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Research Article

Efficiency Frontier Ideal Distance Evaluation in Presence of Integer-Values in DEA by Directional Distance Function Approach

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Abstract

In conventional DEA methods, all inputs and outputs are assumed to be continuous. However, the implied presumption of continuous data may not maintain an acceptable level of precision in practical data. Many practical situations such as number of teachers in a school, number of students in a college, or research papers only take integer values. Presented work presumes subsets of input and output variables in traditional five-pronged axioms in DEA model to be integer values for the first time in the field. An additional effort in this work expands axioms and a new model is presented. This paper presents a novel two phase model that in the first phase produces more precise efficiency values and the best benchmark in its second phase, i.e. the nearest integer point to the efficiency boundary is selected based on constant returns to scale. A case study is presented that demonstrates a comparison of efficiency values obtained with this model compared to prior models that is show in table 1. And in Table 2, the values of the slack are compared with each other.

Keywords : Integer values; Efficiency; Frontier; Distance; Benchmark.

1 Introduction

Data Envelopment Analysis (DEA) is a linear programming and a nonparametric technique developed by Charnes et al. [4] and further improved by Banker et al. [1]. DEA is utilized in operations research for measuring efficiency and productivity of Decision Making Units (DMUs) with several input and outputs. DEA has been a preferred method of estimating and benchmarking relative efficiency of DMUs with multiple inputs and multiple outputs [21].

And that is a mathematic technique for evaluating the relative efficiency of a set Decision Making Units (DMUs) [19]. The relative efficiency of a DMU is the result of comparing the inputs and outputs of the DMU and those of other DMUs in the PPS (Production Possibility Set) [7]. There are several methods for evaluating DMUs with restricted multiplier [8] .

Choosing an appropriate model in DEA to calculate efficiency of DMUs has always been an objective of researchers [10].

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Conventional DEA models may suggest targets for inefficient DMUs with corresponding components having non-integer values. DEA is used to produce a reference technology where it can relatively estimate efficiency of individual DMUs.

Estimating returns-to-scale is a most important topic in DEA [9] and now we work with CRS models.

Farrells radial measurement technique [6] is the preferred model utilized in either an output or an input- oriented Farrell model. Chambers et al. introduced directional distance functions [3]. These functions have been recently shown to be an important tool in production theories that yield more familiar Shepard type output and input distance functions in special cases.

Researchers have already tried to modify the traditional DEA model and remove the weak points as well [2].

Distance functions simultaneously seek to expand output and contract input. Functions evaluation of any decision- making unit will depend on the directional vector and its corresponding production possibility set [5].

All DMUs usually have input and output values where some can only be integer numerals such as number of magazines, passengers, universities and such in a practical situation.

Lozano and Villa were the pioneers in paying attention to this case [18]. They introduced integer constraints into DEA models and proposed a mixed integer linear programming (MILP) model for evaluating the efficiency of DMUs. Kuosmanen and Kazemi Matin noted limitations in Lozano and Villas work and proposed new axioms and a model to address the noted deficiencies [16].

Jie proposed a new model for mixed objective integer DEA [12].

Contributions of Kazemi Matin & Kuosmanen [16] and Kuosmanen & Kazemi Matin [14] were based on explanation of integrality axioms in the same integer-valued DEA. The axiomatic of their work consisted of developing production possibility sets that were used in proposing a variety of Mixed Integer Linear Programming (MILP) models. The Mixed Integer Linear Programming (MILP) models are proposed to estimate the performance of DMUs including both integer and real values in DEA [15].

Lozano and Villa developed a MILP DEA formulation to extent the efficiency of DMUs relative to integer DEA technology by using Farrell's radial input-oriented extension Kuosmanen & Kazemi Matin rationalized that the classic Farrell extent necessitated a correction. They proposed to extent efficiency as the radial distance to monotonic hull of the integer DEA technology [17]. The objective of their models was to extent the efficiency of DMUs. The main advantage of prevalent DEA models with the above integer-valued DEA models is in their reference point. In fact, reference point in integer-valued DEA models is an integer.

Jie et al. proposed a model that amended the original Kuosmanen and Kazemi Matin model that is known as the surmounted Kuosmanen and Kazemi Matin model [11]. Kazemi Matin and Emrouznejad offered a mixed integer programming model for efficient evaluation of integers under the constraints of bounded outputs [13].

Fare and Grosskopf formulated an addition of directional distance function extent of efficiency. They proposed a new model that improved upon Kuosmanen & Kazemi Matin models [20].

Current work is organized in several sections. section 2 describes Lozano & Villa and Kuosmanen& Kazemi Matin efforts.

Section 3 describes two phases of our model. The first phase concerns calculation of efficiency values and in the second phase, a model is presented based on which the projection of point is first drawn in a direction toward vector g, finds the point on the frontier of efficiency and then according to constant returns to scale, multiplies the projected point by times, and the minimum distance between this point and the integer point in the PPS is produced.

Presented work seeks to improve upon previous work by employing minimum distance while the previous efforts are not concerned with distance. A case study is given in section 4 that illustrates the novelty of this work, and conclusions are presented in the last section of this paper.

2 Integer-valued data envelopment analysis

Let us consider n decision making units (DMUs) that are interpreted by

$$\{(x_j, y_j) : j = 1, \cdots, n\}$$

It is assumed each DMU produces outputs by consuming corresponding inputs. For each DMU_j , where $j = 1, \cdots, n$, non-negative input and output vectors are interpreted as

$$X_j = (x_{1j}, \cdots, x_{mj})^T$$

and

$$Y_j = (y_{1j}, \cdots, y_{sj})^T$$

where, T is the sign of transposition.

$$X_j = (x_{1j}, \cdots, x_{mj})^T$$

and

$$Y_j = (y_{1j}, \cdots, y_{sj})^T$$

and are used to interpret, $m \times n$ input and output matrixes $s \times n$, respectively.

Conventional DEA models assume all data to be positive real values. In many practical situations however, some inputs and/or outputs can only hold integer values.

Lozano & Villa incorporated integer constraints into DEA models and suggested a MILP model for measuring efficiency of DMUs for the first time. The assumption of constant returns to scale was employed in their model [13].

The idea here is that each DMU with an input of x and an output of y, (x_j, y_j) is projected in frontier

$$(\sum \lambda_j x_j, \sum \lambda_j y_j)$$

is integer, where λ is an intensity value that must be an integer. Lozano & Villa defined an integer-valued production possibility set as:

$$T_{CRS} = \{(\widehat{x}, \widehat{y} : \exists(\lambda_1, \cdots, \lambda_n)$$

$$\lambda_j \geq o \forall j \widehat{x}_i \geq \sum_j \lambda_j x_{ij}, \widehat{y}_k$$

$$\leq \sum_j \lambda_j y_{kj} \quad \widehat{x}_i \quad integer$$

$$\forall i \in I',$$

$$\widehat{y}_k \quad integer \quad \forall k \in O'\}$$

That

$$I' \subseteq I = 1, 2, \cdots, m$$

and

$$O' \subseteq O = 1, 2, \cdots, s.$$

And a model was suggested as such:

$$Min\theta_j - \varepsilon(\sum_i s_i^- + \sum_k s_k^+)$$

$$\text{s.t.} \quad \textstyle\sum_j \lambda_j x_j = x_i \quad \forall i$$

$$xi = \theta_j x_{ij} - s_i^- \forall i$$

$$\textstyle\sum_j \lambda_j y_{kj} = y_k \quad \forall k$$

$$y_k = y_{kj} + s_k^+ \forall k$$

$$\lambda_j \geq 0 \forall j$$

$$s_i^-, x_i \geq 0 \forall i$$

$$s_k^+, y_k \geq 0 \forall k$$

$$\theta_j \quad free$$

$$x_i \quad integer \quad \forall i \in I'$$

$$y_k \quad integer \quad \forall k \in O' \ (2.1)$$

Kuosmanen & Kazemi Matin corrected Lozano & Villa's axioms further and suggested a new axiomatic foundation for integer-valued DEA models. Kuosmanen & Kazemi Matin also revised the Farrell input efficiency measurement to consider integer-valued measures and introduced a MILP problem. Their presented axioms were constant returns to scale.

1. Envelopment:

$$(x_j, y_j) \in T \quad : \forall j \quad \in 1, \cdots, n$$

2. Natural disposability:

$$(x_j, y_j) \in T \& (u, v) \in R^{m+s} :$$

$$y \geq v \Longrightarrow (x+u, y-v) \in T$$

3. Natural convexity:

If

$$(x^{'}, y^{'}), (x^{''}, y^{''}) \in T,$$
$$(x, y) = \lambda(x^{'}, y^{'}) + (1-\lambda)(x^{''}, y^{''})$$

that

$$\lambda \in [0,1] \;\; then \;\; (x, y) \in R^{m+s}$$
$$\Longrightarrow (x, y) \in T$$

4. Natural divisibility:

$$(x, y) \in T, \exists \lambda \in [0,1]:$$
$$(\lambda x, \lambda y) \in R^{m+s}$$
$$\longrightarrow (\lambda x, \lambda y) \in T$$

5. Natural augmentation:

$$(x, y) \in T, \exists \lambda \geq 1:$$
$$(\lambda x, \lambda y) \in R^{m+s}$$
$$\longrightarrow (\lambda x, \lambda y) \in T$$

They introduced their production possibility set as:

$$T_{IDEA} = \{(x, y) \in R^{m+s} | \\ x \geq \sum_j X_j \lambda_j, y \leq \sum_j Y_j \lambda_j, \\ \lambda_j \geq 0 \forall j\}$$

Due to the drawbacks of the Lozano and Villa models, Kuosmanen and Kazemi Matin corrected the axioms of their model and proposed a model for Farrell's efficiency measure according to the axioms of the integer-valued.

Their main idea is that every point in the production possibility set projected toward efficiency frontier must be dominated with an integer point. Kuosmanen & Kazemi Matin suggested the following:

$$\begin{aligned}
Min \quad & \theta_j - \varepsilon(\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^- + \sum_{i=1}^{p} s_i^I) \\
\text{s.t.} \quad & y_{ro} + s_r^+ = \textstyle\sum_{j=1}^n y_{rj}\lambda_j \quad r \in O \\
& \theta x_{io} - s_i^- = \textstyle\sum_{j=1}^n \lambda_j x_{ij} \quad i \in I^{NI} \\
& \widetilde{x}_i - s_i^- = \textstyle\sum_{j=1}^n \lambda_j x_{ij} \quad i \in I^I \\
& \theta x_{io} - s_i^I = \widetilde{x}_i \quad i \in I^I \\
& \widetilde{x}_i \in Z_+ \quad i \in I^I \\
& \lambda_j \geq 0 \quad j \in J \\
& s_r^+ \geq 0, s_i^- \geq 0, s_j^I \geq 0, \quad i \in I, j \in I^I
\end{aligned} \tag{2.2}$$

Where ε is a small non-Archimedean positive numeral, variables s_r^+, s_i^-, s_r^I indicate non radial slacks, and $\widetilde{x} \in Z^p$ is the integer-valued reference point for inputs. their model is radial efficient must be noted that it. I^I.

The next section concerns a new proposed model and its characteristics.

3 Proposed Model

Consider a technology involving n observed DMUs. Each DMU uses m inputs to produce s outputs, where some inputs and outputs are restricted to integer values.

According to the axioms and production technology mentioned in the previous section, a model in two phases is presented.

In the first phase, it calculates the efficiency of each DMU using the directional distance function in a specific direction under a CCR set and in the second phase, it obtains the minimum value of integer slack. That is, it obtains the minimum distance of the integer point from the point on the CCR boundary whereas this integer point dominates the point on the boundary. Under constant return to scales, inputs and outputs are multiplied by φ in models toward a CCR set, in the second phase, the point that is in a specific direction on the efficiency boundary is multiplied φ times according to the assumption of constant return to scale, and on the boundary goes to the point that is dominated by a integer point inside the PPS. Of course, we are looking for the nearest point to the efficiency boundary that dominates

Table 1: Outputs of four models

DMU	L&V	KKM	DEA CRS	PROPOSED MODEL
1	0.881	0.881	0.88	0.912
2	0.977	0.964	0.956	0.977
3	0.947	0.943	0.94	0.945
4	0.964	0.942	0.941	0.963
5	1	1	1	1
10	0.904	0.902	0.902	0.896
11	0.758	0.758	0.758	0.696
12	0.274	0.266	0.264	0
13	0.885	0.883	0.882	0.876
14	1	1	1	1
15	0.764	0.758	0.758	0.688
20	0.998	0.893	0.892	0.989
21	0.88	0.88	0.879	0.876
22	0.875	0.875	0.874	0.866
23	0.841	0.84	0.84	0.818
24	0.758	0.758	0.756	0.688
25	0.74	0.74	0.738	0.667
30	0.498	0.468	0.466	0
31	1	1	1	1
32	0.78	0.776	0.776	0.715
33	0.819	0.817	0.816	0.779
34	0.955	0.951	0.949	1
35	1	1	1	1
40	0.986	0.971	0.971	1
41	0.834	0.78	0.777	0.927
42	0.868	0.73	0.729	0.708

the φ -fold point on the frontier. In fact, it finds the most real benchmark point in PPS.

In other words, in the first phase, we plot point (x_o, y_o) inside the set using the directional distance function on the boundary and reach point $(x_o - s_i^{-*}, y_o + s_r^{+*} + \beta^* g_r^+)$ on the efficiency boundary. The value obtained from the objective function in this phase is s_i^-, s_r^+, β^* , The value of β^* for each DMU is replaced in the second phase.

Then in the second phase, under the constant return to scale of point $(x_o - s_i^{-*}, y_o + s_r^{+*} + \beta^* g_r^+)$ on the efficiency boundary increases φ times and reaches point $\varphi^*(x_o - s_i^{-*}, y_o + s_r^{+*} + \beta^* g_r^+)$, which is dominated by the integer point inside the production possibility set. As mentioned, we are looking for the least distance between point $\varphi^*(x_o - s_i^{-*}, y_o + s_r^{+*} + \beta^* g_r^+)$ and the dominant point.

φ is the coefficient of development of point $\varphi^*(x_o - s_i^{-*}, y_o + s_r^{+*} + \beta^* g_r^+)$ on the boundary of CCR.

Model is proposed as such for the first phase:

$$Max \quad \beta + \varepsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s^+)$$

$$\text{s.t.} \quad x_{io} - s_i^- = \textstyle\sum_{j=1}^n \lambda_j x_{ij} \quad i \in I$$

$$y_{ro} + s_r^+ + \beta g_r^+ = \textstyle\sum_{j=1}^n \lambda_j y_{rj}$$

$$j \in O^{NI}$$

$$y_{ro} + s_r^+ + \beta g_r^+ = \widehat{y}_r \quad r \in O^I$$

$$\widehat{y}_r \in Z_+ \quad r \in O^I$$

$$\lambda_j \geq 0 \quad \forall j \in 1, \cdots, J$$

$$s_r^+ \geq 0, s_i^- \geq 0, r \in O, i \in I \tag{3.3}$$

Where ε is a small non-Archimedean positive numeric value, variable s_r^+ , s_i^- indicate non radial slacks, and $\widehat{y}_r \in Z_+$ is the integer-valued reference point for outputs, g_r^+ and O^I is direction

Table 2: Efficiency frontier distance to the nearest point to the frontierInteger slacks

DMU	slack variable of the KKM model			slack variable of the proposed model			
1	0	0	0	0	0	0	0
2	0	0.928	0	0	0	0	0
3	0	0.942	0	0	0	0	0
4	0	0.086	0	0	0	0	0
5	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0
12	0	0.873	0	0	0	0	0
13	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0
33	0	0.855	0	0	0	0	0
34	0	0	0.266	0	0	0	0
35	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0

and distance function for the output vector.

From the above model (first phase), the value of efficiency (β^*) and s_r^+, s_i^- slacks are obtained and then the obtained values are placed in the second phase.

$$Min \sum_{r=1}^{p} s_r^I$$

s.t. $\varphi(x_{io}) \geq \sum_{j=1}^{n} \lambda_j x_{ij} \quad i \in I$

$\varphi(y_{ro} + \beta^* g_r^+) \leq \sum_{j=1}^{n} \lambda_j y_{rj}$

j $\in O^{NI}$

$\widehat{y}_r + s_r^I = \sum_{j=1}^{n} \lambda_j y_{rj} \quad r \in O^I$

$\varphi \geq 1$

$\widehat{y}_r \in Z_+ \quad r \in O^I$

$\lambda_j \geq 0 \quad \forall j \in 1, \cdots, J$

s˙rˆI $\geq$ 0 $r \in O$(3.4)Note that integer slack is minimized in the second phase model above.

When constant returns to scale is considered on the frontier of CCR, there is no difference in the point (x_o, y_o) with φ multiples of this point.

Also, the shortest distance to the efficiency frontier or the most real benchmark can be seen, the benchmark point is closer to the frontier.

The following is a theorem in which the distance obtained in the second phase, s_r^I (distance from point $\varphi(x_o, y_o + \beta g_r)$ to the efficiency boundary) is less than the distance from point $(x_o, y_o + \beta g_r)$ to the efficiency boundary, ie s_r^+ . In other words, the best integer benchmark is obtained in the second phase of the proposed model.

Theorem 3.1. $s_r^I \leq s_r^+$.

Proof. See in appendix.

In the next section, a case study is given in which the efficiency values obtained from the previous models and the proposed model in this paper can be compared with each other.

4 Case Study

The presented model is applied to 42 university segments of IAUK per Kuosmanen & Kazemi Matin [11]. Each department is characterized by three input variables are the number of post graduate, the number of bachelor students and the number of master students.

The output variables are the number of graduations, the number of scholarships, the number of research products and the level of manager satisfaction.

The results of the new model in 3 , 3 are compared with results of Lozano and Villa in 2, CCR model, and Kuosmanen & Kazemi Matin integer DEA model in 2.

The following two tables are given in the first table of efficiency values in the previous models, the CCR model and the first phase of the model presented in this paper. In the second table, the values of slack in KKM model and the second phase of this model are compared.

The first column signifies order of DMUs. The second column labeled with L&V is the results of Lozano and Villa model. In the third column, KKM refers to Kuosmanen and Kazemi Matin model. The fourth column is the outputs of CRS model labeled as DEA-CRS and the last column refers to the first phase of the proposed model. Tables 1 and 2 show many of the values obtained from models 3 and 3 that were written using Lingo and GAMs software.

As evident in Table 1 above, the model presented in this work exhibits better performance in some units. In Table 1, the values obtained from the first phase with Lingo and GAMs software are given and the efficiency values can be compared with previous models.

As seen in the objective function of the second phase, the minimum distance between the point $\varphi^*(x_o - s_i^{-*}, y_o + s_r^{+*} + \beta^* g_r^+)$ and the integer point is calculated, where that point on the boundary is dominated by the integer point inside the PPS.

Nearest benchmark to efficiency frontier is chosen in this phase and minimum distance to integer point is calculated. Table 2 shows a comparison between this distance in KKM model and the second phase in the proposed model.

The three columns on the left in Table 2 show the integer slack values of Kuosmanen and Kazemi Matin in model 2 and the four columns on the right indicate the integer slack value of the presented model in model 3 .

Integer slacks in the second phase of model 3 are calculated by Lingo software as shown in Table 2. As you can see, in some units in the KKM model, the slacks have positive values, while the values of these slacks in the second phase of the model proposed in this paper are zero. In fact, there is no distance between the point $\varphi^*(x_o - s_i^{-*}, y_o + s_r^{+*} + \beta^* g_r^+)$ and the dominant integer point in the PPS. In other words, there is radial inefficiency.

Noting Table 1 where efficiency values are presented, that the efficiency values of the units are obtained according to models of LV, KKM 2 and DEA CRS. Additionally, in the last column of the table, the amount of efficiencies obtained from the first phase of the model 3 presented in this paper are compared.

In Table 2, where the second phase slacks are shown, integer slack values in Kuosmanen & Kazemi Matin model 2 and the presented model 3 are compared. Presented model 3 produces integer slacks that are equal to zero. This illustrates the novelty of the introduced mode as a good method of numerating real and nearest benchmarks.

As shown in Tables 1 and 2, in DMUs where the efficieny in models KKM and proposed model are equal to one, and their integer slack value is zero are considered as strong efficiency or Parato-Koopmans efficiency.

5 Conclusions

In conventional DEA models assume that the DMUs have real-valued, however there are many practical situations where inputs and outputs must only integer-valued. Integer-valued DEA has motivated many researchers in recent years to produce methods of integer values. As an example, for the first time Lozano & Villa established a method of Integer Linear Programming. Kuosmanen & Kazemi Matin further modified axioms and extended DEA models to produce more accuracy in calculations. The current work attempts to minimize the distance between the point projected on the frontier with the integer point that dominate it. Presented model is organized in two phases. In the first phase, the model computes efficiency values. In the second phase, the distance between the projected point in frontier and the integer point that dominates the point in the frontier are minimized. In the second phase, all integer slacks are zero. That is to say, an integer point is in the CCR frontier.

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Appendix

Proof of Theorem 3.1. Considering $\varphi = 1$ in the second phase and the optimal solution (s_r^+, s_i^-, β^*) in the first phase of the model, we have:

$$x_{io} - s_i^- = \sum_{j=1}^{n} \lambda_j x_{ij} \quad i \in I$$

$$y_{ro} + s_r^+ + \beta g_r^+ = \sum_{j=1}^{n} \lambda_j y_{rj} \quad j \in O^{NI}$$

$$y_{ro} + s_r^+ + \beta g_r^+ = \widehat{y}_r \quad r \in O^I$$

As a result, point (s_r^+, s_i^-, β^*) is a feasible solution for the second phase of the model. Since the second phase of the model is minimization, the optimal solution of the model is less than the feasible solution, so: $s_r^I \leq s_r^+$.

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