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An algorithm based on Mean-CVaR for selecting efficient portfolio with cardinality constraints

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Abstract
Investors usually hold only a small number of stocks to construct portfolio because of the cardinality constrained portfolio selection problem which arises due to the transaction cost and other market frictions. The cardinality constrained portfolio selection with the traditional mean-variance criteria (Mixed-integer and quadratic programming) and mean-CVaR (linear Mixed-integer) are an NP-Hard optimization problem. To solve this mixed-integer nonlinear programming (NP-Hard), a corresponding genetic algorithm (GA) is utilized. In this paper, we presented an algorithm that implements the model mean-CVaR as a linear model for solving this problem. Furthermore, this algorithm can be suggested all possible optimal and find the exact solution. Additionally, a numerical example, which includes an application of the algorithm by considering the stock's price of the 15 stocks, during the period from 8/16/2019 to 8/14/2020 that obtained from a real dataset, is presented in order to demonstrate that the algorithm is useful for portfolio detection.

Keywords: Portfolio selection, Cardinality Constraint, Conditional Value-at-Risk.

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1. Introduction

In financial studies, both portfolio performance assessment and efficient portfolio selection are important areas. The purpose of creating a portfolio is the risk reduction of the investment, so that the profit of an asset compensates loss of another asset. The portfolio selection was first introduced by Markowitz in [1]. This model is known as the Markowitz model or the mean-variance model. He believed that the investor should maximize return and minimize risk.

One problem pertaining to Markowitz model is its computational complexity. Tobin [2] in 1958, Hanoch, and Levy [3] took steps to improve the Markowitz model. Sharp [4] Trainer [5] and Jensen [6] provided benchmarks for portfolio assessment. The Sharpe index is the risk premium per unit of the total risk. The Treynor index is the risk premium per unit of the systematic risk, and the Jensen index is defined as the difference between the actual portfolio return and the estimated benchmark return. Scutellà et al. [7] reviewed several mathematical models, and related algorithmic approaches, that have been proposed to address uncertainty in portfolio asset allocation, focusing on Robust Optimization methodology. They also analyzed the relationship between the concepts of robustness and convex risk measures.

Regulations for finance businesses, formulate some of the risk management requirements in terms of percentiles of loss distributions. An upper percentile of the loss distribution is called Value-at-Risk (VaR). Value at risk is one of the most popular measures that is deeply rooted in its simplicity, which has achieved the high status of being written into industry regulations. VaR can be quite efficiently estimated and managed when underlying risk factors are normally distributed. For instance, 95%-VaR is an upper estimate of losses which is exceeded with 5% probability.

Artzner e t al., [8, 9] proposed, for non-normal distributions, VaR may have undesirable properties such as lack of sub-additivity, i.e., VaR of a portfolio with two instruments may be greater than the sum of individual VaRs of these two instruments. Also, Value at risk is difficult to optimize for discrete distributions, when it is calculated using scenarios. But this risk measure is neither sub-additive, not convex. This risk measure is proposed by Baumol [10]. Glasserman et al. [11] used the Monte Carlo method along with quadratic estimation to measure the portfolio's VaR. Chen and Tang [12] verified other nonparametric approximation of VaR for related financial returns. Rockefeller and Uryasev [13, 14], expressed another risk measure which was named Conditional Value at Risk (CVaR).

CVaR is also called Expected shortfall (ES), Average Value at Risk (AVaR) and expected tail loss (ETL). CVaR is defined as the weighted average of VaR and losses strictly exceeding VaR for general distribution. The CVaR risk measure is proved to be a coherent risk measure (Pflug [15], Ogryczak and Ruszczynski [16]) and researcher use CVaR as a risk measure for portfolio and financial problems. Hong and Liu [17] used the Monte Carlo simulation method to calculate CVaR for portfolio optimization.

In the real investment, due to various frictions including transaction cost and management cost, the investors only hold few stocks to construct the portfolio because of the cardinality constrained portfolio selection problem which arises due to the transaction cost and other market frictions. This phenomenon motives the researchers to study the cardinality constrained portfolio optimization problem. This problem has been explored by several researchers under the mean-variance criteria [18]. In 1993, Speranza [19] presented a more general model with a weighted risk function and also in 1996 [20] proposed a mixed-integer programming, considering realistic characteristics in portfolio selection, such as minimum transaction lots and maximum number of securities. Yoshimoto [21] considered multi period portfolio selection with transaction costs based on Markowitz's model. Speranza and Mansini [22] regarded transaction costs with and without minimum transaction lots but again not based on Markowitz's model. Konno [23] proposed an algorithm for his portfolio optimization problems regarding transaction costs and minimum transaction lots.

The main idea is to use the special structure of the problem to design the efficient solution algorithm for this NP-hard optimization problem. For example, Bienstock [24] showed a way to use surrogate constraint to approximate cardinality constraint. Bertsimas and Shioda [25] developed an exact algorithm based on branch-and-bound method by using the convex relaxation. Gao and Li [18] develop an efficient solution scheme by incorporating the convex conic programming techniques and using the geometrical properties of this problem. On the other hand, Chen et al [26] relax the cardinality constraint by finding the sparse solution of this problem.

Almost all researches on the cardinality constrained portfolio model are based on the mean-variance model for example in [27] a portfolio selection model which is based on Markowitz's portfolio selection problem including three of the most important limitations.is considered. However, the variance is not an ideal term for risk measure, since it penalizes symmetrically for both parts below and above the mean value. Instead of using the traditional mean-variance criteria, used the Conditional Value-at-Risk (CVaR) as the risk measure to build the cardinality constrained portfolio optimization model. Cheng et al [29] proposed to use the reweighed l_1-norm method to find the approximated solution of this problem. In this work, instead of finding the exact solution, they proposed some methods to identify the approximated solution.

This paper presents an algorithm for the purpose of evaluating the cardinality constrained portfolio selection problem. This algorithm is presented using model mean-CVaR that has been a linear model to solve this problem. Furthermore, this algorithm can be suggested all possible optimal and find the exact solution. The remainder of the paper is organized as follows:

Section 2 reviews the basic definitions and briefly describes portfolio efficiency in cardinality constraint mean-variance and conditional value at risk. Section 3 first shows how the selection of portfolio according to the existing restrictions and then presents the proposed model. Section 4 details the applications of the proposed approach. Section 5 concludes the results.

2. Basic definitions

2.1. Markowitz-based portfolio selection

Because investors have contrasting preferences regarding return and risk, Investments are determined according to the investor's preferences. Therefore, risk-taking and returns are two criteria that determine the amount of utility of an investment for selecting a portfolio of investment assets. Markowitz defined a return of capital as the mathematical expectation of the returns in the past. He measured the risk through the variance of the returns and then presented a model by implementing these two criteria and considering the idea that the main goal is always to reduce risk. This model is a non-linear model, then this model is the computational complexity.

The first portfolio theory was published by Markowitz [1]. This model is known for "Markowitz model" or "mean-variance model". Suppose that there are "n" assets available in market at this model that is a quadratic model, the return vector and covariance matrix are given $R=(R_1,\dots,R_n)$ and $G=[\sigma_{ij}]$ expected return of investor is R_{expected}. x_j is the proportion of portfolio's initial value invested in asset j [1].

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} \\
s.t. \quad & \\
& \sum_{j=1}^{n} x_j R_j \ge R_{\text{expected}} && (1.1)\\
& \sum_{j=1}^{n} x_j = 1 && (1.2)\\
& x_j \ge 0 \qquad j=1,\dots,n && (1.3)
\end{aligned}
$$

The cardinality constrained portfolio selection problem arises due to the empirical findings that investors tend to hold a limited number of assets. Let k be the desired number of risky assets in constructing portfolios [28]. To solve this problem, Markowitz's model is a quadratic programming and mixed-integer programming problem. Mixed-integer nonlinear programming is NP-Hard. This model is as follows [28]:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} \\
s.t. \quad & \\
& \sum_{j=1}^{n} x_j R_j \ge R_o && (2.1)\\
& \sum_{j=1}^{n} x_j = 1 && (2.2)\\
& z_j \varepsilon \le x_j \le M z_j \qquad j=1,\dots,n && (2.3)\\
& \sum_{j=1}^{n} z_j = k && (2.4)\\
& x_j \ge 0 \qquad j=1,\dots,n && (2.5)\\
& z_j \in \{0,1\} \qquad j=1,\dots,n && (2.6)
\end{aligned}
$$

Where ε is a non-Archimedean element smaller than any positive real number and M is a very large number.

2.2. CVaR-based portfolio selection

2.2.1. Value at Risk

VaR is defined as maximum quantity of invest that one may lose in a specified time interval, that p_t is initial wealth, and p_{t+k} is secondary wealth after k period time, probability of loss is:

$$p(-\Delta_k p_t < VaR) = \alpha \qquad (3)$$

Where $\Delta_k p_t = p_{t+k} - p_t$ and $1-\alpha$ is margin of error, correspondingly α is confidence level.

There are different methods for computing the VaR, such as Variance-Covariance method, historical simulation, and also Monte Carlo simulation.

The variance-Covariance method only uses for normal distribution. There is no need for normal distribution data in either of historical simulation and Monte Carlo simulation methods.

Historical simulation is no need to know the distribution of data for calculating the VaR. In fact, VaR is computed by putting concentration on an assumptive time series of returns and supposition that changes future data based on historical changes. The convenience of this method is that there is a need for variance and covariance calculation. This method believes that the behavior of returns is the same as before. However, VaR lacks sub-additivity, when analyzed with scenarios. The VaR is nonconvex as well as nondifferentiable, and hence, it is difficult to find a global minimum via conventional optimization techniques. Alternatively, conditional VaR (CVaR), introduced by Rockafellar and Uryasev [13], and further developed in [14].

Let $f(x, \mathrm{R})$ be the loss associated with the decision vector x, to be chosen from a certain subset X of R^n, and the random vector R in R^m. The vector x can be interpreted as a portfolio, with X as the set of available portfolios (subject to various constraints), but other interpretations could be made as well. The vector stands for the uncertainties, e.g., market prices that can affect the loss. Of course, the loss might be negative and thus, in effect, constitutes a gain.

CVaR is defined as the conditional expectation of the portfolio loss exceeding.

$$CVaR_\alpha(\mathrm{x}) = \frac{1}{1-\alpha}\int_{\mathrm{f(x,R)}>\gamma} \mathrm{f(x,R)}\, p(R)\, \mathrm{dR} \qquad (4)$$

Where $p(R)$ is a probability density function of R . For general distributions, including discrete distributions, CVaR is a weighted average of VaR, and the conditional expectation given by (4) (see [13]). To avoid complications caused by an implicitly defined function $VaR_\alpha(\mathrm{x})$, Rockafellar and Uryasev [12] have provided an alternative function given by:

$$\mathrm{F}_\alpha(\mathrm{x},\gamma) = \gamma + \frac{1}{1-\alpha}\int_{\mathrm{f(x,R)}>\gamma} (\mathrm{f(x,R)} - \gamma)\, p(R)\, \mathrm{dR} \qquad (5)$$

For which, they show that minimizing $\mathrm{F}_{\alpha}(\mathrm{x},\gamma)$ with respect to (x,γ) yields the minimum CVaR and its solution. (This statement is again true for general distributions) In case that the probability distribution of R is not available or an analytical solution is difficult, it is possible to exploit price scenarios, which can be obtained from past price data and/or through Monte Carlo simulation. Assuming that this price data is equally identical (e.g., random sampling from a joint price distribution). Given price data $R_{p,s}$ for $s=1,...,\mathrm{S}$, $\mathrm{F}_{\alpha}(\mathrm{x},\gamma)$ can be approximated by

$$\mathrm{F}_{\alpha}(\mathrm{x},\gamma)=\gamma+\frac{1}{(1-\alpha)S}\sum_{s=1}^{S}[R_{p,s}-\gamma]^{+} \qquad (6)$$

Where $[x]^{+}=\max[\mathrm{x},0]$ and $R_{p,s}=\sum_{i=1}^{n}x_i R_{i,s}$

We can find a portfolio that minimizes CVaR by considering the following nondifferentiable optimization (NDO) problem, which is intended to be, solved [13]

$$\min \quad \gamma+\frac{1}{(1-\alpha)s}\sum_{s=1}^{S}R_{\mathrm{s}\,p}^{+}$$

$$s.t. \quad \sum_{i=1}^{n}x_i R_{si}=R_{sp} \qquad s=1,...,\mathrm{S} \qquad (7.1)$$

$$-R_{sp}-\gamma\le R_{\mathrm{s}\,p}^{+} \qquad s=1,...,\mathrm{S} \qquad (7.2)$$

$$\sum_{i=1}^{n}x_i\bar{R}_i\ge R_{\mathrm{exp}eced} \qquad (7.3)$$

$$\sum_{i=1}^{n}x_i=1 \qquad (7.4)$$

$$R_{\mathrm{s}\,p}^{+}\ge 0 \qquad s=1,...,\mathrm{S} \qquad (7.5)$$

$$x_i\ge 0 \qquad i=1,...,n \qquad (7.6)$$

This model can be reformulated as a mixed-integer optimization [29]

$$\min \quad \gamma + \frac{1}{(1-\alpha)s}\sum_{s=1}^{S} R_{sp}^{+}$$

$$s.t. \quad \sum_{i=1}^{n} x_i R_{si} = R_{sp} \qquad s=1,\ldots,S \qquad (8.1)$$

$$-R_{sp} - \gamma \le R_{sp}^{+} \qquad s=1,\ldots,S \qquad (8.2)$$

$$\sum_{i=1}^{n} x_i \bar{R}_i \ge R_{\text{expected}} \qquad (8.3)$$

$$\sum_{i=1}^{n} x_i = 1 \qquad (8.4)$$

$$R_{sp}^{+} \ge 0 \qquad s=1,\ldots,S \qquad (8.5)$$

$$\varepsilon z_i \le x_i \le M z_i \qquad i=1,\ldots,n \qquad (8.6)$$

$$\sum_{i=1}^{n} z_i = k \qquad (8.7)$$

$$z_i \in \{0,1\} \qquad (8.8)$$

$$x_i \ge 0 \qquad i=1,\ldots,n \qquad (8.9)$$

To evaluate the CVaR, it is necessary to generate a large number of scenarios, which significantly increases the size of this problem. In consequence, it is not practical to solve the resulted mixed-integer programming problem (mixed-integer optimization) directly.
This model (7) is linear. In the following section, how to choose an efficient portfolio based on cardinality constraints with the help of the mean-CVaR model, is discussed.

3. Selection of portfolio according to the existing restrictions

The cardinality-constrained portfolio selection problem is a quadratic programming and mixed-integer programming problem. Mixed-integer nonlinear programming is NP-Hard and since solving model (8) directly for the exact solution is difficult when the problem size n and S are large. In this section, it is discussed how to choose an efficient portfolio based on cardinality constraints with the help mean-CVaR model that this model which is a linear model.

In this study, the following algorithm is implemented to solve the cardinality constraints portfolio selection problem.

k is, the number of assets that an investor would like to invest.

Step 1

Model (7) is evaluated for return that expect an investor to invest and $(x^*, \gamma^*, R_{sp}^{+*})$ is optimal model (7) then $G = \{i \mid x_i^* > 0\}$. If the cardinal of sets G is less than k, this set is the optimal answer to the cardinality constrained portfolio selection problem, otherwise go to the step 2.

Step 2

Let $|G| = m$. We know that the number of subsets k members of a set of m members is equal:

$$d = \binom{m}{k}$$

We can show subsets with A_l ($l = 1,...,d$)

Step 3

On the other hand, according to model (7) (constraint (7.3)) sets of assets can be considered to make an efficient portfolio that convex combination of their returns greater than or equal to the expected return on investment. As a result, we eliminate subsets that maximum their return is less than to the expected return of investor or only one of the assets of that subset has a maximum return and that return is equal to the expected return. We show subsets were not eliminated with I_h (let $h = 1,\ldots,q$) and $q \le d$.

$W = \varnothing, c = 0, b = 1$

Step 4

$f = c + 1$

$$\begin{aligned} CVaR(f) = \min \quad & \gamma^f + \frac{1}{(1-\alpha)s}\sum_{s=1}^{S} R_{sp}^{+f} \\ s.t. \quad & \sum_{i \in I_f} x_i R_{si} = R_{sp}^f & s = 1,...,S \quad & (9.1) \\ & -R_{sp}^f - \gamma^f \le R_{sp}^{+f} & s = 1,...,S \quad & (9.2) \\ & \sum_{i \in I_f} x_i \bar{R}_i - S_{slack}^f = R_{\exp eced} & & (9.3) \\ & \sum_{i \in I_f} x_i = 1 & & (9.4) \\ & R_{sp}^{+f} \ge 0 & s = 1,...,S \quad & (9.5) \\ & x_i \ge \varepsilon & i \in I_f \quad & (9.6) \end{aligned}$$

Where:

$\bar{R}_i =$ The expected return of asset i

$R_{si} =$ The returns asset i in scenario s

Step 5

If $f = 1$

$CVaR(b) = CVaR(f)$

$W = W \bigcup \{I_f\}$

If $f > 1$

If $CVaR(f) < CVaR(b)$

$W = \varnothing,\ b = 1$

$J_b = I_f$

$W = \{J_b\}$

$CVaR(b) = CVaR(f)$

Else

If $CVaR(f) = CVaR(b)$

$b = b + 1$

$J_b = I_f$

$W = W \bigcup \{J_b\}$

Step 6

If $f < q$

$c = c + 1$

Go to step 4

Else

If $b > 1$

$T = \{\{J_h\} \subseteq W \mid R_{\exp eced} + S_{slack}^{h*} \geq R_{\exp eced} + S_{slack}^{f*} \quad f = 1,...,b\}$

T is the set of all optimal sets with $(CVaR(J_h), R_{\exp eced} + S_{slack}^{h*})$.

Else If $b = 1$

J_b is set optimal with $(CVaR(J_b), R_{\exp eced} + S_{slack}^{b*})$

End

Theorem 1: In step 3 always $q \geq 1$ or the algorithm ends in the first step

Proof

According to model (7) (constraint (7.3)) sets of assets can be considered to make an efficient portfolio that convex combination of their returns greater than or equal to the expected return on investment then in G there is at least one asset that its return is greater than or equal $R_{\exp ected}$ then if there is only one asset that its return is equal $R_{\exp ected}$ then according constraint (7.3) G has only this asset otherwise $q \geq 1$

The above algorithm tries to reduce the computational volume in each step and this algorithm uses a linear model thus reducing the computation time and also identifies it if there are multiple answers. It is also possible that the problem is solved only by solving a linear model and we do not enter the next steps of the algorithm. Considering that model (9) is the same as model (7) and also according to step 3 then this algorithm is always feasible.

4. Empirical application

The dataset was randomly collected from the stock's price of the 15 stocks, from 8/16/2019 to 8/14/2020. Besides, missing data over holidays estimated through interpolation. The dataset was obtained from "https://finance.yahoo.com/most-active". All of the stock companies are shown by the company symbol, in Table 1. In this example, we have used GAMS and MATLAB software.

Table1. Symbol of stock companies that were used.

Company symbol	Company symbol	Company symbol	Company symbol	Company symbol
AAL (DMU_1)	AMAT (DMU_4)	IQ (DMU_7)	NIO (DMU_{10})	UAL (DMU_{13})
AAPL (DMU_2)	GE (DMU_5)	ITUB (DMU_8)	OXY (DMU_{11})	WFC (DMU_{14})
BAC (DMU_3)	INO (DMU_6)	MESO (DMU_9)	PLUG (DMU_{12})	SRNE (DMU_{15})

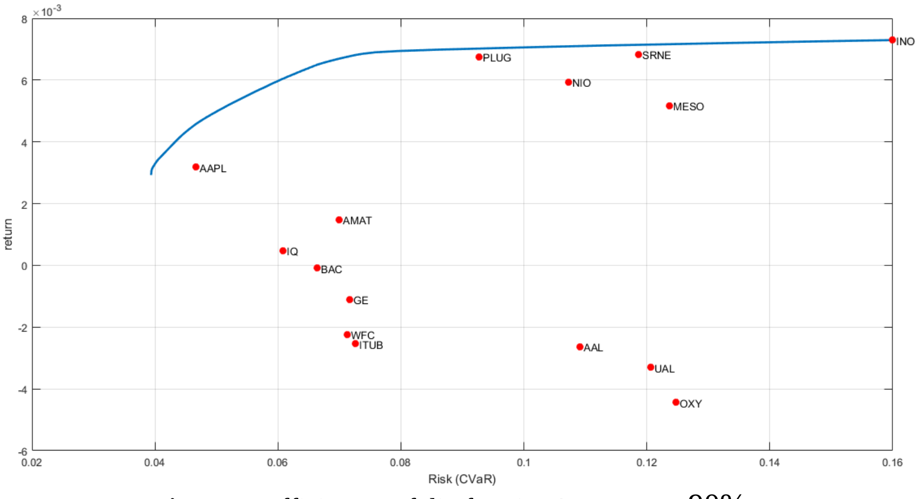

Figure 1. Efficient portfolio frontier CVaR, $\alpha = 90\%$

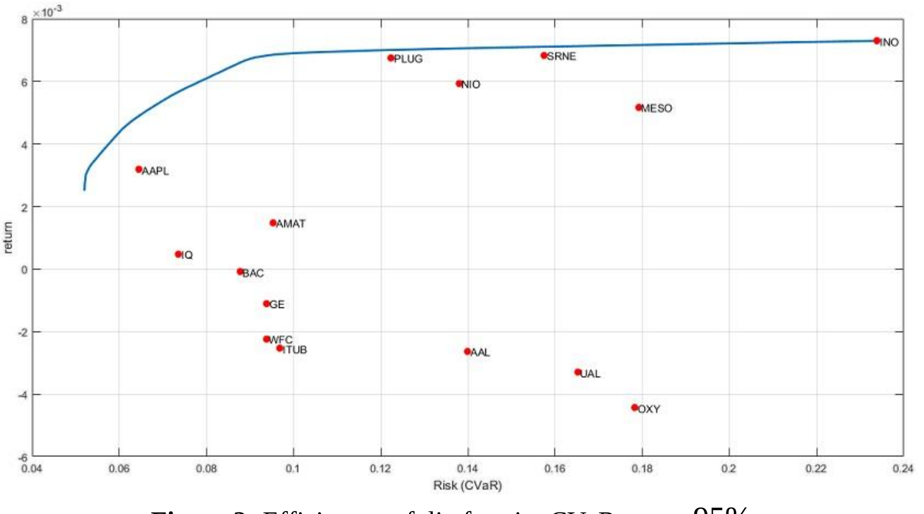

Figure 2. Efficient portfolio frontier CVaR, $\alpha = 95\%$

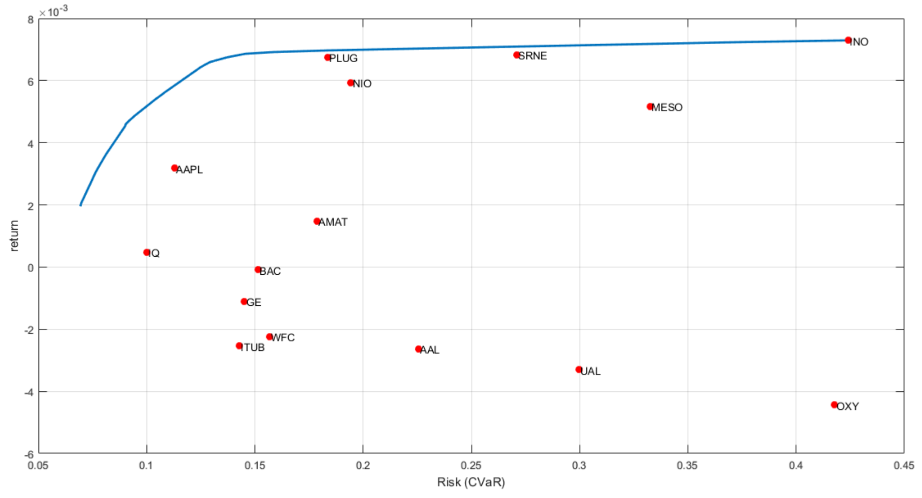

Figure 3. Efficient portfolio frontier CVaR, $\alpha = 99\%$

Suppose k=3 is the number of assets that an investor would like to invest and $R_{expected} = 0.006$.

Table 2. Minimum CVaR model 7, 8 and 9

	Model 7	Model 8 and 9
Min CVaR $\alpha = 90\%$	0.0603268638244	0.0691613115168
Min CVaR $\alpha = 95\%$	0.0785472478613	0.0878562074744
Min CVaR $\alpha = 99\%$	0.1160966677940	0.1258278591750

As can be seen in Table 2, the efficient portfolio frontier is drawn to the right by a new constraint because the optimal value of Model 7 is less than the optimal value of Model 8.
In Table 3, we represent optimal assets which are obtained from model 7 and 8 for all α .

Table 3. Optimal assets

Optimal assets	Model 7	Model 8
CVaR $\alpha = 90\%$	$x_2 = 0.14873545$ $x_6 = 0.15950859$ $x_9 = 0.13892124$ $x_{10} = 0.12393643$ $x_{12} = 0.30886486$ $x_{15} = 0.12003344$	$x_2 = 0.21775348$ $x_{12} = 0.51939371$ $x_{15} = 0.26285281$

CVaR $\alpha = 95\%$	$x_2 = 0.19905731$ $x_6 = 0.15615197$ $x_9 = 0.01822843$ $x_{10} = 0.14361221$ $x_{12} = 0.27522153$ $x_{15} = 0.20772856$	$x_2 = 0.21959218$ $x_{12} = 0.42595438$ $x_{15} = 0.35445345$
CVaR $\alpha = 99\%$	$x_2 = 0.14844514$ $x_6 = 0.13408635$ $x_{10} = 0.37514260$ $x_{12} = 0.24774203$ $x_{15} = 0.09458389$	$x_2 = 0.21762934$ $x_{12} = 0.52570221$ $x_{15} = 0.25666844$

If the example is solved with the help of algorithm for $\alpha = 90\%$, $\alpha = 95\%$, $|G| = 6$ $A_l\ (l = 1,...,20)$, and $I_h\ (h = 1,...,19)$, then $b = 1$ ($J_1 = \{AAPL,\ PLUG,\ SRNE\}$); and for $\alpha = 99\%$ $|G| = 5$ $A_l\ (l = 1,...,10)$, and $I_h\ (h = 1,...,10)$, then $b = 1$ ($J_1 = \{AAPL,\ PLUG,\ SRNE\}$). This example has only an optimal solution.

5. Conclusion

The cardinality constrained portfolio selection problem arises due to the empirical findings that investors tend to hold a limited number of assets. Almost all researches on the cardinality constrained portfolio model are based on the mean-variance model. However, the variance is not an ideal term for risk measure, since it penalizes symmetrically for both parts below and above the mean value. Instead of using the traditional mean-variance criteria, used the Conditional Value-at-Risk (CVaR) as the risk measure to build the cardinality constrained portfolio optimization model. These mixed-integer nonlinear programs are NP-Hard program. The purpose of this study is to develop an algorithm to solve the cardinality constrained portfolio selection problem using a linear model.

This algorithm is due to the fact that nonlinear problems with large volumes of data are an NP-Hard problem, using a linear model and aiming to reduce the data volume; then possible, the complexity of the problems is reduced. This algorithm also detects them if there are multiple answers. In addition, it uses the mean-CVaR model. The CVaR considers only undesirable fluctuations for risk assessment; therefore, it is superior to variance.

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