Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 15, No. 1, 2023 Article ID IJIM-1553, 6 pages
DOI: http://dx.doi.org/10.30495/ijim.2022.63781.1553
Research Article

Science and Research Branch (IAU)

On the Solution of Volterra-Fredholm Integro-Differential Equation by Using New Iterative Method

A. Jafarian *†

Received Date: 2021-10-16 Revised Date: 2021-12-11 Accepted Date: 2022-04-13

Abstract

Integro-differential equations arise in various physical and biological problems. In this paper, a new iterative technique for solving linear Volterra-Fredholm integro-differential equation (VFIDE) has been introduced. The method is discussed in details and it is illustrated by solving some numerical examples. The approximate solution is most easily produced iteratively via the recurrence relation. Results are compared with the exact solutions, which reveal that new iteration method is very effective and convenient.

Keywords : New iterative method; Volterra-Fredholm integro-differential equation; Approximate solution.

1 Introduction

INtegro-differential equations appear in many scientific and physical applications such as glassforming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. In this study, we consider the following linear Volterra-Fredholm integro-differential equation [7]

$$\begin{cases} v'(x) = f(x) + \int_0^x K_1(x,t)\ v(x)\ dt + \\ \int_0^1 K_2(x,t)\ v(x)\ dt, \quad 0 \leq x,t < 1, \\ v(0) = \alpha. \end{cases} \tag{1.1}$$

where $v(x)$ and $f(x)$ are in $L^2([0,1))$ and $K_1(x,t)$ and $K_2(x,t)$ belong to $L^2([0,1) \times [0,1))$. Moreover $K_1(x,t)$, $K_2(x,t)$ and $f(x)$ are known and $v(x)$ is unknown. We assume equation (1.1) has a unique solution.

In the past years, many powerful techniques have been developed to find solutions of linear VFIDE. Rahmani et al. [7] employed the Block Pulse Functions and their operational matrices for solving linear Volterra-Fredholm integro-differential equation. In [1], the homotopy analysis method was used to determine solution to linear integro-differential equation. Homotopy perturbation method and finite difference method for such problem have been proposed by Raftari [6]. Zarebnia and Nikpour utilized Sinc-collocation method for approximate of the linear Volterra integro-differential equations with boundary con-

*Corresponding author. jafarian5594@yahoo.com, Tel:+98(914)1893041.
†Young Researchers and Elite Club, Urmia Branch, Islamic Azad University, Urmia, Iran.

ditions in [19]. In 2006 Daftardar-Gejji and Jafari [5] presented their new iterative method (NIM) for solving linear and nonlinear functional equations. Recently, the applications of new iterative method have appeared in the works of many scientists and researchers [2, 4, 11]. The new iterative method is general and can be applied to solve the linear and nonlinear Volterra-Fredholm integro-differential equations.

The remaining part of the paper is organized as follows. In section 2, the NIM is introduced. In section 3, we extend the application of the method to construct analytical approximate solution to Volterra-Fredholm integro-differential equation. Several examples are employed to illustrate the advantage, accuracy and computational efficiency of this approach in section 4. A short summary are expressed in final.

2 New Iterative Method

Consider the following general functional equation [5, 12, 13]

$$v = N(v) + f \tag{2.2}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function. We are looking for a solution v of equation (2.2) having the series form

$$v = \sum_{i=1}^{\infty} v_i(t), \tag{2.3}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=1}^{n} v_i\right) = N(v_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} v_j\right) - N\left(\sum_{j=0}^{i-1} v_j\right)\right] \tag{2.4}$$

From equation (2.3) and (2.4), equation (2.2) is equivalent to

$$\sum_{i=1}^{n} v_i = f + N(v_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} v_j\right) - N\left(\sum_{j=0}^{i-1} v_j\right)\right] \tag{2.5}$$

We define the recurrence relation

$$\begin{cases} v_0 = f, \\ v_1 = N(v_0), \\ v_{m+1} = N(v_0 + \ldots + v_m) - N(v_0 + \ldots + v_{m-1}), \\ \qquad m = 1, 2, \ldots . \end{cases} \tag{2.6}$$

Then

$$(v_0 + \ldots + v_{m+1}) = N(v_0 + \ldots + v_m), \;\; m = 1, 2, \ldots .$$

and

$$\sum_{i=1}^{\infty} v_i = f + N\left(\sum_{i=0}^{\infty} v_i\right),$$

The k-term approximate solution of (2.2) and (2.3) is given by $v = \sum_{i=0}^{k} v_i$.

3 Analysis of the method for VFIDE

In this section we will use the NIM for VFIDE. First, we integrate from equation (1.1) in the interval $[0, x]$. Therefore, equation (1.1) can be presented as the following simple form

$$v(x) = \alpha + \int_0^x f(\eta)\, d\eta + \int_0^x \int_0^\eta K_1(\eta, t)\, v(\eta)\, dt d\eta + \int_0^x \int_0^1 K_2(\eta, t)\, v(\eta)\, dt d\eta. \tag{3.7}$$

Now, we extend the technique described in previous section to solve equation (3.7). Some first terms of the successive approximation series are as follows: The k-term approximate solution of (2.2) and (2.3) is given by

$$v(x) = v_0(x) + v_1(x) + \ldots + v_k(x).$$

4 Numerical Experiments

In this section, several examples of linear Volterra-Fredholm integro-differential equations are provided to clarify the reliability and effectiveness of the novel technique. For comparison the approximate solution given by NIM with the exact solution, we calculate the absolute error for all of them.

$$\begin{cases} v'(x) = 11 + 17x - 2x^3 - 3x^4 + \int_0^x t\ v(x)\ dt + \int_0^1 (x-t)\ v(x)\ dt, \\ v(0) = 0. \end{cases} \tag{4.9}$$

Example 4.1. *Consider the following linear Volterra-Fredholm integro-differential equations with the exact solution*

$v(x) = 6x + 12x^2$ [10].

In this case, we have $f(x) = 11 + 17x - 2x^3 - 3x^4$, $k_1(x,t) = t$ and $k_2(x,t) = x - t$. Following the algorithm given in previous section, some first terms of the successive approximation series are as follows

$v_0(x) = (x(-6x^4 - 5x^3 + 85x + 110))/10,$

$v_1(x) = (x(-9x^7 - 10x^6 + 357x^4 + 770x^3 + 3416x - 4723))/840,$

$v_2(x) = -(x(324x^{10} + 440x^9 - 25245x^7 - 72600x^6 - 676368x^4 + 1558590x^3 + 2002935x - 2150324))/3326400,$

$\vdots$

Hence the series solution of equation (4.9) is given by

$v(x) = (x(-324x^{10} - 440x^9 - 10395x^7 + 33000x^6 + 94248x^4 - 172590x^3 + 39798825x + 20037644))/3326400 + \ldots.$

Table 1 show the absolute errors for differences between the exact solutions and the approximate solutions obtained by NIM at some points.

Example 4.2. *We consider the following linear Fredholm integro-differential equations with the exact solution*

$v(x) = e^{3x}$ [1].

$$\begin{cases} v'(x) = 3e^x - \frac{1}{3}(2e^3 + 1)x + \int_0^1 3xt\ v(x)\ dt, \\ v(0) = 1. \end{cases} \tag{4.10}$$

In this case, we have $f(x) = 3e^x - \frac{1}{3}(2e^3 + 1)x$, $k_1(x,t) = 0$ and $k_2(x,t) = 3xt$. By the algorithm given in previous section, we can see that, some first terms of NIM series are as follows

$v_0(x) = e^{3x} - (1043x^2)/152,$

$v_1(x) = x^2(e^3/3 - 2729/1134),$

$v_2(x) = x^2((11e^3)/24 - 996/301) - x^2(e^3/3 - 2729/1134),$

$\vdots$

Hence the series solution of equation (4.10) is given by

$v(x) = e^{3x} + (11x^2e^3)/24 - (2858x^2)/281 + \ldots.$

Table 2 displays the values of absolute errors at some x's.

Example 4.3. *Let us consider the following linear Volterra integro-differential equations with the exact solution*

$v(x) = sin(x)$ [7, 14].

$$\begin{cases} v'(x) = 1 - \int_0^x v(x)\ dt, \\ v(0) = 0. \end{cases} \tag{4.11}$$

In this case, we have $f(x) = 1$, $k_1(x,t) = -1$ and $k_2(x,t) = 0$. Following the algorithm given in previous section, some first terms of the successive approximation series are as follows

$v_0(x) = x,$

Table 1: Absolute errors between the exact solution and approximate solution

x	*Absolute Error for* $k = 10$	*Absolute Error for* $k = 16$
0.1	1.1223e-010	1.5700e-016
0.2	2.1402e-010	2.5896e-016
0.3	3.0662e-010	2.9601e-016
0.4	3.9125e-010	2.5477e-016
0.5	4.6818e-010	1.2151e-016
0.6	5.3555e-010	1.1421e-016
0.7	5.8813e-010	4.5516e-016
0.8	6.1612e-010	8.9176e-016
0.9	6.0427e-010	1.3971e-015
1.0	5.3186e-010	1.9226e-015

Table 2: Absolute errors between the exact solution and approximate solution

x	*Absolute Error for* $k = 30$	*Absolute Error for* $k = 36$
0.1	1.1400e-014	2.6584e-017
0.2	4.5601e-014	1.0634e-016
0.3	1.0260e-013	2.3926e-016
0.4	1.8240e-013	4.2535e-016
0.5	2.8500e-013	6.6461e-016
0.6	4.1041e-013	9.5703e-016
0.7	5.5861e-013	1.3026e-015
0.8	7.2961e-013	1.7014e-015
0.9	9.2341e-013	2.1533e-015
1.0	1.1400e-012	2.6584e-015

$$v_1(x) = -x^3/6,$$

$$v_2(x) = x^5/120,$$

$$v_3(x) = -x^7/5040,$$

$$v_4(x) = x^9/362880,$$

$$\vdots$$

Hence the series solution of equation (4.11) is given by

$$v(x) = x - x^3/6 + x^5/120 - x^7/5040 + x^9/362880 - x^{11}/39916800+$$

$$x^{13}/6227020800 - \ldots .$$

The solution $v(x)$ in a closed form is $v(x) = sin(x)$, which is the exact solution. The behavior of the exact and approximate solutions are illustrated in figure 1.

5 Conclusion

In this letter, we used an application of new iterative method for solving linear Volterra-Fredholm integro-differential equation. From the given numerical examples, tables 1-2 and figure 1, we conclude that the method is accurate and efficient to implement for solving linear Volterra-Fredholm integro-differential equation. The approximate solutions obtained by NIM are compared with exact solutions.

$$v\dot{}0(x) = \alpha + \int_0^x f(\eta)\, d\eta,$$

$$v_1(x) = \int_0^x\int_0^\eta K_1(\eta,t)\, v_0(\eta)\, dtd\eta + \int_0^x\int_0^1 K_2(\eta,t)\, v_0(\eta)\, dtd\eta,$$

$$v_2(x) = \left(\int_0^x\int_0^\eta K_1(\eta,t)\left(v_0(\eta)+v_1(\eta)\right)\, dtd\eta + \int_0^x\int_0^1 K_2(\eta,t)\left(v_0(\eta)+v_1(\eta)\right)\, dtd\eta\right) - \left(\int_0^x\int_0^\eta K_1(\eta,t)v_0(\eta)\, dtd\eta + \int_0^x\int_0^1 K_2(\eta,t)v_0(\eta)\, dtd\eta\right)$$

$$\ldots$$

$$y_{m+1}(x) = \left(\int_0^x\int_0^\eta K_1(\eta,t)\left(\sum_{i=1}^m v_i(\eta)\right)\, dtd\eta + \int_0^x\int_0^1 K_2(\eta,t)\left(\sum_{i=1}^m v_i(\eta)\right)\, dtd\eta\right) - \left(\int_0^x\int_0^\eta K_1(\eta,t)\left(\sum_{i=1}^{m-1} v_i(\eta)\right)\, dtd\eta + \int_0^x\int_0^1 K_2(\eta,t)\left(\sum_{i=1}^{m-1} v_i(\eta)\right)\, dtd\eta\right) \quad (3.8)$$

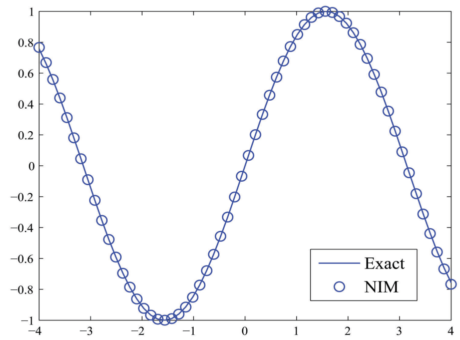

Figure 1: Comparison between exact results and the 5th-order NIM approximate results for Example 4.3.

References

[1] Z. Abbas, S. Vahdati, F. Ismail, A. Karimi Dizicheh, Application of homotopy analysis method for linear integro-differential equations,*International Mathematical Forum* 5 (2010) 237-249.

[2] S. Bhalekar, V. Daftardar-Gejji, New iterative method: application to partial differential equations, *Applied Mathematics and Computation* 203 (2008) 778-783.

[3] S. Bhalekar, V. Daftardar-Gejji, Convergence of the new iterative method, *International Journal of Differential Equations* 2011 (2011) 1-10.

[4] V. Daftardar-Gejji, S. Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, *Computers & Mathematcs with Applications* 59 (2010) 1801-1809.

[5] V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations, *Journal of Mathematical Analysis and Applications* 316 (2006) 753-763.

[6] B. Raftari, Numerical solutions of the linear Volterra integro-differential equations: homotopy perturbation method and finite difference method, *World Applied Sciences Journal* 9 (2010) 07-12.

[7] L. Rahmani, B. Rahimi, M. Mordad, Numerical solution of Volterra-Fredholm integro-differential equation by block pulse functions and operational matrices, *General Mathematics Notes* 4 (2011) 37-48.

[8] S. Shahmorad, Numerical solution of the general from linear Fredholm Volterra integro-differential equations by the Tau method with an error estimation, *Applied Mathematics and Computation* 167 (2005) 1418-1429.

[9] A. M. Wazwaz, A First Course in Integral Equations, *New Jersey*, 1997.

[10] A. M. Wazwaz, Linear and nonlinear integral equations: methods and applications, *Publisher: Higher Education Press and Springer Verlag*, (2011).

[11] M. Yaseen, M. Samraiz, The Modified New Iterative Method for Solving Linear and Nonlinear Klein-Gordon Equations,*Applied Mathematical Sciences* 6 (2012) 2979-2987.

[12] K. Asghari, M. Masdari, F. S. Gharehchopogh, R. Saneifard, Multi-swarm and chaotic whale-particle swarm optimization algorithm with a selection method based on roulette wheel, *Expert Systems* 38 (8) (2021) 1-44. https://doi.org/10.1111/exsy.12779.

[13] K. Asghari, M. Masdari, F. S. Gharehchopogh, R. Saneifard, A fixed structure learning automata-based optimization algorithm for structure learning of Bayesian networks, *Expert Systems* 38 (7) (2021) 1-19. https://doi.org/10.1111/exsy.12734.

[14] K. Asghari, M. Masdari, F. S. Gharehchopogh, R. Saneifard, 7. A chaotic and hybrid gray wolf-whale algorithm for solving continuous optimization problems, *Progress in Artificial Intelligence* 10 (2021) 349-374. https://doi.org/10.1007/s13748-021-00244-4.

[15] R. Saneifard, A. Jafarian, N. Ghalami, SM. Nia, Extended artificial neural networks approach for solving two-dimensional fractional-order Volterra-type integro-differential equations, *Information Sciences* 612 (2022) 887-897. https://doi.org/10.1016/j.ins.2022.09.017.

[16] A. Jafarian, R. Rezaei, A. Khalili Golmankhaneh, On Solving Fractional Higher-Order Equations via Artificial Neural Networks, *Iranian Journal of Science and Technology, Transactions A: Science* 46 (2022) 487-497.

[17] A. Jafarian, F. Rostami, AK. Golmankhaneh, On the solving fractional Volterra-type differential equations by using artificial neural networks approach, *Prograsive Fractal Differntioal Application* 5 (2019).

[18] A. Jafarian, M. Mokhtarpour, D. Baleanu, Artificial neural network approach for a class of fractional ordinary differential equation, *Neural Computing and Applications* 28 (2017) 765-773

[19] M. Zarebnia, Z. Nikpour, Solution of linear Volterra integro-differential equations via Sinc functions, *International Journal of Applied Mathematics and Computation* 2 (2010) 1-10.

Ahmad Jafarian was born in 1978 in Tehran, Iran. He received B. Sc (1997) in Applied mathematics and M.Sc. in applied mathematics from Islamic Azad University Lahijan Branh to Lahijan, Ferdowsi University of Mashhad respectively. He is an Associate Prof. in the department of mathematics at Islamic Azad University, Urmia Branch, Urmia, in Iran. His current interest is in artificial intelligence, solving nonlinear problem and fuzzy mathematics.