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Research Article

An algorithm for weak defining hyperplanes of the PPS in Data Envelopment Analysis

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Abstract

Border points of the production possibility set (PPS) have particular importance in data envelopment analysis (DEA). The present study aims to provide a method to find weak defining hyperplanes of DEA models by the anchor points. Anchor points are an important subset of the set of extreme efficient points of the PPS in DEA. They are directly related to weak defining hyperplanes in the production possibility set. Therefore, we used this feature and offered a practical algorithm to find weak defining hyperplanes of PPS. Eventually, we illustrate our algorithm using two numerical examples.

Keywords : Data Envelopment Analysis (DEA); Anchor point; Production Possibility Set (PPS); Weak defining hyperplane; Frontier.

1 Introduction

Data envelopment analysis (DEA), introduced by Farrel [12] and Charnes, et al. [10] is a mathematical non-parametric programming, for evaluating the relative efficiency of decision making units (DMUs) from a set of n DMUs. The original DEA model considered by Charnes, et al. [10] with constant returns to scale (CRS) was extended Banker, et al. [7] for variable returns to scale (VRS) technologies. DEA is a useful tool for evaluating decision making units with multiple inputs and multiple outputs. In recent decades, DEA has rapidly expanded towards new applications. DEAs principal application is efficiency measurement, estimating production possibility sets (PPS), and approximating super efficiency measurement. Making efficient DEA hyperplanes will provide an in-depth analysis of production efficiencies. Moreover, identifying the efficient frontiers of the PPS is a step toward determining the rate of change of outputs with change in inputs.

One problem in DEA is finding the equation of the strong and weak defining hyperplanes. There are many PPS of DEA models that do not have any strong defining hyperplanes. Also, by describing the weak defining hyperplanes of PPS, one can check if a particular DMU is weak DEA efficiency or not. Wei et al. [22] studied the char-

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acteristics and structure of the weak surface of the production possibility set. They applied the techniques and methods of transferring a polyhedral cone from its intersection form to its sum form, identify an intersection representation of the production possibility set. Olesen and Petersen [18] studied the characteristics of the production possibility set and discussed utilizations of the given surface structure information. Yu et al. [23] studied the structural properties of DEA efficient surfaces of the PPS under the Generalized DEA model. Jahanshahloo et al. [14] provided method for determining weak defining hyperplanes of BCC model by super-efficiency models. Also, Davtalab Olyaie et al. [11] presented another method for determining weak defining hyperplanes of DEA for models with variable returns to scale.

This research presents a method to find weak defining hyperplanes of PPS of DEA models by the anchor points. An anchor point in DEA is an extreme efficient DMU for which some inputs can be increased and/or outputs decreased without penetrating the interior of the production possibility set [1, 2, 5, 6, 8, 9, 16, 17]. An anchor point is, therefore, an extreme element of the production possibility set that lies on the transition between the strong efficient frontier and the free-disposability"(unbounded face) part of the boundary. They are directly related to the weak defining hyperplanes in the production possibility set. Therefore, we used this feature and we offered a simple algorithm to find weak defining hyperplanes of PPS.

2 Preliminaries

Consider a set of n DMUs, associated with m inputs and s outputs. We apply the notation (x_j, y_j) $(j \in J = \{1, \dots, n\})$ for the observed DMUs, in which, the first component is the vector of inputs $(x_{ij} \geq 0,\ i = 1, \dots, m)$ and the second component is the vector of outputs $(y_{rj} \geq 0,\ r = 1, \dots, s)$. The Production Possibility Set (PPS) of FDH under different Returns to Scale (RTS) assumptions is as

$$P^{\Gamma} = \left\{ (x, y) \;\middle|\; \begin{array}{c} \sum_{j=1}^{n} \lambda_j x_j \leq x,\ \sum_{j=1}^{n} \lambda_j y_j \geq y, \\ \lambda_j = \delta w_j;\ j \in J, \\ \sum_{j=1}^{n} w_j = 1,\ \delta \in \Gamma \end{array} \right\}$$

where $\Gamma \in \{\text{CRS, VRS, NIRS, NDRS}\}$ and CRS, VRS, NIRS and NDRS stand for Constant, Variable, Nonincreasing and Nondecreasing RTS, respectively. For the CRS, VRS, NIRS and NDRS cases, Γ is $\{\delta : \delta \geq 0\}, \{\delta : \delta = 1\}, \{\delta : 0 \leq \delta \leq 1\}$, and $\{\delta : \delta \geq 1\}$, respectively.

$$\begin{aligned} \theta_o^{\Gamma} = min \quad & \theta - \varepsilon\Big(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\Big) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}, \quad i = 1, \dots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \dots, s, \\ & \lambda_j = \delta w_j, \quad j \in J, \\ & \delta \in \Gamma,\ \sum_{j=1}^{n} w_j = 1, \\ & s_i^- \geq 0, \quad i = 1, \dots, m, \\ & s_r^+ \geq 0, \quad r = 1, \dots, s, \\ & \theta \ is\ free. \end{aligned} \tag{2.1}$$

$$\begin{aligned} \varphi_o^{\Gamma} = max \quad & \varphi + \varepsilon\Big(\sum_{i=1}^{m} t_i^- + \sum_{r=1}^{s} t_r^+\Big) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + t_i^- = x_{io}, \quad i = 1, \dots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - t_r^+ = \varphi y_{ro}, \quad r = 1, \dots, s, \\ & \lambda_j = \delta w_j, \quad j \in J, \\ & \delta \in \Gamma,\ \sum_{j=1}^{n} w_j = 1, \\ & t_i^- \geq 0, \quad i = 1, \dots, m, \\ & t_r^+ \geq 0, \quad r = 1, \dots, s, \\ & \varphi \ is\ free. \end{aligned} \tag{2.2}$$

where ε is non-Archimedean small and positive number and s_i^-, s_r^+, t_i^- and $t_r^+, i = 1, \ldots, m, r = 1, \ldots, s$ are called slack variables belonging to $\Re^{\geq 0}$. Note that s_i^- and t_i^- represent input excesses; also s_r^+ and t_r^+ represent output shortfalls. In here, θ, φ and $\lambda_j, j \in J$ are real numbers and Models (2.1) and (2.2) are called envelopment forms (with non-Archimedean number).
BCC model, proposed by Banker et al. (1984), is among the most representative DEA models for evaluating the relative efficiency of a set of DMUs. We express our algorithm for VRS model and the same can be said for other cases (CRS, NIRS, NDRS). The production possibility set of the BCC model can be defined as follows:

$$T_v = \left\{ (x,y) \;\middle|\; \begin{array}{c} \sum_{j=1}^{n} \lambda_j x_j \leq x, \sum_{j=1}^{n} \lambda_j y_j \geq y, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \geq 0, j \in J, \end{array} \right\}$$

A face of a polyhedral set is the support set of a supporting hyperplane. The PPS of the BCC model has bounded and unbounded faces. The unbounded faces make up the free-disposability part of the frontier. A facet of a k-dimensional polyhedral set is a k - 1 dimensional face. The input-oriented BCC model, corresponds to $DMU_k, k \in J$, is given by:

$$\begin{aligned} & min \quad \theta - \varepsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ik}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rk}, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \;\; j \in J, \\ & s_i^- \geq 0, \quad i = 1, \ldots, m, \\ & s_r^+ \geq 0, \quad r = 1, \ldots, s, \\ & \theta \; is \; free. \end{aligned} \tag{2.3}$$

Also, the output-oriented BCC model, corresponds to $DMU_k, k \in J$, is as follows:

$$\begin{aligned} & max \quad \varphi + \varepsilon(\sum_{i=1}^{m} t_i^- + \sum_{r=1}^{s} t_r^+) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + t_i^- = x_{ik}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - t_r^+ = \varphi y_{rk}, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \;\; j \in J, \\ & t_i^- \geq 0, \quad i = 1, \ldots, m, \\ & t_r^+ \geq 0, \quad r = 1, \ldots, s, \\ & \varphi \; is \; free. \end{aligned} \tag{2.4}$$

DMU_k is said to be strong efficient (BCC-efficient) if and only if either (i) or (ii) happen:
(i) $\theta^* = 1$ and $(s^{+*}, s^{-*}) = (0, 0)$,
(ii) $\varphi^* = 1$ and $(t^{+*}, t^{-*}) = (0, 0)$.
DMU_k is said to be weak efficient if and only if either (iii) or (iv) happen:
(iii) $\theta^* = 1$ and $(s^{+*}, s^{-*}) \neq (0, 0)$,
(iv) $\varphi^* = 1$ and $(t^{+*}, t^{-*}) \neq (0, 0)$.
Note that if $\theta^* < 1$ and $\varphi^* > 1$ then DMU_k is an interior point of the PPS. Each interior DMU and weak efficient DMU in the BCC model is said to be a BCC-inefficient DMU. We denote the set of BCC-inefficient DMUs, non-extreme DMUs and extreme DMUs as F, E and E^*, respectively. These three subsets partition the set J. The set E^* is also called the frame of J. The frames are important in DEA because the PPS of the DEA models are constructed by them and the exclusion each of them alters the shape of the PPS. The PPS of the BCC model with one input and one output is depicted in Fig. 1. In Fig. 1. $j = \{DMU_1, DMU_2, DMU_3, DMU_4, DMU_5, DMU_6\}$, $F = \{DMU_4, DMU_5\}$, $E = \{DMU_6\}$ and $E^* = \{DMU_1, DMU_2, DMU_3\}$.

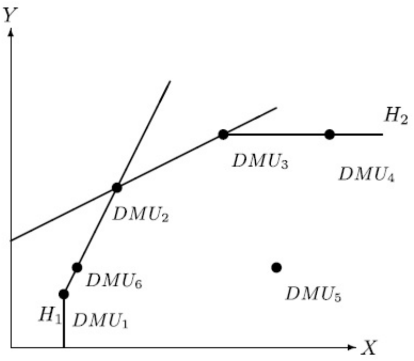

Fig. 1. Extreme, non-extreme and anchor DMUs.

The dual of models (2.3) and (2.4) (without ε i.e. $\varepsilon = 0$), which are called multiplier forms, are as models (2.5) and (2.6), respectively:

$$\begin{aligned} max \quad & \sum_{r=1}^{s} u_r y_{rk} + u_0 \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \leq 0,\ j \in J, \\ & \sum_{i=1}^{m} v_i x_{ik} = 1, \\ & u_r \geq 0, \quad r = 1, \ldots, s, \\ & v_i \geq 0, \quad i = 1, \ldots, m, \\ & u_0 \ is\ free. \end{aligned} \tag{2.5}$$

$$\begin{aligned} min \quad & \sum_{i=1}^{m} v_i x_{ik} + u_0 \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} + u_0 \geq 0,\ j \in J, \\ & \sum_{r=1}^{s} u_r y_{rk} = 1, \\ & u_r \geq 0, \quad r = 1, \ldots, s, \\ & v_i \geq 0, \quad i = 1, \ldots, m, \\ & u_0 \ is\ free. \end{aligned} \tag{2.6}$$

Note 1. DMU_k is said to be strong efficient DMU if and only if either (i) or (ii) happen:
(i) $\bar{u}^{*t} y_k + \bar{u}_0^* = 1$ and $(\bar{u}^{*t}, \bar{v}^{*t}) > 0$ for some optimal solutions of (2.5).
(ii) $\hat{v}^{*t} x_k + \hat{u}_0^* = 1$ and $(\hat{u}^{*t}, \hat{v}^{*t}) > 0$ for some optimal solutions of (2.6).
Note 2. DMU_k is said to be weak efficient DMU if and only if either (iii) or (iv) happen:
(iii) $\bar{u}^{*t} y_k + \bar{u}_0^* = 1$ and $(\bar{u}^{*t}, \bar{v}^{*t}) \ngtr 0$ for all optimal solutions of (2.5).
(iv) $\hat{v}^{*t} x_k + \hat{u}_0^* = 1$ and $(\hat{u}^{*t}, \hat{v}^{*t}) \ngtr 0$ for all optimal solutions of (2.6).

Corresponding to each BCC-efficient DMU $DMU_j = (x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{sj})$ we define virtual DMUs $DMU_j^l = (x_{1j}, \ldots, x_{lj} + \alpha, \ldots, x_{mj}, y_{1j}, \ldots, y_{sj})$ and $DMU_j^q = (x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{qj} - \gamma, \ldots, y_{sj})$ in which $\alpha, \gamma > 0$. These virtual DMUs are either interior point of the PPS of the BCC model or lie on the some free-disposability faces (unbounded faces). In the latter case; we call these virtual DMUs as weak efficient virtual DMUs, or WEV DMUs, hereafter. Evidently, $WEVDMU_j^l(DMU_j^q)$ lies on the unbounded face with $v_l = 0 (u_q = 0)$.

Definition 2.1. *The supporting hyperplane* $H = \{(x, y) | \bar{u}^t y - \bar{v}^t x + \bar{u}_0 = 0, (\bar{u}, \bar{v}) \geq 0, (\bar{u}, \bar{v}) \neq 0\}$ *of the PPS of the BCC model is weak defining hyperplane if and only if at least* $m + s$ *extreme efficient and weak efficient (virtual or real) DMUs of the PPS lie on H. (In this case at least one component of its gradient (normal vector) is zero). In Fig. 1.* H_1 *and* H_2 *are weak defining hyperplanes.*

Bougnol and Dul (2009) defined the anchor DMUs as follows [9]:

Definition 2.2. *The* $DMU_k \in E^*$ *is an anchor DMU if it belongs to an unbounded face of the PPS of the BCC model.*

Remark 2.1. *By definition of anchor DMU it is obvious that* $DMU_k \in E^*$ *is anchor DMU if and only if there exist some l (or q) so that* DMU_k^l *(or* DMU_k^q *) is WEV DMU (i.e.* $\bar{v}_l = 0$ *(or* $\bar{u}_q = 0$*)).*

In Fig. 1. DMU_1 and DMU_3 are anchor DMUs. They lie on the unbounded faces with $\bar{u} = 0$ and $\bar{v} = 0$, respectively.
We denote the set of extreme DMUs and anchor DMUs as E^* and A^* respectively.

3 Identifying the weak defining hyperplanes of the PPS

In this section, we identify the weak defining hyperplanes of the PPS of the BCC model in the following manner. First, we evaluate each $DMU_k (k \in J)$, using available models. Next, we put all extreme efficient DMUs in E^*, and all anchor DMUs in A^* (see [1, 2, 5, 6, 8, 9, 16, 17]).

Step 1. Corresponding to each anchor point $DMU_o \in A^*$, we consider m+s virtual DMUs as follows:

$$DMU_o^i = (\acute{x}_o, \acute{y}_o)_i =$$

$$(x_{o1}, \dots, x_{oi}+1, \dots, x_{om}, y_{o1}, \dots, y_{os}) \ 1 \leq i \leq m,$$

$$DMU_o^{m+r} = (\acute{x}_o, \acute{y}_o)_{m+r} =$$

$$(x_{o1}, \dots, x_{om}, y_{o1}, \dots, y_{or} - \frac{y_{or}}{2}, \dots, y_{os}) \ 1 \leq r \leq s.$$

Step 2. Evaluate virtual DMUs. If one of the following conditions holds, then we put DMU_o^i,$1 \leq i \leq m$ or DMU_o^{m+r},$1 \leq r \leq s$ in set X.
i) DMU_o^i is output-oriented BCC-weakly efficient, $1 \leq i \leq m$.
ii) DMU_o^{m+r} is input-oriented BCC-weakly efficient, $1 \leq r \leq s$.
Step 3. Steps 1 and 2 are iterated for all anchor points, and set X contains all created virtual DMUs on the weak frontier.
Step 4. Define $T = X \cup A^*$. Suppose $|T| = Q$.
Step 5. Calculate $\binom{Q}{m+s}$=k. Define $B_l \subseteq T, l = 1, \dots, k, |B_l| = m+s$ and set $W = \{B_1, B_2, \dots, B_k\}$.
Step 6. Choose $B_p \in W$, as the affine independent.
Step 7. Set $DMU_p = (x_p, y_p) = \{(\frac{1}{m+s}\sum_{j=1}^{m+s} x_j \, , \, \frac{1}{m+s}\sum_{j=1}^{m+s} y_j) | (x_j, y_j) \in B_p\}, p = 1, \dots, k'$.
Step 8. Evaluate DMU_p by the following models $(p = 1, \dots, k')$.

$$\begin{aligned} max \quad & \sum_{r=1}^{s} u_r y_{rp} + u_0 \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \leq 0, \ j \in J, \\ & \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & u_r \geq 0, \quad r = 1, \dots, s, \\ & v_i \geq 0, \quad i = 1, \dots, m, \\ & u_0 \ is \ free. \end{aligned} \tag{3.7}$$

$$\begin{aligned} min \quad & \sum_{i=1}^{m} v_i x_{ip} + u_0 \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} + u_0 \geq 0, \ j \in J, \\ & \sum_{r=1}^{s} u_r y_{rp} = 1, \\ & u_r \geq 0, \quad r = 1, \dots, s, \\ & v_i \geq 0, \quad i = 1, \dots, m, \\ & u_0 \ is \ free. \end{aligned} \tag{3.8}$$

Step 9.
(a) If both models (3.7) and (3.8) are inefficient, then DMU_p is the interior unit. In this case, we ignore DMU_p and move on to the next unit.
(b) If both models (3.7) and (3.8) are efficient, then DMU_p is a strong efficient unit belonging to a strong defining hyperplane. In this case, no weak defining hyperplane passes through it.
(c) If one of models (3.7) or (3.8) is efficient and one of them is inefficient, then DMU_p is on the weak frontier, and models (3.7) and (3.8) give weak defining hyperplane passing through DMU_p.

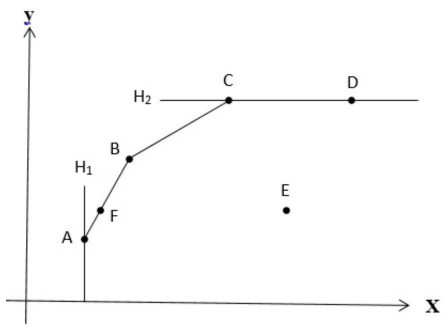

Fig. 2. DMUs with one input and one output.

4 Examples

In this section, we explain the model suggested in the previous section by presenting two numerical examples.

Example 4.1. *(DMUs with one input and one output)*

Table 1 shows data for 6 DMUs with one input and one output. Using the mentioned algorithm in the previous section the extreme DMUs and the anchor points are $E^* = \{A, B, C\}$ *and* $A^* = \{A, C\}$ *respectively. Also, we have two weak defining hyperplanes as follows (see Fig. 2.):*

$$H_1 = \{(x, y)|x = 1\}\ ,\ H_2 = \{(x, y)|y = 4\}$$

Table 1: Data of Example 4.1.

DMU	A	B	C	D	E	F
Input	1	2	4	6	5	1.2
Output	1	3	4	4	1.4	1.4

Example 4.2. *(DMUs with two inputs and single output)*

Consider a system of four DMUs as in Table 2. The extreme units and the anchor points are $E^* = \{D_1, D_2, D_3\}$ *and* $A^* = \{D_1, D_2, D_3\}$ *respectively. Now, we run the steps of our algorithm. First, let us consider m+s virtual DMUs as follows:*

$$D_1^1 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix},\ D_1^2 = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix},$$

$$D_1^3 = \begin{pmatrix} 1 \\ 4 \\ 0.5 \end{pmatrix},$$

$$D_2^1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},\ D_2^2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix},$$

$$D_2^3 = \begin{pmatrix} 2 \\ 2 \\ 0.5 \end{pmatrix},$$

$$D_3^1 = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix},\ D_3^2 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix},$$

$$D_3^3 = \begin{pmatrix} 5 \\ 1 \\ 0.5 \end{pmatrix}.$$

All units are evaluated according to Step 2, Step 3 and Step 4, and sets X and T are created.

Table 2: Data of Example 4.2.

	D_1	D_2	D_3	D_4
Input 1	1	2	5	6
Input 2	4	2	1	5
Output	1	1	1	1

$$X = \{D_1^2, D_1^3, D_2{}^3, D_3^1, D_3^3\}$$

$$T = \{D_1, D_1^2, D_1^3, D_2, D_2^3, D_3, D_3^1, D_3^3\}.$$

All m+s affine independent compositions are formed according to Step 6, and DMUs created for their convex composition (Step 7) are evaluated by models (3.7) and (3.8). In the last step, the weak defining hyperplanes are expressed as follows (see Fig. 3.):

$$H_1^* = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \middle| \, x_1 = 1 \right\},$$

$$H_2^* = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \middle| \, x_2 = 1 \right\},$$

$$H_3^* = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \middle| \, y = 1 \right\},$$

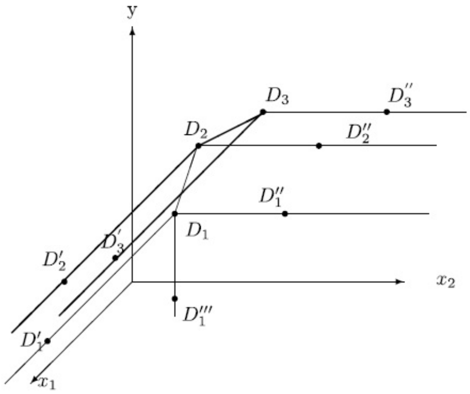

Fig. 3. DMUs with two input and single output.

5 Conclusion

By having the equations of defining hyperplanes of production possibility sets, a lot of information about the position of the DMUs in the PPS will be reflected. Of course, there are many PPS of DEA models that do not have any strong defining hyperplanes. Also, by describing the weak defining hyperplanes of PPS, one can check if a particular DMU is weak DEA efficiency or not. Among the existing scientific and technical works for finding the weak defining hyperplanes, we can mention [11, 14, 18, 22, 23]. Our research offers a method to find this hyperplanes of PPS of DEA models by the anchor points. An anchor point in DEA is an extreme efficient DMU for which some inputs can be increased and/or outputs decreased without penetrating the interior of the production possibility set. They are directly related to the weak defining hyperplanes of the PPS. Therefore, we used this feature and presented a practical algorithm to find them in DEA. Also, it seems that the genetic programming (see [13, 15, 19, 20, 21, 3, 4]) can be used to search the weak defining hyperplanes of PPS, which is a field for future research.

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