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Research Article

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An Inverse Dynamic FDH Approach to Estimate Outputs

F. Asadi *, S. Kordrostami †‡, AR. Amirteimoori §, M. Bazrafshan ¶

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Abstract

In many situations, the performance and the changes of outputs related to dynamic systems should be estimated while the convexity property is relaxed. Accordingly, first, a dynamic free disposal hull (FDH) model is proposed in this paper to address the efficiency of processes in multiple period of time while the convexity assumption is unsatisfied. Also, wo problems, including a mixed integer linear programming and a linear programming model are provided to compute the dynamic FDH model that is a mixed integer non-linear programming problem. Then the changes of multi-period outputs are dealt with for changes of inputs related to several periods using the proposed inverse dynamic FDH model while the efficiency levels are preserved. A case study of gas industry is, moreover, presented to demonstrate the introduced models. The results show the proposed technique is useful to analyze the performance and to estimate outputs in dynamic processes without including convexity.

Keywords : Data envelopment analysis (DEA); Network DEA; Inverse DEA; Free disposal hull (FDH); Dynamic DEA.

1 Introduction

THe performance analysis is an important aspect for organizations to be aware from their strengths and weaknesses. Data envelopment analysis (DEA), firstly rendered by Charnes et al. [8] (1987), is a popular non-parametric approach to evaluale the relative efficiency of decision making units (DMUs) with multiple inputs and outputs. In the DEA literature, it can be found many DEA models to analyze the performance that have been presented for different applications [22, 1, 34, 19, 42, 5, 30, 9, 47, 51, 3, 14, 34, 35].

Dynamic DEA approaches [23, 30, 14, 46, 39, 25, 45, 50, 30, 44, 53, 31, 7, 25] are among DEA methods to address the efficiency of DMUs, in several periods of time. Kao (2013) [30] proposed a dynamic DEA model to evaluate the period and system efficiency values of multi-period processes, simultaneously. Mariz et al. (2018) [40] provided a review of the existing dynamic DEA models from 1996 to 2016. Omrani and Soltanzadeh (2016) [43] presented a relational dynamic DEA model to evaluate the energy efficiency of European Union countries. Foladi et al. (2019) [17] introduced inverse dynamic DEA models to access faculties of universities where

*Department of Industrial Engineering, Lahijan Branch, Islamic Azad University, Lahijan, Iran.
†Corresponding author. kordrostami@liau.ac.ir, Tel:+98(13)41229076.
‡Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran.
§Department of Applied Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran.
¶Department of Industrial Engineering, Lahijan Branch, Islamic Azad University, Lahijan, Iran.

quasi-fixed inputs are presented. In some inverse DEA problems, the number of outputs (inputs) are increased or decreased and the input (output) values are obtained so that the efficiency value remains at the same level. Inverse DEA has founed wide applications todays, one of its applications is in decision-making issues [12, 20]. Wei et al. (2000) [51] primarily presented an inverse DEA model to access the performance measures. Lertworasirikul et al. (2011) [35] considered the inverse DEA approach under the variable returns to scale assumption for a resource allocation problem when the expansion of some outputs and the reduction of others are contemplated. Ghiyasi and Zhu (2020) [19] presented an inverse semi-oriented DEA approach when there are negative measures. Hassanzadeh at al. (2018) [21] evaluated the sustainability of countries using their inverse DEA models, containing input-oriented and output-oriented inverse semi-oriented radial measures. Jahanshahloo et al. (2014) [26] introduced inverse nonradial enhanced Russell models to estimate inputs, outputs and both of them. An et al. (2019) [5] dealt with planning the resources of commericial banks applying their two-stage inverse DEA approach where undesirable outputs are presented. Kalantary and Farzipoor Saen (2019) [28] suggested an inverse network dynamic slacks based measure model accompanied by evaluating the sustainability of supply chains. Kalantary et al. (2018) [29] considered the sustainability of supply chains using their inverse network dynamic slacks based measure adjust measure approach. To the best of our knowledge, all existing inverse dynamic DEA models are based upon the convex technology that sometimes DMUs are compared with artificial DMUs that are absurd thenefore, in this study, a dynamic DEA approach based on the free disposal hull (FDH) model, orginally introduced by deprins et al. [10], is rendered to measure the efficiency of multi-period systems with quasi-fixed inputs. Then an inverse dynamic FDH approach is presented to estimate outputs under the non-convex technology. Two approaches, including a mixed integer linear programming problem and a linear programming model, are formulated to solve the proposed non-linear inverse dynamic FDH model. An application of Iranian gas companies is given to explain the proposed approach.

The rest of this paper is organized as follows: in Section 2 the prerequisites for other sections are described. In Section 3, a dynamic FDH model and an inverse dynamic FDH approach are provided to assess the efficiency of dynamic processes under the non-convex technology and also the estimation of outputs. A real case study of Iranian gas companies is given in Section 4 to clarify the suggested models. Finally, conclusions and suggestions are presented in Section 5.

2 Preliminaries

In this section, the necessary preparations for areas of this research are reviewed. First, the FDH model is described, which is the main infrastructure of the suggested model. Afterward, some information about dynamic models and their general structure are mentioned, and finally, the inverse dynamic DEA approach introduced by Foladi et al. [17] is examined.

2.1 Free Disposal Hull

One of the problems with DEA models based on the convex technology is that a unit is measured by a convex combination of other units, however sometimes accepting this aspect for a unit may not be logical or practical. The FDH model in which each unit is compared to exactly another unit, so it is more appropriate for these units.

FDH models relax the convexity assumption of basic DEA models. The production possibility set of the FDH model formulated on DEA principles and with the observations (X_j, Y_j) $j = 1, 2, ..., n$ is as follows:

$$T = \Big\{(X,Y)\ :\ \sum_{j=1}^{n} \lambda_j \omega X_j \leq X,\\ \sum_{j=1}^{n} \lambda_j \omega Y_j \geq Y, \omega \in \mathbb{R}_+,\\ \lambda_j \in \{0,1\}, \sum_{j=1}^{n} \lambda_j = 1\Big\}$$

Considering T, the FDH model becomes as follows:

$$\begin{aligned} max \quad & \phi \qquad (2.1) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j \omega x_{ij} \leq x_{io}, \quad i = 1, 2, ..., m \\ & \sum_{j=1}^{n} \lambda_j \omega y_{rj} \geq \phi y_{ro}, \quad r = 1, 2, ..., s \\ & \lambda_j \in \{0, 1\}, \quad j = 1, 2, ..., n \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \omega \geq 0 \end{aligned}$$

Problem (2.1) is a mixed-integer nonlinear programming problem and it can be rewritten as to the following linear programming [32]:

$$\begin{aligned} max \quad & \sum_{j=1}^{n} \phi_j \qquad (2.2) \\ s.t. \quad & (\lambda_j + \omega_j) x_{ij} \leq \lambda_j x_{io}, \\ & \qquad i = 1, 2, ..., m, \; j = 1, 2, ..., n \\ & (\lambda_j + \omega_j) y_{rj} \geq \phi_j y_{io}, \\ & \qquad r = 1, 2, ..., s, \; j = 1, 2, ..., n \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \; j = 1, 2, ..., n \end{aligned}$$

If ϕ^* is the optimal value of (2.1) and ϕ_j^* are the optimal values of the problem (2.2), then $\phi^* = \sum_{j=1}^{n} \phi_j$. Note that, there are other forms of FDH that are very useful, but the proposed model in this study is based on the problem (2.1) so they will not be mentioned.

2.2 Dynamic DEA

In this subsection, dynamic DEA models are discussed. Suppose we have n DMUs, each with m inputs and s outputs. We also have information about these units in p different periods. One strategy is to calculate the efficiency in each period separately, another strategy is to consider each period as an independent unit and calculate the efficiency in all periods, in which case the number of units is equal to np. Another strategy is to make the sum of the inputs and outputs of different periods into one unit and measure the efficiency of these new units. All previous strategies have major drawbacks, but a suitable strategy is to consider the units with all their periods as a unit in which different periods are considered. Apart from the fact that each unit has its inputs and outputs, some outputs in one period are used as the inputs of the next period, which are called links or intermediate measures. Suppose we have n DMUs, with m inputs $x_{ij}^t \in \mathbb{R}_+$, s outputs $y_{rj}^t \in \mathbb{R}_+$ and P periods, and $z_{fj}^t \in \mathbb{R}_+$ is the link between period t and $t+1$ ($Z_j^t \in \mathbb{R}^g$). Consider $x_{ij} = \sum_{t=1}^{P} x_{ij}^t$ and $y_{rj} = \sum_{t=1}^{P} y_{rj}^t$ as the sum of ith input and rth output during p periods ($t = 1, 2, ..., P$), respectively. See Figure 1 for a better understanding of the dynamic processes. Kao [30] in 2013 proposed the following

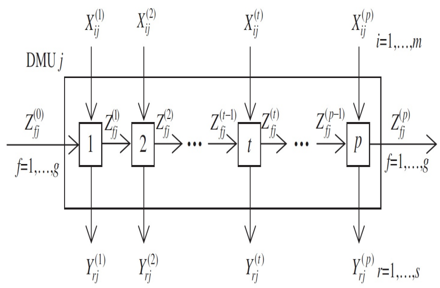

Figure 1: The graphic form of a process with P different time period [17]

dynamic DEA model to address the efficiency of multi-period processes with quasi-fixed inputs:

$$\begin{aligned} min \quad & \sum_{i=1}^{m} v_i X_{io} + \sum_{f=1}^{g} w_f Z_{fo}^0 \qquad (2.3) \\ s.t. \quad & \sum_{r=1}^{s} u_r Y_{ro} + \sum_{f=1}^{g} w_f Z_{fo}^p = 1 \\ & \left(\sum_{i=1}^{m} v_i X_{ij} + \sum_{f=1}^{g} w_f Z_{fj}^0 \right) \\ & \quad - \left(\sum_{r=1}^{s} u_r Y_{rj} + \sum_{f=1}^{g} w_f Z_{fj}^p \right) \geq 0 \\ & \qquad (2.4) \end{aligned}$$

$$j = 1, 2, ..., n$$

$$\left(\sum_{i=1}^{m} v_i X_{ij}^t + \sum_{f=1}^{g} w_f Z_{fj}^{t-1}\right) - - \left(\sum_{r=1}^{s} u_r Y_{rj}^t + \sum_{f=1}^{g} w_f Z_{fj}^t\right) \geq 0$$

$$j = 1, 2, ..., n, t = 1, 2, ..., p$$

$$u_r \geq 0 \quad r = 1, 2, ..., s$$

$$v_i \geq 0 \quad i = 1, 2, ..., m$$

$$w_f \geq 0 \quad f = 1, 2, ..., g$$

The second constraints state that the ratio of total outputs to total inputs must be less than or equal to one, the third constraints show that the output-to-input ratio in period t must be less than or equal to one. In the basic models, there must have been a unit or units that are efficient, i.e. the efficiency value is equal to one, but in this type of model, no unit may be efficient, because there may be a period in which the unit that does not perform well. The dual of model (2.3) is as follows:

$$\begin{aligned} max \quad & \phi && (2.5) \\ s.t. \quad & \sum_{t=1}^{p}\sum_{j=1}^{n} \lambda_j^t X_{ij}^t + \sum_{j=1}^{n} \lambda_j X_{ij} \leq X_{io} \\ & \qquad i = 1, 2, ..., m \\ & \sum_{t=1}^{p}\sum_{j=1}^{n} \lambda_j^t Y_{rj}^t + \sum_{j=1}^{n} \lambda_j Y_{rj} \geq \phi Y_{ro} \\ & \qquad j = 1, 2, ..., s \\ & \sum_{t=1}^{p}\sum_{j=1}^{n} \lambda_j^t \left(Z_{fj}^t - Z_{fj}^{t-1}\right) + \sum_{j=1}^{n} \lambda_j \left(Z_{fj}^p - Z_{fj}^0\right) \\ & \quad + Z_{fj}^0 \geq \phi Z_{fo}^p \qquad f = 1, 2, ..., g \\ & \lambda_j^t \geq 0, \quad \lambda_j \geq 0 \quad j = 1, 2, ..., n, \quad t = 1, 2, ..., p \end{aligned}$$

The optimal value $\frac{1}{\phi^*}$ is considered as the efficiency level.

In the next subsection, the inverse dynamic DEA model provided by Foladi et al. (2020) is represented [17].

2.3 Inverse Dynamic DEA's Foladi et al.

Inverse DEA is a category of DEA, which includes approaches that make changes to the overall structure of units to create a new unit, for example, (X, Y) changes to $(\alpha, \beta) = (X+\Delta X, Y+\Delta Y)$ where $(\Delta X, \Delta Y)$ are the perturbation and usually are not very large. The new unit (α, β) has the same efficiency of (X, Y). To describe inverse dynamic problems, the inverse dynamic approach proposed by Foladi et al. [17] to estimate outputs is illustrated. First, the problem (2.5) is solved, assuming that the optimal value is ϕ^*. Then values to X_o $(\alpha_o = X_o \pm \Delta X)$ are added (subtracted) and the input vector β that is not greater (less) than Y_o is looked for so that (α, β) has the same efficiency as ϕ^*. For this purpose, the following model is computed:

$$\begin{aligned} max \quad & \beta_{ro} && (2.6) \\ s.t. \quad & \sum_{t=1}^{P}\sum_{j=1}^{n} \lambda_j^{(t)} x_{ij}^{(t)} + \sum_{j=1}^{n} \lambda_j x_{ij} \leq \alpha_{io} \\ & \qquad i = 1, 2, ..., m \\ & \sum_{t=1}^{P}\sum_{j=1}^{n} \lambda_j^{(t)} y_{rj}^{(t)} + \sum_{j=1}^{n} \lambda_j y_{rj} \geq \phi_o \beta_{ro} \\ & \qquad r = 1, 2, ..., s \\ & \sum_{t=1}^{P}\sum_{j=1}^{n} \lambda_j^{(t)} \left(z_{fj}^{(t)} - z_{fj}^{(t-1)}\right) \\ & + \sum_{j=1}^{n} \lambda_j \left(z_{fj}^{(P)} - z_{fj}^{(0)}\right) + z_{fo}^{(0)} \geq \phi_o z_{fo}^{(p)} \\ & \qquad f = 1, 2, ..., g \\ & \beta_{ro} \geq y_{ro} \\ & \lambda_j^{(t)}, \lambda_j \geq 0, \quad t = 1, 2, ..., P, \quad j = 1, 2, ..., n \end{aligned}$$

Problem (2.6) is a multi-objective programming problem that is usually solved by the weighted sum method, and weights can be calculated based on the importance of the factors.

3 Proposed Approach

In this section, an alternative model to estimate the performance of dynamic processes is presented and then its inverse model is rendered. Our approaches are based on the FDH model and a non-convex technology. Most FDH models are expressed as mixed-integer linear programming forms while in the case addressed in this research, the problems are as mixed integer nonlinear programming models and we try to reformulate them as linear.

Models (2.5) and (2.6) are under the convex technology. Nevertheless, the efficiency analysis of multi-period systems and the estimation of changes of measures under the non-convex technology is major due to this fact that a more flexible insight of frontier is procided [38].

3.1 Relational Dynamic FDH Model

Similar to Figure 1, suppose there are n DMUs, with m inputs $x_{ij}^t \in \mathbb{R}_+$, s outputs $y_{rj}^t \in \mathbb{R}_+$ and g links between the period t and $t+1$, $z_{fj}^t \in \mathbb{R}_+$ that should be evaluated in P periods (t=1,...,P). Also, $x_{ij} = \sum_{t=1}^P x_{ij}^t$ and $y_{rj} = \sum_{t=1}^P y_{rj}^t$ are considered as the sum of ith input and rth output during P periods, respectively. Under the non-convex technology, the following dynamic FDH model is proposed to evaluate the efficiency of multi-period systems with quasi-fixed inputs.

$$max \quad \phi \tag{3.7a}$$

$$s.t. \quad \sum_{t=1}^{p}\sum_{j=1}^{n}\omega^t\lambda_j^t X_{ij}^t + \sum_{j=1}^{n}\omega\lambda_j X_{ij} \leq X_{io} \quad i=1,2,...,m \tag{3.7b}$$

$$\sum_{t=1}^{p}\sum_{j=1}^{n}\omega^t\lambda_j^t Y_{rj}^t + \sum_{j=1}^{n}\omega\lambda_j Y_{rj} \geq \phi Y_{ro} \quad r=1,2,...,s \tag{3.7c}$$

$$\sum_{t=1}^{p}\sum_{j=1}^{n}\omega^t\lambda_j^t\left(Z_{fj}^t - Z_{fj}^{t-1}\right) + \sum_{j=1}^{n}\omega\lambda_j\left(Z_{fj}^p - Z_{fj}^0\right) + Z_{fj}^0 \geq \phi Z_{fo}^p \quad f=1,2,...,g \tag{3.7d}$$

$$\tag{3.7e}$$

$$\sum_{j=1}^{n}\lambda_j^t = 1 \quad t=1,2,...,p \tag{3.7f}$$

$$\sum_{j=1}^{n}\lambda_j = 1 \tag{3.7g}$$

$$\lambda_j^t \in \{0,1\}, \quad \lambda_j \in \{0,1\} \quad j=1,2,...,n, \quad t=1,2,...,p \tag{3.7h}$$

$$\omega^t \geq 0, \ \omega \geq 0, t=1,2,...,p \tag{3.7i}$$

The constraints (3.7f) to (3.7h) state that exactly one unit must be selected in each period. The first constraint (3.7b) shows that in each period, only one unit is used to compare the inputs, the second constraint also denotes (3.7c) that the same unit used in the constraint should be considered in the constraint (3.7c) to compare the outputs. The constraint (3.7d) also states that the same link and the previous link should be used, the corresponding unit of which is selected in the first and second constraints i.e. (3.7b) and (3.7c). Constraints $\omega^t \geq 0$, and $\omega \geq 0$ have been added to the model since the returns to scale status is assumed to be constant. The optimal value $\frac{1}{\phi^*}$ resulted from model (3.7) shows the multi-period efficiency level.

As can be seen, the proposed model is a mixed-integer nonlinear programming problem. Note that if the number of periods and DMUs are not large, problem (3.7) can be divided into several linear programming problems as follows by replacing different choices of $\lambda_j^t, \lambda_j \in \{0,1\}$, but the number of cases may be too high.

$$\begin{aligned}
max \quad & \phi_{j^1,j^2,...,j^p,j} \\
s.t. \quad & \omega^1 X_{ij^1}^1 + \omega^2 X_{ij^2}^2 + ... + \omega^p X_{ij^p}^p + \omega X_{ij} \\
& \quad \leq X_{io} \quad i=1,2,...,m \\
& \omega^1 Y_{rj^1}^1 + \omega^2 Y_{rj^2}^2 + ... + \omega^p Y_{rj^p}^p + \omega Y_{rj} \\
& \quad \geq \phi Y_{ro} \quad r=1,2,...,s \\
& \omega^1\left(Z_{fj^1}^1 - Z_{fj^1}^0\right) + \omega^2\left(Z_{fj^2}^2 - Z_{fj^2}^1\right) \\
& \quad + ... + \omega^p\left(Z_{fj^p}^p - Z_{fj^p}^{p-1}\right) \\
& \quad + \omega\left(Z_{fj}^p - Z_{fj}^0\right) + Z_{fj}^0 \geq \phi Z_{fo}^p \\
& \quad f=1,2,...,g \\
& \omega^t \geq 0, \quad \omega \geq 0, \quad t=1,2,...,p
\end{aligned}$$

where $j^1, j^2, ..., j^p, j \in \{1, 2, ..., n\}$ and the optimal value of problem (3.7) is equal to

$$\phi = max\{\phi_{j^1,j^2,...,j^p,j} \ : \ j^1, j^2, ..., j^p, \\ j \in \{1, 2, ..., n\}\}$$

Thus, in the next subsections, two methods are presented to solve this problem. The first model is a mixed-integer linear programming problem and the second method is a linear programming form. In appearance, the linear programming probelm is much easier to solve than mixed-integer programming, but because the number of variables and constraints increases in the linear method, sometimes the computational time is high and it is recommended to use mixed-integer programming forms.

3.2 Mixed Integer LP Model Related to Model (3.7)

In this subsection, using the big M method, problem (3.7) is converted into a mixed-integer programming problem. Therefore, we have the following the mixed-integer programming form:

$$
\begin{aligned}
max \quad & \phi \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.8)\\
s.t. \quad & \sum_{t=1}^{p}\sum_{j=1}^{n}\pi_j^t X_{ij}^t + \sum_{j=1}^{n}\pi_j X_{ij} \leq X_{io} \\
& \qquad i = 1, 2, ..., m \\
& \sum_{t=1}^{p}\sum_{j=1}^{n}\pi_j^t Y_{rj}^t + \sum_{j=1}^{n}\pi_j Y_{rj} \geq \phi Y_{ro} \\
& \qquad r = 1, 2, ..., s \\
& \sum_{t=1}^{p}\sum_{j=1}^{n}\pi_j^t \left(Z_{fj}^t - Z_{fj}^{t-1}\right) + \sum_{j=1}^{n}\pi_j \left(Z_{fj}^p - Z_{fj}^0\right) \\
& \quad Z_{fj}^0 \geq \phi Z_{fo}^p \qquad f = 1, 2, ..., g \\
& \sum_{j=1}^{n}\lambda_j^t = 1 \ \ t = 1, 2, ..., p \\
& \sum_{j=1}^{n}\lambda_j = 1 \\
& \lambda_j^t \in \{0, 1\}, \quad \lambda_j \in \{0, 1\} \quad j = 1, 2, ..., n \\
& \qquad t = 1, 2, ..., p \\
& 0 \leq \pi_j^t \leq M\lambda_j^t, \quad 0 \leq \pi_j \leq M\lambda_j \\
& \qquad j = 1, 2, ..., n, \quad t = 1, 2, ..., p
\end{aligned}
$$

where M is a sufficiently large number. Problems (3.7) and (3.8) are equivalent. There are two cases for λ_j^t or λ_j (consider λ_j). If $\lambda_j = 0$, then $\pi_j = 0$ and if $\lambda_j = 1$, then $\pi_j \geq 0$ and π_j is equal to w_j. Of course, different numbers can be used instead of an M. If M is chosen as a large value, we may have a computational error, so the value M is important.

3.3 LP From of Model

In this subsection, the problem (3.7) is transformed into a linear programming problem. The syntax shown in this problem is a bit complicated, consider the following problem:

$$
\begin{aligned}
max \quad & \sum_{j^1,j^2,...,j^p,j} \phi_{j^1,j^2,...,j^p,j} \qquad\qquad\qquad\qquad (3.9)\\
s.t. \quad & (\omega^1_{j^1,j^2,...,j^p,j} + \lambda_{j^1,j^2,...,j^p,j})X^1_{ij^1} \\
& + (\omega^2_{j^1,j^2,...,j^p,j} + \lambda_{j^1,j^2,...,j^p,j})X^2_{ij^2} \\
& + ... \\
& + (\omega^p_{j^1,j^2,...,j^p,j} + \lambda_{j^1,j^2,...,j^p,j})X^p_{ij^p} \\
& + (\omega_{j^1,j^2,...,j^p,j} + \lambda_{j^1,j^2,...,j^p,j})X_{ij} \\
& \leq \lambda_{j^1,j^2,...,j^p,j}X_{io} \quad \forall i, j^1, j^2, ..., j^p, j \\
& \\
& (\omega^1_{j^1,j^2,...,j^p,j} + \lambda_{j^1,j^2,...,j^p,j})Y^1_{rj^1} \\
& + (\omega^2_{j^1,j^2,...,j^p,j} + \lambda_{j^1,j^2,...,j^p,j})Y^2_{rj^2} \\
& + ... \\
& + (\omega^p_{j^1,j^2,...,j^p,j} + \lambda_{j^1,j^2,...,j^p,j})Y^p_{rj^p} \\
& + (\omega_{j^1,j^2,...,j^p,j} + \lambda_{j^1,j^2,...,j^p,j})Y_{rj} \\
& \quad \geq \phi_{j^1,j^2,...,j^p,j}Y_{ro} \quad \forall r, j^1, j^2, ..., j^p, j \\
& (\omega^1_{j^1}, j^2, ..., j^p, j + \lambda_{j^1,j^2,...,j^p,j})(Z^1_{fj^1} - Z^0_{fj^1}) \\
& +(\omega^2_{j^1}, j^2, ..., j^p, j + \lambda_{j^1,j^2,...,j^p,j})(Z^2_{fj^2} - Z^1_{fj^2}) \\
& + ... \\
& +(\omega^p_{j^1}, j^2, ..., j^p, j + \lambda_{j^1,j^2,...,j^p,j})(Z^p_{fj^3} - Z^{p-1}_{fj^3}) \\
& +(\omega_{j^1}, j^2, ..., j^p, j + \lambda_{j^1,j^2,...,j^p,j})(Z^p_{fj^p} - Z^0_{fj^p}) \\
& +\lambda_{j^1}, j^2, ..., j^p, jZ^0_{fj} \geq \phi_{j^1,j^2,...,j^p,j}Z^p_{fo} \\
& \quad \forall f, j^1, j^2, ..., j^p, j \qquad\qquad\qquad\qquad (3.10)\\
& \sum_{j^1}, j^2, ..., j^p, j\lambda_{j^1,j^2,...,j^p,j} = 1 \\
& \quad \lambda_{j^1,j^2,...,j^p,j} \geq 0 \qquad\qquad\qquad\qquad (3.11)\\
& \omega^t_{j^1,j^2,...,j^p,j} + \lambda_{j^1,j^2,...,j^p,j} \geq 0
\end{aligned}
$$

Theorem 3.1. *Problems (3.7) and (3.9) are equivalent.*

Proof. For each $\lambda_{j^1,j^2,\dots,j^p,j}$ three cases are presented:

1. $\lambda_{j^1,j^2,\dots,j^p,j} = 0$, in this case, the constraint corresponding to $\lambda_{j^1,j^2,\dots,j^p,j}$ will become worthless, the two sides will be equal to zero.

2. $\lambda_{j^1,j^2,\dots,j^p,j} = 1$, in this case, the other λ must be zero, and only one of the constraints is active and the result is obvious.

3. $0 < \lambda_{j^1,j^2,\dots,j^p,j} < 1$, in this case, the sides of the inequality can be divided by $\lambda_{j^1,j^2,\dots,j^p,j}$. So a new solution is obtained. The optimal value is $\frac{\phi_{j^1,j^2,\dots,j^p,j}}{\lambda_{j^1,j^2,\dots,j^p,j}}$, but the convexity property implies that these solutions cannot be optimal unless all solutions are equal.

□

3.4 An Inverse Dynamic FDH Model

In this part, an inverse dynamic model under the non-convex technology and considering model (3.7) is proposed. In the inverse dynamic FDH model, values are added (subtracted) to the inputs (outputs) and the outputs (inputs) are found in such a way that the new unit has the same efficiency. Actually, we change the values of the inputs and calculate the values of the outputs. Set $\beta = X \pm \Delta X$ where ΔX is the perturbed nonnegative vector. The following model is presented to estimate outputs for changes of inputs:

$$\max \quad (\alpha_{1o}, \alpha_{2o}, \dots, \alpha_{so}) \tag{3.12}$$

$$\begin{aligned}
s.t. \quad & \sum_{t=1}^{p}\sum_{j=1}^{n} \omega^t \lambda_j^t X_{ij}^t + \sum_{j=1}^{n} \omega \lambda_j X_{ij} \le \beta_{io} \\
& \quad i = 1, 2, \dots, m \\
& \sum_{t=1}^{p}\sum_{j=1}^{n} \omega^t \lambda_j^t Y_{rj}^t + \sum_{j=1}^{n} \omega \lambda_j Y_{rj} \ge \phi^* \alpha_{ro} \\
& \quad r = 1, 2, \dots, s \\
& \sum_{t=1}^{p}\sum_{j=1}^{n} \omega^t \lambda_j^t \left(Z_{fj}^t - Z_{fj}^{t-1} \right) \\
& \quad + \sum_{j=1}^{n} \omega \lambda_j \left(Z_{fj}^p - Z_{fj}^0 \right) + Z_{fj}^0 \ge \phi^* Z_{fo}^p \\
& \quad f = 1, 2, \dots, g \\
& \sum_{j=1}^{n} \lambda_j^t = 1, t = 1, 2, \dots, p \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j^t \in \{0, 1\}, \quad \lambda_j \in \{0, 1\} \\
& \quad j = 1, 2, \dots, n, \quad t = 1, 2, \dots, p \\
& \omega^t \ge 0, \quad \omega \ge 0, \quad t = 1, 2, \dots, p \\
& \alpha_{ro} \ge Y_{ro} \quad (\alpha_{ro} \le Y_{ro}), \qquad r = 1, 2, \dots, s
\end{aligned} \tag{3.13}$$

where ϕ^* is the optimal value of problem (3.7). If $\beta = X + \Delta X$, the condition $\alpha_{ro} \ge Y_{ro}$ is added to model (3.12) and if $\beta = X - \Delta X$, $\alpha_{ro} \le Y_{ro}$ is included in model (3.12). This problem can be solved in the same way described in the previous areas. Of course, this problem is multi-objective programming and transforms into a single-objective problem using the weighted sum method.

Notice that, the changes of outputs for the perturbation of inputs are addressed in this study. Nevertheless, the approaches proposed can be extended to assess the changes of inputs for the modifications of outputs while the multi-period efficiency remain unchanged. In the next section. the introduced approaches are used to analyze the efficiency of gas companies deemed as dynamic systems and also the estimation of output measures.

3.5 Feasibility discussion

In this subsection, we discuss the feasibility of problem (3.12). We consider two cases. Either we increase the inputs or we decrease them. In the first case, it is always feasible. Suppose we have added some value to the inputs, the feasible solutions of problem (3.7) can be used for this problem, so the problem is feasible. But in the second case, it may be impossible. First, we prove that it may become an infeasible. Suppose we reduce the inputs to a significant amount, in this case we have to make ω and ω^t very small, and in this case the problem is feasible when $Z^0_{fj} \geq \phi^* Z^p_{fo}$, otherwise the problem may be infeasible. Therefore, we offer a model that is always feasible. Consider the following model:

$$
\begin{aligned}
max \quad & (\alpha_{1o}, \alpha_{2o}, ..., \alpha_{so}) && (3.14)\\
s.t. \quad & \sum_{t=1}^{p}\sum_{j=1}^{n} \omega^t \lambda_j^t X_{ij}^t + \sum_{j=1}^{n} \omega \lambda_j X_{ij} \leq \beta_{io} \\
& \quad i = 1, 2, ..., m \\
& \sum_{t=1}^{p}\sum_{j=1}^{n} \omega^t \lambda_j^t Y_{rj}^t + \sum_{j=1}^{n} \omega \lambda_j Y_{rj} \geq \phi^* \alpha_{ro} \\
& \quad r = 1, 2, ..., s \\
& \sum_{t=1}^{p}\sum_{j=1}^{n} \omega^t \lambda_j^t \left(Z_{fj}^t - Z_{fj}^{t-1}\right) \\
& + \sum_{j=1}^{n} \omega \lambda_j \left(Z_{fj}^p - Z_{fj}^0\right) && (3.15)\\
& \quad + \omega^1 Z_{fj}^0 \geq \phi^* Z_{fo}^p \quad f = 1, 2, ..., g \\
& \sum_{j=1}^{n} \lambda_j^t = 1 \quad t = 1, 2, ..., p \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j^t \in \{0,1\}, \lambda_j \in \{0,1\}, j = 1, 2, ..., n, t = 1, 2, ..., p \\
& \omega^t \geq 0, \quad \omega \geq 0, \quad t = 1, 2, ..., p \\
& \alpha_{ro} \geq Y_{ro} \quad (\alpha_{ro} \leq Y_{ro}), \quad r = 1, 2, ..., s
\end{aligned}
$$

In this model, only the coefficient ω^1 is added to Z^0_{fj}. Considering this model, the change is always feasible.

Theorem 3.2. *Model (3.14) is always feasible.*

Proof. We know that each input has decreased by a percentage. Suppose the lowest percentage is equal to Υ. Also consider a feasible solution to problem (3.7) (take ω^* and ω^{t*} as a optimal solution). Set $\omega^t = \Upsilon \omega^{t*}$ and $\omega = \Upsilon \omega^*$. Clearly, a feasible solution has been found. □

4 Neumerical Result

In this section, a case study of Iranian gas companies is considered. Actually, the performance and changes of outputs of 29 gas companies are examined. The results of the proposed model is also compared with the findings of model (2.5).

4.1 An Application of Gas Companies

As Bansal et al. [7] (2018) mentioned, the gas industry is among the largest industries. Accordingly, the performance analysis of gas companies is an important aspect for nations and has a major role in their economy. In this section, the multi-period efficiency of 29 Iranian gas companies is assessed and also the changes of outputs for the perturbations of inputs are investigated using the models proposed. After reviewing the literature and consulting the authorities, performance measures have been considered as follows (see Figure 2):

- Two independent inputs (staff, cost)
- Three independent outputs (subscriptions, amount of gas delivered, number of installed branches)
- One link (income)

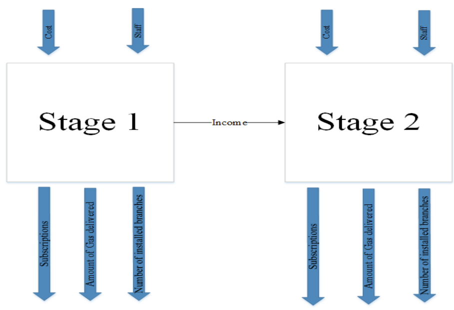

Figure 2: Graphic form of example

The data are related to years 2013-2014. The statistical information of these companies is give in Table 1. The two period efficiency of these companies are evaluated by using the proposed model (3.7). The results are shown in the column 3 of Table 2. Companies placed in Bushehr, Fars and Kermanshah are efficient. Also, the company located in Khuzestan has obtained the least efficiency score by 0.3375.

The findings of model (2.5) are presented in the column 2 of Table 2. As can be seen, in this model

Table 1: Statistical information of gas companies

	2013		2014		2013		
	X_1^1	X_2^1	X_1^2	X_2^2	Y_1^1	Y_2^1	Y_3^1
Min	126	1847.2	123	2446.6	1052	398.2	1890
Max	3462	4035879	3145	5892482	277425	23748	41577
avarage	853.6	877517.9	843.62	1290421.1	49109.1	5017.0	15617.8
S.D.	820.5	868343.3	773.1	1303805.6	53415.9	5205.6	11094.5

Table 1. Continue

	2014			2013-2014		
	Y_1^2	Y_2^2	Y_3^2	Z^0	Z^1	Z^3
Min	1165	449.5	1121	171430	312674	600479
Max	278533	25533	29546	14426760	20778626	20208906
avarage	45065.9	5635.8	13292.6	2343164.1	3534333.4	4917151.3
S.D.	52436.2	5799	8368.2	3203315.9	5148778.3	5482697.9

Table 2: The efficiency of two models

Province	Model (2.5)	Model (3.7)
East Azarbaijan	0.4913	0.5001
Western Azerbaijan	0.7528	0.7729
Ardabil	0.5658	0.5658
Esfahan	0.4016	0.4137
Ilam	0.6896	0.7421
Bushehr	1	1
Tehran and Alborz	0.8021	0.8076
Chahar Mahal Bakhtiari	0.3710	0.3956
Southern Khorasan	0.7863	0.8481
Khorasan Razavi	0.8555	0.8780
North Khorasan	0.6854	0.7118
Khuzestan	0.3301	0.3375
Zanjan	0.5363	0.5883
Semnan	0.4085	0.4085
Fars	1	1
Qazvin	0.5702	0.5931
Qom	0.6931	0.7298
Kurdistan	0.6745	0.6877
Kermanshah	1	1
Kerman	0.5914	0.6392
Kohgiluyeh and Boyer-Ahmad	0.3119	0.3880
Golestan	0.3753	0.4045
Gilan	0.6124	0.6627
Lorestan	0.7355	0.7922
Mazandaran	0.6549	0.7243
Markazi	0.7003	0.7035
Hormozgan	0.8685	0.8685
Hamadan	0.4390	0.4469
Yazd	0.5938	0.6139

Table 3: ΔY for model (2.6) and the proposed method in reducing inputs

Province	$\Delta y_1^{model(2.6)}$	$\Delta y_2^{model(2.6)}$	$\Delta y_3^{model(2.6)}$	$\Delta y_1^{Proposed}$	$\Delta y_2^{Proposed}$	$\Delta y_3^{Proposed}$
East Azarbaijan	-8693.03	-4199.40	0	-12469.23	-2495.29	0
Western Azerbaijan	-7505.38	-2863.30	0	-5145.91	-2598.50	0
Ardabil	-6181.09	0	0	-6181.09	0	0
Esfahan	0	-8846.74	0	-317.53	-9662.55	0
Ilam	0	0	-2550.56	0	0	-1926.52
Bushehr	0	-3326.75	0	0	-3326.75	0
Tehran and Alborz	-47686.15	-7987.35	0	-39181.33	-14611.38	0
Chahar Mahal Bakhtiari	-2756.32	-407.55	0	-3184.98	-116.08	0
Southern Khorasan	-4126.92	0	-963.39	-2899.35	0	-2189.43
Khorasan Razavi	-15488.02	-5735.30	0	-18277.16	-4125.08	0
North Khorasan	-1989.42	-1933.42	0	-2172.59	-1172.57	0
Khuzestan	0	-6904.80	0	0	-8158.19	0
Zanjan	-3330.35	-665.51	0	-4677.29	0	0
Semnan	-1489.22	-1633.02	0	-1489.22	-1633.02	0
Fars	-13985.30	-1806.77	-2018.58	-13991.70	-1807.13	-2030.55
Qazvin	-254.05	-3195.03	0	-790.13	-2203.11	0
Qom	-3474.21	-682.47	0	-4301.68	-404.18	0
Kurdistan	-5246.82	-1319.39	0	-6175.65	-992.03	0
Kermanshah	-10276.14	0	0	-8346.56	-476.90	-4591.91
Kerman	-5442.87	-1591.76	0	-8478.15	-77.97	0
Kohgiluyeh and Boyer-Ahmad	-853.83	0	-2539.01	0	0	-1198.26
Golestan	-6732.71	-430.80	0	-8937.71	0	0
Gilan	-11944.05	-1265.64	0	-14014.91	0	0
Lorestan	-6405.69	0	0	-7352.16	0	0
Mazandaran	-16056.82	0	-6255.12	-15693.87	0	-5907.13
Markazi	0	-2155.29	0	0	-3000.16	0
Hormozgan	0	-832.50	0	0	-832.50	0
Hamadan	-5921.89	-1151.41	0	-7565.11	-732.29	0
Yazd	0	-2412.92	0	0	-2613.78	0

also, three companies Bushehr, Fars and Kermanshah are efficient. But, there are differences between the efficiency scores obtained from two approaches. For example, see companies Ilam, Kohgiluyeeh and Boyer-Ahmad ad Mazandaran. Now, to estimate the changes of outputs for the perturbation of inputs, model (3.12) is calculated.

Actually, we reduce some inputs and calculate outputs. To estimate the outputs, 10 percent of inputs are subtracted, i.e., $\Delta X_{io} = -0.1X_{io}$. Note that different values can be subtracted from each company. We have subtracted a constant ratio from all the units here. The columns 5-7 of Table 3 show the obtained values $\Delta Y = \alpha - Y$ for the proposed model. It should be mentioned that all problems are feasible. For more explain, consider company located at Ardabil. For these changes of inputs, only y_1 has decreased by 6181.09. Furthermore, For Fars company, all outputs including, income, shareability and number of selec-

Table 4: ΔY for model (2.6) and the proposed method in increasing inputs

Province	$\Delta y_1^{model(2.6)}$	$\Delta y_2^{model(2.6)}$	$\Delta y_3^{model(2.6)}$	$\Delta y_1^{Proposed}$	$\Delta y_2^{Proposed}$	$\Delta y_3^{Proposed}$
East Azarbaijan	3129.86	5685.25	64051.43	5191.10	4273.43	53027.31
Western Azerbaijanarbi	5764.49	2177.24	41765.68	8444.27	1102.06	34323.61
Ardabil	0	4662.33	21095.59	1526.62	2922.06	19847.42
Esfahan	24781.89	0	70140.44	23793.75	0	53590.31
Ilam	5087.13	2338.70	1806.79	7388.87	2324.53	1565.49
Bushehr	9570.17	696.80	11670.63	9570.17	696.80	11670.63
Tehran and Alborz	52287.20	5417.75	358607.30	55596.01	5765.01	359762.70
Chahar Mahal Bakhtiari	0	1195.46	9357.02	1710.37	826.05	6462.27
Southern Khorasan	0	1735.25	6654.94	6385.77	712.60	1746.85
Khorasan Razavi	7841.67	7831.89	82411.11	14886.81	5469.69	64438.37
North Khorasan	4506.80	263.06	12759.33	5174.29	0	7992.12
Khuzestan	19582.27	0	31929.52	18147.67	0	19477.10
Zanjan	452.22	1560.51	9461.41	4256.08	1166.52	6122.35
Semnan	0	2029.52	15461.13	0	2527.90	14206.68
Fars	10229.63	1321.87	1490.73	10229.63	1321.87	1490.73
Qazvin	6252.13	0	24280.80	6430	0	18651.49
Qom	2218.03	1063.31	16387.89	3931.15	541.18	11548.53
Kurdistan	3749.22	1820.88	20825.25	4769.60	721.15	12901.43
Kermanshah	1868.69	4454.72	13173.03	4499.81	2381.07	8648.90
Kerman	2090.37	2562.85	32973.25	8086.85	1728.71	26276.66
Kohgiluyeh and Boyer-Ahmad	2102.81	2041.03	1312.43	7401.95	2208.46	645.48
Golestan	0	2632	16020.37	3686.83	2093.83	10354.11
Gilan	2521.72	4234.79	25699.90	12402.76	2751.98	14575.14
Lorestan	0	3217.42	14017.02	6883.78	2527.85	7663.82
Mazandaran	2286.84	8599.17	20048.41	15693.85	7513.41	7291.55
Markazi	12036.40	0	16129.17	9861.15	0	10857.01
Hormozgan	16959.97	289.52	17159.84	16959.97	289.52	17159.84
Hamadan	1472.27	2360.38	24557.97	2706.78	1299.59	17042.01
Yazd	6701.96	0	19686.39	6691.18	0	14987.84

tions have decreased for the reductions of staff and cost.

To compare the results of the proposed inverse dynamic FDH model with the available approaches, the models Foladi et al. [17] is solved. The consequences are denoted in column 2-4 of Tables 3. As can be found, there are considerable differences between gains resulted from two models. For instance, take Esfahan into account. By using model (2.6), only subscriptions has decreased by -8846.74,; but two outputs, amount of gas delivered, and number of selections, have decreased by -317.53 and -9662.55 using the proposed inverse dynamic FDH model.

Now we add some values to the inputs and calculate outputs. Actually 10 percents of inputs are added to inputs, i.e., $\Delta X_{io} = 0.1X_{io}$. The aim is to find outputs for these changes. Table 4 shows the obtained values $\Delta Y = \alpha - Y$ for model (2.6) and the proposed method. In this case, most of the solutions are non-zero and this helps the decision-maker to make decisions.

As can be seen, there are substantial differences, between the findings achieved from two approaches. For more illustration, consider the company situated in Ardabil. For these increase of inputs, all outputs, subscriptions, shareability and number of selections, increase using the introduced model while applying model (2.6) shareability and number of selections are increased by 4662.33 and 21095.59, respectively. Similarly, outcomes of other companies can be analyzed and compared. To sum up, the approaches provided are beneficial to assess the multi-period efficiency and to address the changes of outputs while the convexity assumption is violated.

5 Conclusion

In the presence of inter-relations between of a span of time and also the violation of convexity property, developing a dynamic model and its inverse problem is significant for making better decisions and analyzing investment problems. In this research, a dynamic FDH model under the non-convex technology and also an inverse dynamic FDH model have been proposed. The main models are mixed-integer nonlinear programing problems, thus, several methods, a linear problem and a mixed-integer linear model, have been presented to solve them. An application of gas companies have been provided to clarify the techniques developed. The results show the models presented are suitable to measure the performance of dynamic processes and the changes of outputs when the convexity assumption is violated.

This study includes the overall performance of dynamic processes. Thus, a next study can be conducted to estimate outputs of multi-period systems when the overall and multi-period efficiencies are maintained and the inputs are perturbed. The extension of the planned technique for situation that undesirable outputs are presented is also an interesting topic for future research. The investigation of the changes of performance measures of dynamic network systems under the non-convex technology can, moreover, be considered as further exploration.

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Farzaneh Asadi received her Ph.D. degree in Islamic Azad University of Lahijan branch in 2023 in industrial engineering. Her research interests include performance analysis, Data Envelopment Analysis (DEA) and simulation.

Sohrab Kordrostami is a full professor in applied mathematics (operations research field) department in Islamic Azad University, Lahijan branch. He completed his Ph.D. degree in Islamic Azad University of Tehran, Iran. His research interests include performance management with special emphasis on the quantitative methods of performance measurement, and especially those based on the broad set of methods known as Data Envelopment Analysis, (DEA). Kordrostami's papers have appeared in a wide series of journals such as Applied mathematics and computation, Journal of the operations research society of Japan, Journal of Applied mathematics, International journal of advanced manufacturing technology, International journal of production economics, Optimization, International Journal of Mathematics in Operational research, Journal global optimization, etc.

Alireza Amirteimoori is a full professor in applied mathematics operations research group in Islamic Azad University in Iran. He completed his Ph.D degree in Islamic Azad University in Tehran, Iran. His research interests lie in the broad area of performance management with special emphasis on the quantitative methods of performance measurement, and especially those based on the broad set of methods known as Data Envelopment Analysis, (DEA). Amirteimoori's papers appear in journals such as Applied mathematics and computation, Journal of the operations research society of Japan, Journal of Applied mathematics, RAIRO-Operations research, International journal of advanced manufacturing technology, International Journal of Production Economics, Optimization, Expert Systems with Applications, Central European Journal of Operational Research, International Journal of Mathematics in Operational Research, Decision Support Systems, Journal of Global Optimization and etc.

Morteza Bazrafshan is assistant professor of industrial engineering, faculty of Islamic Azad University, Lahijan branch. His research interests include BPM, simulation and also knowledge management. He received his Ph.D from Tarbiat Modares university of Tehran, Iran, 2015