\alpha_f = 0$ and $\beta_f = 1$, the fourth constraint of model (2.6) is redundant and third constraint namely $\sum_{j=1}^{n} \lambda_j z_{fj} \leq \theta^{SF} z_{fo} + M\alpha_f$ is satisfied. In this way, z_{fo} is considered as input for the unit under evaluation, i.e. $DMU_o = (X_o, Z_o, Y_o)$. Similarly, in model (2.6) by choosing $\alpha_f = 1$ and $\beta_f = 0$, the third constraint of model (2.6) is redundant and fourth constraint namely $\sum_{j=1}^{n} \lambda_j z_{fj} \geq z_{fo} - M\beta_f$ is established. In this way, z_{fo} for the unit under evaluation, i.e. $DMU_o = (X_o, Z_o, Y_o)$ is considered as output. In model (2.6), only one of the constraints third and fourth will be satisfied, and these two constraints are not simultaneously satisfied.

Theorem 2.3. *Model (2.6) is always feasible and has a bounded optimal solution.*

Proof: If we put $\tilde{\theta^{SF}} = 1$, $\tilde{\lambda}_o = 1$, $\tilde{\lambda}_j = 0$, $j = 1,\cdots,n$, $j \neq o$, $\tilde{\alpha_f} = 1$, $\tilde{\beta_f} = 0$, $f = 1,\cdots,D$. We can provide a feasible solution for model (2.6). With this choice, the third constraint of model(2.6) will be a redundant constraint. Considering that $\theta^{\tilde{SF}} \geq 0$ and $(\tilde{\theta}^{SF}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$ is a feasible solution for model (2.6), from the point of view of optimization, because model (2.6) is a model is minimization, so model (2.6) has a bounded optimal solution score and, the proof is complete.□

Therefore, to evaluate all the DMUs using the design model based on T_{SF}, we must solve this model 2^n times, while to evaluate all the DMUs using the design model based on the T_{SF}^{new} namely model (2.6), then model (2.6) must be solved n times. If the number of observed DMUs is large, solving the design model according to set T_{SF} is not appropriate from a computational point of view. An alternative approach to model (2.6) is based on the entire collection of DMUs. In order to determine designation (Both input and output) for each flexible variable which maximizes the efficiency of the entire set of DMUs. In determining whether flexible measures are selected as inputs or outputs for the DMU under evaluation is to use a new aggregation unit instead of the unit under evaluation in model (2.6). We consider the new aggregation unit based on the all observed DMUs as follows.

$$D\hat{M}U = (\sum_{j=1}^{n} x_{ij}, \sum_{j=1}^{n} z_{fj}, \sum_{j=1}^{n} y_{rj} : \\ i=1,\cdots,m, \ \ f=1,\cdots,D, \ \ r=1,\cdots,s).$$

In model (2.6), instead of evaluating the unit under evaluation, i.e. DMU_o, we evaluate the aggregation unit, i.e. $D\hat{M}U$. In this case, model (2.6) becomes as follows.

$$\begin{aligned} \theta^{SF*} = \min \ & \theta^{SF} \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta^{SF}(\sum_{j=1}^{n} x_{ij}), \\ & i = 1, \cdots, m, \\ & \sum_{j=1}^{n} \lambda_j z_{fj} \leq \theta^{SF}(\sum_{j=1}^{n} z_{fj}) + M\alpha_f, \\ & f = 1, \cdots, D, \\ & \sum_{j=1}^{n} \lambda_j z_{fj} \geq (\sum_{j=1}^{n} z_{fj}) - M\beta_f, \\ & f = 1, \cdots, D, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq (\sum_{j=1}^{n} y_{rj}), \\ & r = 1, \cdots, s, \\ & \sum_{j=1}^{n} \lambda_j \geq 1, \ 0 \leq \lambda_j \leq 1, \ j = 1, \cdots, n, \\ & \alpha_f + \beta_f = 1, \quad f = 1, \cdots, D, \\ & \alpha_f, \beta_f \in \{0, 1\}, \quad f = 1, \cdots, D. \end{aligned} \tag{2.7}$$

Model (2.7) includes flexible measures components as input for some components of the aggregation unit and as output for some others, thus maximizing the efficiency of the aggregation unit $D\hat{M}U$.

Now we present another approach to measure the efficiency of DMUs in the presence of flexible measures as follows.

$$\begin{aligned} EF_{CL}^{SF*} = \min \ & (\theta_{CL}^{SF} / \varphi_{CL}^{SF}) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_{CL}^{SF} \, x_{io}, \ i = 1, \cdots, m, \\ & \sum_{j=1}^{n} \lambda_j z_{fj} \leq \theta_{CL}^{SF} \, z_{fo} + M d_f, \\ & f = 1, \cdots, D, \\ & \sum_{j=1}^{n} \lambda_j z_{fj} \geq \varphi_{CL}^{SF} \, z_{fo} - M(1 - d_f), \\ & f = 1, \cdots, D, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi_{CL}^{SF} \, y_{ro}, \ r = 1, \cdots, s, \\ & \sum_{j=1}^{n} \lambda_j \geq 1, \ 0 \leq \lambda_j \leq 1, \ j = 1, \cdots, n, \\ & d_f \in \{0, 1\}, \ f = 1, \cdots, D, \\ & \theta_{CL}^{SF} \geq 0, \ \varphi_{CL}^{SF} \geq 0. \end{aligned} \tag{2.8}$$

Model (2.8) is a model without essence in DEA. This model is a mixed fractional correct programming problem that we expressed based on the concept of most productive scale (MPS). Model (2.8) obtains the efficiency of the unit under evaluation in the presence of flexible measures in semi-additive technology. θ_{CL}^{SF}, and φ_{CL}^{SF}, are variables considered in model (2.8) to decrease and increase the input and output components, respectively. The vector $\lambda = (\lambda_1, \cdots, \lambda_n)$ is the intensity vector corresponding to the observation DMUs. d_f, $f = 1, \cdots, D$, are binary variables. In model (2.8), by choosing M as a large number, only one of the second and third constraints will be satisfied and the other constraint will be redundant. If in model (2.8) the value of $d_f = 0$ in an optimal solution, then second constraint will be valid and third constraint will be redundant. In this way, z_{fo} is considered as an input for the unit under evaluation, i.e. $DMU_o = (X_o, Z_o, Y_o)$. Similarly, if the model (2.8) obtains the value of $d_f = 1$, in an optimal solution, then third constraint will be valid and second constraint will be redundant. In this way, z_{fo} for the unit under evaluation, i.e. $DMU_o = (X_o, Z_o, Y_o)$ is considered as output. In model (2.8), only one of the second and third constraints will be satisfied, and these two constraints are not simultaneously satisfied.

Theorem 2.4. *Model (2.8) is always feasible and has a bounded optimal solution. Proof: If we put* $\tilde{\theta_{CL}^{SF}} = 1$, $\tilde{\varphi_{CL}^{SF}} = 1$, $\tilde{\lambda}_o = 1$, $\tilde{\lambda}_j = 0$, $j = 1, \cdots, n$, $j \neq o$, $\tilde{d}_f = 0$, $f = 1, \cdots, D$, *we can provide a feasible solution for model (2.8). With this choice, third constraint will be a redundant constraint. Considering that* $\tilde{\theta_{CL}^{SF}} \geq 0$, $\tilde{\varphi_{CL}^{SF}} \geq 0$. *Then* $(\tilde{\theta_{CL}^{SF}}, \tilde{\varphi_{CL}^{SF}}, \tilde{\lambda}, \tilde{d})$ *is a feasible solution for model (2.8), from the point optimization opinion, because model (2.8) is a minimization model, therefore model (2.8) has a bounded optimal solution and and, the proof is complete.*□

Model (2.8) includes flexible measure components for some components of the aggregate unit as input and for others as output, thus maximizing the efficiency of the aggregate unit $D\hat{M}U$. In model (2.8), we can determine the

Table 1: An illustrative example.

DMU	Input	Output	Flexible
1	1	3	6
2	2	5	6
3	1	6	5
4	1	7	6
5	1	8	5
6	2	5	6
7	7	7	2
8	9	3	2

Table 2: The result based on the proposed approaches.

	Results from model (2.6)			Results from model (2.8)	
DMU	Efficiency	δ_1	δ_2	Efficiency	d
1	1	1	0	0.375	0
2	0.5	1	0	0.497	0
3	1	1	0	0.75	0
4	1	1	0	0.875	0
5	1	1	0	1	0
6	0.5	1	0	0.497	0
7	0.14	1	0	0.125	1
8	0.11	1	0	0.0427	1
Aggregated DMU	0.561	1	0	0.2639	1

Table 3: The result based on the previous proposed approaches in the VRS technology.

Amirteimoori and Emrouznejad [1]				Cook and Zhu [12]	
DMU	Efficiency	δ_1	δ_2	Efficiency	d
1	1	1	0	1	0
2	0.5	1	0	0.79	0
3	1	1	0	1	0
4	1	1	0	1	0
5	1	1	0	1	0
6	0.5	1	0	0.79	0
7	0.14	1	0	1	0
8	0.11	1	0	1	0

Table 4: Inputoutput data (Ghiyasi and Cook [19]).

DMU	Input1	Input2	Felexible	Output1	Output2
A	2	5	2	9	4
B	5	7	5	5	5
C	4	4	4	4	6
D	8	1	7	1	2
E	3	9	9	2	8
F	5	8	8	6	4
DMU	27	34	35	27	29

returns to scale class of unit under evaluation based on an optimal solution of the model. Assume that $(\theta_{CL}^{SF*}, \varphi_{CL}^{SF*}, \lambda^*, d^*)$ is an arbitrary optimal solution of model (2.8). If we have A) $\theta_{CL}^{SF*} = \varphi_{CL}^{SF*} = 1$, in this case, the unit under evaluation, i.e. $DMU_o = (X_o, Z_o, Y_o)$

Table 5: Classification results assuming semi additive (considering the flexible measure).

	Results from model (2.6)			Results from model (2.8)	
DMU	Efficiency	δ_1	δ_2	Efficiency	d
A	1	1	0	1	0
B	0.6296	0	1	0.6122	0
C	1	1	0	1	0
D	1	1	0	1	0
E	1	1	0	1	0
F	0.5618	0	1	0.4864	0
Aggregated DMU	0.8933	0	1	0.6721	0

Table 6: The results of Ghiyasia and Cook [19] in the VRS technology

DMU	Efficiency	d
A	1	0
B	1	1
C	1	0
D	1	1
E	1	1
F	1	1
Aggregated DMU	0.946	1

will have a CRS class. (Banker et al. [5]). b) $\theta_{CL}^{SF*} > \varphi_{CL}^{SF*} > 1$, in this case, the unit under evaluation, i.e. $DMU_o = (X_o, Z_o, Y_o)$ will have the IRS (increasing returns to scale) class. Due to the fact that in the semi-additive production technology, we remove the region with decreasing scale efficiency (DRS), therefore the unit under evaluation, i.e. $DMU_o = (X_o, Z_o, Y_o)$ does not have the DRS class. (Ghiyasi and Cook [20]).

Theorem 2.5. *The optimal objective function score of the model (2.8) will not be greater than the optimal objective function score of the model (2.6).*
Proof: Assume that $(\theta^{SF*}, \alpha^*, \beta^*, \lambda^*)$ *be an optimal solution for model (2.6). Then* $(\check{\theta}_{CL}^{SF} = \theta^{SF*}, \check{\varphi}_{CL}^{SF} = 1, \check{\lambda} = \lambda_j^* / \sum_{j=1}^{n} \lambda_j^*, \check{d} = \alpha^*)$, *is a feasible solution for model (2.8). Therefore, the optimal objective function score of model (2.8) will be less than or equal to optimal objective function score of model (2.6)and, the proof is complete.*□

An alternative approach for model (2.8) in determining whether flexible measures are selected as inputs or outputs for the unit under evaluation is to create a new aggregation unit based on all observed DMUs instead of the unit under evaluation. We present the aggregation unit based on all new observed DMUs based on the model (2.8) as follows.

$$
\begin{aligned}
EF_{CL}^{SF*} = \min \ & (\theta_{CL}^{SF} / \varphi_{CL}^{SF}) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_{CL}^{SF} \, (\sum_{j=1}^{n} x_{ij}), \\
& i = 1, \cdots, m, \\
& \sum_{j=1}^{n} \lambda_j z_{fj} \le \theta_{CL}^{SF} \, (\sum_{j=1}^{n} z_{fj}) + M d_f, \\
& f = 1, \cdots, D, \\
& \sum_{j=1}^{n} \lambda_j z_{fj} \ge \varphi_{CL}^{SF} \, (\sum_{j=1}^{n} z_{fj}) - M(1 - d_f), \\
& f = 1, \cdots, D, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge \varphi_{CL}^{SF} \, (\sum_{j=1}^{n} y_{rj}), \\
& r = 1, \cdots, s, \\
& \sum_{j=1}^{n} \lambda_j \ge 1, \ 0 \le \lambda_j \le 1, \ j = 1, \cdots, n, \\
& d_f \in \{0, 1\}, \ f = 1, \cdots, D, \\
& \theta_{CL}^{SF} \ge 0, \ \ \varphi_{CL}^{SF} \ge 0.
\end{aligned}
\tag{2.9}
$$

In this paper, we present two different approaches to measure efficiency in semi-additive production

Table 7: University data (Beasley [6]).

University	Input1	Input2	Output1	Output2	Output3	Flexible
U1	528	64	145	0	26	254
U2	2605	301	381	16	54	1485
U3	304	23	44	3	3	45
U4	1620	485	287	0	48	940
U5	490	90	91	8	22	106
U6	2675	767	352	4	166	2967
U7	422	0	70	12	19	298
U8	986	126	203	0	32	776
U9	523	32	60	0	17	39
U10	585	87	80	17	27	353
U11	931	161	191	0	20	293
U12	1060	91	139	0	37	781
U13	500	109	104	0	19	215
U14	714	77	132	0	24	269
U15	923	121	135	10	31	392
U16	1267	128	169	0	31	546
U17	891	116	125	0	24	925
U18	1395	571	176	14	27	764
U19	990	83	28	36	57	615
U20	3512	267	511	23	153	3182
U21	1451	226	198	0	53	791
U22	1018	81	161	5	29	741
U23	1115	450	148	4	32	347
U24	2055	112	207	1	47	2945
U25	440	74	115	0	9	453
U26	3897	841	353	28	65	2331
U27	836	81	129	0	37	695
U28	1007	50	174	7	23	98
U29	1188	170	253	0	38	879
U30	4630	628	544	0	217	4838
U31	977	77	94	26	26	490
U32	829	61	128	17	25	291
U33	898	39	190	1	18	327
U34	901	131	168	9	50	956
U35	924	119	119	37	48	512
U36	1251	62	193	13	43	563
U37	1011	235	217	0	36	714
U38	732	94	151	3	23	297
U39	444	46	49	2	19	277
U40	308	28	57	0	7	154
U41	483	40	115	0	23	531
U42	515	68	79	7	23	305
U43	593	82	101	1	9	85
U44	570	26	71	20	11	130
U45	1317	123	293	1	39	1043
U46	2013	149	403	2	51	1523
U47	992	89	161	1	30	743
U48	1038	82	151	13	47	513
U49	206	1	16	0	6	72
U50	1193	95	240	0	32	485

technology in the presence of flexible measures. Both approaches can have their advantages. The differences between the two models are as follows.

D1. The first approach, i.e. model (2.6) is in the input oriented, while the second approach, i.e. model (2.8) is without orientation.

D2. Model (2.6) is a mixed integer programming

Table 8: Classification results assuming semi additive (considering the flexible measure).

	Results from model (2.6)			Results from model (2.8)	
DMU	Efficiency	δ_1	δ_2	Efficiency	d
U1	1	1	0	1	0
U2	0.653	1	0	0.6147	0
U3	0.968	1	0	0.6627	1
U4	0.6893	1	0	0.6451	0
U5	0.951	1	0	0.8928	1
U6	1	1	0	1	1
U7	1	1	0	1	1
U8	0.7666	0	1	0.7497	0
U9	0.7296	1	0	0.6576	1
U10	0.9449	1	0	0.8921	0
U11	0.7562	1	0	0.747	1
U12	0.711	1	0	0.6907	0
U13	0.851	1	0	0.7726	1
U14	0.7074	1	0	0.7037	1
U15	0.6892	1	0	0.6892	1
U16	0.5237	1	0	0.5237	1
U17	0.5552	0	1	0.5359	0
U18	0.5925	0	1	0.5925	0
U19	1	1	0	1	0
U20	0.9019	0	1	0.8575	0
U21	0.6693	1	0	0.6693	1
U22	0.664	0	1	0.664	0
U23	0.5603	1	0	0.5603	1
U24	0.4855	0	1	0.4841	0
U25	1	1	0	0.9517	0
U26	0.4344	0	1	0.4251	0
U27	0.856	1	0	0.8534	0
U28	0.8088	1	0	0.8088	1
U29	0.8014	0	1	0.7755	0
U30	0.852	0	1	0.8315	0
U31	0.7283	0	1	0.7283	0
U32	0.8407	1	0	0.8407	1
U33	1	1	0	1	1
U34	1	1	0	1	0
U35	1	1	0	1	0
U36	0.7467	1	0	0.7349	1
U37	0.808	0	1	0.7816	0
U38	0.8064	1	0	0.8064	1
U39	0.9202	0	1	0.7887	1
U40	0.9896	1	0	0.7401	0
U41	0.9979	0	1	0.9935	0
U42	0.9248	0	1	0.8355	1
U43	0.7011	1	0	0.6426	1
U44	1	1	0	1	1
U45	0.8913	0	1	0.8832	0
U46	0.8516	0	1	0.8477	0
U47	0.6573	0	1	0.6573	0
U48	0.8835	1	0	0.8835	1
U49	1	1	0	0.6366	1
U50	0.8369	1	0	0.8369	1
Aggregated DMU	0.9352	1	0	0.6592	0

problem, while model (2.8) is a mixed fractional integer programming problem.

D3. Models (2.6) and (2.8) have $m+s+3D+n+1$ and $m+s+2D+n+1$ constraint respectively,

Table 9: Classification results assuming VRS (considering the flexible measure).

	Ghiyasi and Cook [19]		Tohidi and Matroud [31]	
DMU	Efficiency	*d*	Efficiency	*d*
U1	1	0	1	0
U2	1	0	0.61	0
U3	1	0	0.66	1
U4	0.9559	0	0.65	0
U5	1	0	0.89	1
U6	1	0	0.79	1
U7	1	1	1	1
U8	0.9106	1	0.75	0
U9	1	0	0.53	1
U10	0.9492	0	0.89	0
U11	1	0	0.75	1
U12	0.7368	1	0.58	0
U13	0.865	0	0.77	1
U14	0.7074	1	0.7	1
U15	0.735	0	0.69	1
U16	0.6242	0	0.52	1
U17	0.83	1	0.52	0
U18	0.6726	1	0.59	0
U19	1	1	1	0
U20	1	0	0.73	0
U21	0.8752	0	0.59	1
U22	0.79106	1	0.66	0
U23	0.7786	0	0.55	1
U24	1	0	0.45	0
U25	1	1	0.95	0
U26	1	1	0.43	0
U27	0.8686	1	0.7	0
U28	1	0	0.81	1
U29	0.9584	0	0.78	0
U30	1	0	0.63	0
U31	0.8609	1	0.73	0
U32	0.9497	0	0.84	1
U33	1	1	1	1
U34	1	1	0.92	0
U35	1	1	1	1
U36	1	1	0.73	1
U37	0.9419	0	0.78	0
U38	0.8707	0	0.81	1
U39	0.9209	1	0.62	0
U40	0.9969	0	0.74	0
U41	1	1	1	0
U42	0.9263	1	0.8	1
U43	0.9828	0	0.64	1
U44	1	1	1	1
U45	1	1	0.88	0
U46	1	1	0.85	0
U47	0.7526	1	0.66	0
U48	1	0	0.79	1
U49	1	1	0.45	0
U50	1	0	0.84	1
Aggregated DMU	1	0	-	-

and model (2.6) has more D than model (2.8) . Therefore, if the number of flexible sizes is large, the solution of model (2.8) is more appropriate from the computational point of view.

D4. Models (2.6) and (2.8) have $n + 2D + 1$ and $n + D + 2$ variables, respectively. Therefore, if the number of flexible measures is large, the solution of model (2.8) is more appropriate from the computational point of view.
D5. The objective function of the model (2.6) is linear, while the objective function of the model (2.8) is nonlinear.
D6. Based on the model (2.8), we can also determine the class of returns to scale of the DMUs, which is constant or IRS in the semi-additive production technology in the presence of flexible measures.
According to the above comparison, model (2.8) has fewer constraints and variables than model (2.6). Also, the model (2.8) is a non-orientation model that can make a correct choice to determine whether a flexible measure is selected as input or output.

3 Numerical examples

In this section, in order to check the results of the two proposed approaches in this paper and compare them, we present two numerical examples.

3.1 Numerical example 1

Consider 8 DMUs, each of which uses one input to produce one output, according to Table 1. Each DMUs has a flexible measure.
First, we examine the results based on model (2.6). As can be seen, in the evaluation of the efficiency of the DMUs based on model (2.6), the value of $\alpha^*_f = 0\ f = 1, \cdots, D.$ is obtained, that is, for all the DMUs of model (2.6) the flexible measure considered as output. The second column of Table2 shows the efficiency of each of the measure based on model (2.6). As can be seen, the DMUs, 1, 3, 4, and 5 are efficient units and other units are inefficient. If we use the model (2.7) for the aggregation unit corresponding to all the observed DMUs, then the model (2.7) includes the flexible measure for the aggregation unit as output and the efficiency score of this unit is 0.561. These results are in the last row of Table 2.
Now let's compare the results in semi-additive technology according to model (2.6) with the results obtained from the approach provided by Amirteimoori and Emrouznejad [1] in VRS technology according to Table 3. The results of the two approaches are the same. The approach presented by Amirteimoori and Emrouznejad [1] considers the flexible measure as output for all DMUs. We examine the results of the second proposed approach in this paper to deal with flexible measures in semi-additive production technology according to model (2.8). The results are in the last two columns of Table 2. As can be seen, this model considers flexible measure for units 1, 2, 3, 4, 5, and 6 as input and for units 7 and 8 as output. According to model (2.8), the only efficient unit is unit 5.

If we apply model (2.9) to the aggregation unit corresponding to all the observed DMUs, then model (2.9) includes the flexible measure for the aggregation unit as output and obtains the efficiency score of this unit as 0.2639. Now let's compare the results in semi-additive technology according to model (2.8) with the results obtained from the approach provided by Cook and Zhu [12] in VRS technology. The results are in the last two columns of Table 3. The results of the two approaches are the same. The approach presented by Cook and Zhu [12] takes the flexible measure as the output for all DMUs. Based on the model of Cook and Zhu [12], all DMUs except DMUs 2 and 6 are efficient. These results show the difference between the efficiency scores of the DMUs in flexible measures in VRS and semi-additive production technology. DMUs, 1, 3, 4, 7, and 8 are efficient in the presence of flexible measure in VRS technology, but these DMUs are inefficient in the presence of flexible measure in semi-additive technology.

3.2 Numerical example 2

Now we present another numerical example in order to compare the results of the proposed approaches in this paper in the semi-additive production technology and the previous approaches

in the VRS technology. Table 4 show data related to seven DMUs, each of which has two inputs and two outputs and a flexible measure. The data is given in Table 4.
Now we examine the results of the first approach presented in this paper to deal with flexible measures in semi-additive production technology according to model (2.6). The results are in the second to fourth columns of Table 5. As can be seen, model (2.6) introduces units A, C, D, and E as efficient units and obtains units B and F as inefficient. Model (2.8) introduces flexible measure for units A, C, D, and E as output and for units, B and F as input. If we use model (2.7) for the aggregated unit corresponding to all the observed units, then model (2.7) includes the flexible measure for the aggregated unit as an input and the efficiency score of this unit is 0.8933. These results are in the last row of Table 5.
Now we will examine the results of model (2.8) in semi-additive production technology. The results are in the last two columns of Table 5). As can be seen, model (2.8) introduces units A, C, D, and E as efficient units and obtains units B and F as inefficient. Model (2.8) introduces flexible measure for all units as input.

Now we will examine the results of model (2.8) in semi-additive production technology with the results of the approach provided by Ghiyasia and Cook [19] in VRS technology.
The results are shown in Tables 5 and 6 respectively.

The results related to the approach provided by Ghiyasia and Cook [19] in VRS technology are shown in Table 6. This approach introduces all DMUs as efficient. It also introduces flexible measure for units B, D, F, and E as outputs and for units A and C as inputs.
If we use model (2.9) for the aggregated unit corresponding to all the observed DMUs, then model (2.9) includes the flexible measure for the aggregated unit as an input and the efficiency score of this unit is 0.6741 according to last row of Table 5. But the approach provided by Ghiyasia and Cook [19] in VRS technology considers the flexible measure for the aggregated unit as output and obtains the efficiency score of this unit as 0.946 according to last row of Table 6. As can be seen, the results of the proposed approaches in this paper in the presence of flexible measure and semi-additive production technology are different from the previous approaches in the face of flexible measures in VRS technology. A unit may be efficient in semi-additive production technology, while it is inefficient in VRS technology, and vice versa.

4 An application for university units

In this section, we present an application of the proposed approaches in this paper in the presence of flexible measures and semi-additive production technology. Data are from Beasley [6] and Cook and Zhu [12]. The data is given in Table 7. The data are related to university units. Each academic unit has two entrances included: General Expenditure and Equipment Expenditure and three outputs, consisting of three types of students. The "flexible measure" here is the Research Income.

Now we examine the results of the first proposed approach in this paper to deal with flexible measures in semi-additive production technology according to model (2.6). The results are in the second to fourth columns of Table 8. As can be seen, model (2.6) introduce units 1, 6, 7, 19, 25, 33, 34, 35, 44, 49 as efficient and other units inefficient. According to Table 8, model (2.6) introduces flexible measure for 64 percent of units as output and for 34 percent of units as input. Now we examine the results of model (2.8) in semi-additive production technology. The results are shown in the fifth and sixth columns of Table 8. As can be seen, model (2.8) introduce units 1, 6, 7, 19, 33, 34, 35, 44 as efficient and other units inefficient. According to Table 8, model (2.6) introduces flexible measure for 48 percent of units as output and for 52 percent of units as input.
Now we examine the results of the approaches provided by Tohidi and Matroud [31] and Ghiya-

sia and Cook [19]in VRS technology. The results are shown in Table 9.
The results of the approach provided by Tohidi and Matroud [31] are shown in the fourth and fifth columns of Table 9. Based on this approach, units 1, 7, 19, 33, 36, 41, 44 are efficient and other units are inefficient. This approach introduces the flexible measure for 46 percent of the units as output and for 54 percent of the units as input.
The results of the approach of Ghiyasia and Cook [19] are shown in the second and third columns of Table 8. Based on this approach, units 1, 2, 3, 5, 6, 7, 9, 11, 19, 20, 24, 25, 26, 28, 33, 35, 36, 41, 44, 45, 46, 48, 49, 50, are efficient unit and other units as inefficient. This approach introduces the flexible measure for 52 percent of the units as output and for 48 percent of the units as input.

According to Table 9, if in the approach provided by Tohidi and Matroud [31] and Ghiyasia and Cook [19] the value of $d_f^* = 0$ is obtained, then the flexible measure is obtained as the output and if the value of $d_f^* = 1$ is obtained, then we take the flexible measure as input. Based on the results obtained in this paper, we can find out that the flexible measure may be considered as an input for a special DMU in semi-additive production technology and as an output in VRS technology. This measure can be considered as input or output in order to maximize the efficiency score of the unit under evaluation.
In this study, efficiency measurement models in semi-additive production technology in the presence of flexible measures were presented. As stated in the paper and we showed it geometrically in section 2, VRS and semi-additive technologies are different. So far, no model has been presented to measure efficiency in semi-additive production technology in the presence of flexible measures. Two approaches presented in this paper can be useful. In this paper, we discussed the results of previous models presented to measure efficiency in VRS technology and the results of the models presented in this paper in semi-additive production technology. Although the two technologies are different, comparing the results of the models in these two technologies can help to better understand the models and their validity. In this paper, we have shown that a flexible measure may be considered as an input in the semi-additive production technology, while it is considered as an output in a VRS technology. It is not important to choose a flexible measure as input or output in the model, and the goal is to maximize the efficiency of the DMU under evaluation. As we have shown in the paper, the results related to the two models introduced in this paper to measure efficiency in semi-additive production technology in the presence of flexible measures, i.e. models (2.6) and (2.8), may be different.

5 Conclusion

As we know, the efficiency score corresponding to each DMUs in DEA depends on their input and output vector and the technology in which this unit is evaluated. In traditional DEA models, the input and output variables are determined in advance by the DM. But in the real world, some certain variables can play the role of input or output. It is important to determine whether these variables are included in the evaluation model as input or output. In this paper, we presented the semi-additive production technology in DEA in the presence of flexible measures.
This technology is created based on the observed DMUs and their corresponding aggregations DMUs. We have shown that we can propose semi-additive production technology in the presence of flexible measures based only on the observed DMUs by removing the region with DRS property from the PPS. We also present two different approaches to deal with flexible measures in semi-additive production technology in order to evaluate the DMUs. These two approaches allocate flexible measure as input or output in such a way that the efficiency score of the unit under evaluation is maximized. We have shown that the proposed second presented in this paper is more suitable compared to the first approach because

this approach, in addition to obtaining the efficiency score of the unit under evaluation in the presence of flexible measures, also provides the returns to scale class of the unit under evaluation in semi-additive production technology. We have shown that the class of returns to scale of this unit can be constant or increasing. We have shown that the models presented in this paper are always feasible and have a bounded optimal solution. Then we can say that the advantages of the presented study over the existing is as follows.
a. We present the semi-additive production technology in the presence of flexible measures based on the observed DMUs and their corresponding aggregation DMUs which has not been provided so far.
b. The major problem includes the semi-additive production technology is computational complexity when there are many DMUs. (Observed DMUs and the aggregation of DMU). This paper develops two model that reduces the computational complexity of the semi-additive production technology by reducing the number of variables in the presence of flexible measures. This paper deals with computational complexity issues in traditional DEA model that the consideration of the power set corresponding to the observed DMUs as aggregations DMUs.
c. We present two new approaches for calculating the efficiency of DMUs in the presence of flexible measures in semi-additive production technology. Both presented models are always feasible and have a bounded optimal solution.
d. The presented models obtain the efficiency of the DMU under evaluation only based on the observed DMUs, however, in semi-additive production technology we also have aggregate DMUs corresponding to the observed DMUs.
e. We determine the type of flexible factors, for this propose, two models were presented in this paper, in the direction that flexible factors used in the best situation. Then factors in both output and input roles will maximum efficiency of under evaluation DMU.
f. Most of the previous suggested approaches for determining the status of flexible measures are oriented. The second approach in this paper have not oriented. Therefore, the right choice to choose the flexible measure as input or output is made correctly and the efficiency value of the unit under evaluation is maximized in semi-additive production technology.
g. The models presented in this paper are always feasible and have a bounded optimal solution.
The limitations of proposed models are as follows. The two proposed models are nonlinear programming. The firs model is mixed integer programming model and second model is a mixed fractional correct programming problem.
To solve them, we must use software for solving non-linear models.
In semi-additive production technology, the targets for inefficient DMUs may be selected based on aggregation DMUs. These targets may not be created in some situations.
As future work, inverse DEA models can be developed in the presence of flexible measures. Also, the models presented in this paper were developed in the presence of flexible measures in semi-additive production technology for two-stage network structure.

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