Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 13, No. 3, 2021 Article ID IJIM-1351, 10 pages
Research Article

Science and Research Branch (IAU)

Multiplication-Like Modules

M. Ahmadi *, J. Moghaderi †‡

Received Date: 2019-07-11 Revised Date: 2021-04-09 Accepted Date: 2021-08-22

Abstract

In this paper we introduce the concept of multiplication-like modules and we obtain some related results. We show that an R-module M is multiplication-like if and only if for each ideal I of R, $I = (IM :_R M)$. We prove that any multiplication-like module is faithful and r-multiplication. So we get that any flat and multiplication-like module is faithfully flat.

Keywords : Modules; Free modules, Flat modules, Multiplication modules; Multiplication-like modules.

1 Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary. Let M be an R-module. For a submodule N of M, let $(N :_R M)$ denote the set of all elements r in R such that $rM \subseteq N$. The annihilator of M, denoted by $Ann_R(M)$, is $(0 :_R M)$. A proper submodule N of M is called prime (*primary*) if $rx \in N$, for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r \in (N :_R M)$ ($r^n \in (N :_R M)$, for some $n \in \mathbb{N}$). We denote the set of prime submodules of M by Spec(M). For a submodule N of M, $V(N)$ denotes $\{P \in Spec(M) | N \subseteq P\}$, and $rad(N) = \bigcap V(N)$, is called the radical of N and was introduced in [9], [10] and [11]. A proper submodule N of M is said to be primary-like if $rm \in N$, for $r \in R$ and $m \in M$, implies that either $m \in rad(N)$ or $r \in (N :_R M)$ (see [7]).
It is said that M is a multiplication module, if for each submodule N of M, there is an ideal I of R, such that $N = IM$. Equivalently, M is a multiplication module if and only if for each submodule N of M, we have $N = (N :_R M)M$ [5] and [6].
In [3] the notion of a comultiplication module was introduced as a dual of the concept of a multiplication module. An R-module M is called comultiplication, if for every submodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$. For example, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a comultiplication module since all of its proper submodules are of the form $(0 :_M P^i\mathbb{Z})$ for $i = 0, 1, \ldots$. It is clear that M is comultiplication if and only if for every submodule N of M, we have $N = (0 :_M (0 :_R N))$. An R-module M is said to be strong comultiplication, if for every submodule N of M there is exactly one ideal I of R with $N = (0 :_M I)$ (*see*[4]).
M is said to be an r-multiplication module, when $IM \neq M$ for every proper ideal I of R (see [12]).
A non-zero submodule N of M is said to be

*Department of Mathematics, University of Hormozgan, Bandar Abbas, Hormozgan, Iran.
†Corresponding author. j.moghaderi@hormozgan.ac.ir, Tel:+98(913)1405183.
‡Department of Mathematics, University of Hormozgan, Bandar Abbas, Hormozgan, Iran.

second, if for each $a \in R$, the homomorphism $N \overrightarrow{a} N$ is either surjective or zero [15]. An R-module M is said to be distributive, if the lattice of its submodule is distributive, i.e. $(X+Y) \cap Z = (X \cap Z) + (Y \cap Z)$, for any of its submodules X, Y and Z. A non-zero module M over a ring R is said to be prime, if the annihilator of M is the same as the annihilator of N for every non-zero submodule N of M (see [2]).

In this article, we introduce multiplication-like module and obtain some basic results and characterizations.

2 Multiplication-Like Modules

Definition 2.1. *An R-module M is said multiplication-like, if for any ideal I of R, there exists a submodule N of M such that $I = (N :_R M)$.*

Example 2.1. *(i) Every vector space is multiplication-like.*
(ii) $R[X]$ is a multiplication-like R-module.
(iii) $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module are not multiplication-like.

It is clear that every free module is multiplication-like; but $M = \mathbb{Z} \oplus \mathbb{Z}_2$ is a multiplication-like $\mathbb{Z}$-module, which is not free.

Lemma 2.1. *An R-module M is multiplication-like if and only if for each ideal I of R, $I = (IM :_R M)$.*

Proof. The sufficiency is clear. Conversely, suppose that M is a multiplication-like. Then there exists a submodule N of M such that $I = (N :_R M)$. So we have $IM \subseteq N$. Hence $I \subseteq (IM :_R M) \subseteq (N :_R M) = I$. This implies that $I = (IM :_R M)$ as desired. □

Proposition 2.1. *Let M be an R-module. Then M is multiplication-like if and only if for every ideal I of R, there exist submodules N_i of M ($i \in J$), such that $I = \sum_{i \in J}(N_i :_R M) = (\sum_{i\in J} N_i :_R M)$.*

Proof. Let M be multiplication-like and let I be an ideal of R. Then $I = (IM :_R M)$. On the other hand, $I = \sum_{r_i \in I} Rr_i$ and for each $r_i \in I$, $Rr_i = (r_iM :_R M)$. So we have $I = \sum_{r_i\in I} Rr_i = \sum_{r_i \in I}(r_iM :_R M) = (\sum_{r_i\in I} r_iM :_R M)$.

Hence the proof is completed. □

Theorem 2.1. *Let M be an R-module. Then the following statements are equivalent.*

(i) M is multiplication-like.

(ii) For every ideal I of R and each submodule N of M with $I \subset (N :_R M)$, there exists a submodule L of M such that $L \subset N$ and $I = (L :_R M)$.

(iii) For every ideal I of R and each submodule N of M with $I \subset (N :_R M)$, there exists a submodule L of M such that $L \subset N$ and $I \subseteq (L :_R M)$.

Proof. (i)$\Longrightarrow$ (ii) Let $I \subset (N :_R M)$. Since M is multiplication-like, $I = (IM :_R M)$. Put $L = IM \cap N$. Since $I = (IM :_R M) \subset (N :_R M)$, hence $L \subset N$ and we have
$(L :_R M) = (IM \cap N :_R M) = (IM :_R M) \cap (N :_R M) = I$.

$(ii) \Longrightarrow (iii)$ It is obvious.

$(iii) \Longrightarrow (i)$ Let I be an ideal of R and put
$H = \{L : L \textit{ is a submodule of } M \textit{ and } I \subseteq (L :_R M)\}$.

Clearly H is a non-empty set, so by Zorn's Lemma, H has a minimal member like K and so $I \subseteq (K :_R M)$. Assume that $I \neq (K :_R M)$. Then by part (iii), there exists a submodule U of M with $U \subset K$ and $I \subseteq (U :_R M)$. But this is a contradiction by the choice of K. Thus we have $I = (K :_R M)$. This shows that M is multiplication-like. □

Example 2.2. *Let $M = \mathbb{Z}_6$ and $R = \mathbb{Z}_6$. Then M is multiplication-like but, $\bar{2}\mathbb{Z}_6$ and $\bar{3}\mathbb{Z}_6$ are not multiplication-like modules.*

Let M be a torsion-free R-module. Clearly, every non-zero cyclic submodule of M is a multiplication-like R-module. But, if every non-zero cyclic submodule of an R-module M is multiplication-like, then M is not necessarily multiplication-like. As the following example Shows:

Example 2.3. *Let $M = \mathbb{Q}$ and $R = \mathbb{Z}$. Then every non-zero cyclic submodule of M is free and so multiplication-like; but $\mathbb{Q}$ is not a multiplication-like R-module.*

Theorem 2.2. *Let R be a comultiplication ring and M be a faithful R-module. Then M is a multiplication-like R-module.*

Proof. Assume that I is a proper ideal of R and $rM \subseteq IM$, for $r \in R$. Then $rAnn_R(I)M = 0$. Since M is faithful and R is a comultiplication ring, we have $r \in I$. Thus M is a multiplication-like module. □

It is straightforward to prove that R is a comultiplication ring if and only if $(I :_R J) = (Ann_R(J) :_R Ann_R(I))$, for each ideals I and J of R. So by Theorem 2.2, we have:

Corollary 2.1. *Let R be a ring such that for every ideal I and J of R, $(I :_R J) = (Ann_R(J) :_R Ann_R(I))$. Then every faithful R-module is multiplication-like module.*

By Example 3.8 [3] and Theorem 2.2, we obtain the following corollary.

Corollary 2.2. *Let R be a semi-simple ring. Then every faithful R-module is multiplication-like.*

Corollary 2.3. *Let M be a strong comultiplication module which has a maximal submodule over a reduced ring R (recall that a reduced ring is one with no nilpotents). Then M is a multiplication-like R-module.*

Proof. As M is strong comultiplicatin, then $Ann_R(M) = 0$. Now it follows easily from Corollary 4.5 [12] and Corollary 2.2. □

By Proposition 4.3 [12] and Theorem 2.2, we get the following corollary.

Corollary 2.4. *Let M be a non-zero multiplication and strong comultiplication R-module. Then M is a multiplication-like R-module.*

Clearly, if M' is a multiplication-like R-module and $\rho : M \longrightarrow M'$ is an R-epimorphism, then M is a multiplication-like module.

Also, let M be an R-module and N be a submodule of M. If $\frac{M}{N}$ is a multiplication-like R-module, then we can conclude that M is a multiplication-like R-module. But, the converse is not true in general, as the following example shows:

Example 2.4. *$\mathbb{Z}$ as $\mathbb{Z}$-module is a multiplication-like R-module, but for submodule $n\mathbb{Z}$, $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is not a multiplication-like $\mathbb{Z}$-module.*

Lemma 2.2. *Let M be a multiplication-like R-module.*

(i) If for submodule N of M, $N \subseteq IM$ for each non-zero ideal I of R and $\frac{M}{N}$ is faithful, then $\frac{M}{N}$ is a multiplication-like R-module.

(ii) If M' is a faithful R-module, $\rho : M \longrightarrow M'$ is an epimorphism and for any non-zero ideal I of R, $ker(\rho) \subseteq IM$, then M' is a multiplication-like R-module.

Proof. We have $I(\frac{M}{N}) = \frac{IM}{N}$. Hence $\frac{M}{N}$ is a multiplication-like R-module.

(ii) This is clear by part (i). □

Corollary 2.5. *Let N be a faithful second submodule of a multiplication-like R-module M. Then for every non-zero ideal I of R, there is a submodule L of $\frac{M}{N}$ such that $I = (L :_R \frac{M}{N})$.*

Proof. Since N is second and faithful, we have that $IN = N$, for each non-zero ideal I of R. So $N \subseteq IM$. By Lemma 2.2 (i), the proof is complete. □

Proposition 2.2. *Let M be a multiplication-like R-module and I be an ideal of R. Then $\frac{M}{IM}$ is a multiplication-like $\frac{R}{I}$-module.*

Proof. It is enough to prove that for each ideal J of R containing I, $(\frac{J}{I}(\frac{M}{IM}) :_{\frac{R}{I}} \frac{M}{IM}) \subseteq \frac{J}{I}$. Since M is multiplication-like, we have $J = (JM :_R M)$. If $(r + I) \in (\frac{J}{I}(\frac{M}{IM}) :_{\frac{R}{I}} \frac{M}{IM})$, then for every $x \in M$, $(r + I)(x + IM) \in \frac{J}{I}(\frac{M}{IM}) = \frac{JM}{IM}$.

This implies that $rx \in JM$. So we have that $r \in J$. It follows $r + I \in \frac{J}{I}$. □

Corollary 2.6. *Let M be a multiplication-like R-module. Then for any ideal I of R such that $I \subseteq Ann_R(M)$, M is a multiplication-like $\frac{R}{I}$-module.*

Remark 2.1. The converse of previous corollary is not true in general. For example, $\mathbb{Z}_n$ is a multiplication-like $\mathbb{Z}_n$-module, while $\mathbb{Z}_n$ as $\mathbb{Z}$-module is not multiplication-like.

The following proposition shows the behavior of modules that are multiplication-like module over localizations.

Proposition 2.3. *Let M be an R-module and S be a multiplicative closed subset of R.*

(i) If M is a finitely generated multiplication-like R-module, then M_S is a multiplication-like R_S-module.

(ii) If M_S is a multiplication-like R_S-module and for any ideal I of R and any $r \notin I$, $S \cap (I :_R r) = \emptyset$, then M is a multiplication-like R-module.

Proof. (i) Since M is a multiplication-like module, $I = (IM :_R M)$ for any ideal I of R. So we have $I_S = (I_S M_S :_{R_S} M_S)$, as M is finitely generated.

(ii) Let I be an ideal of R and $r \in (IM :_R M)$. So $\frac{r}{1} M_S \subseteq I_S M_S$. Since M_S is a multiplication-like R_S-module, $\frac{r}{1} \in I_S$. So there exists $u \in S$ such that $ur \in I$. If $r \notin I$, then $u \in S \cap (I :_R r)$ which is a contradiction. Hence $r \in I$. □

We now give an example to show that in Proposition 2.3 (ii), the condition is necessary.

Example 2.5. *Let $M = \mathbb{Q}$, $R = \mathbb{Z}$ and $S = \mathbb{Z} - \{0\}$. Then M_S is a vector space on field $R_S = \mathbb{Q}$. So M_S is a multiplication-like R_S-module; but M is not a multiplication-like R-module.*

Corollary 2.7. *Let (R, m) be a local ring and M be a finitely generated R-module. Then M is a multiplication-like R-module if and only if M_m is a multiplication-like R_m-module.*

Proposition 2.4. *Let M and N be R-modules and $M \otimes_R N$ be a multiplication-like module. Then M and N are multiplication-like R-modules.*

Proof. Let I be an ideal of R and $r \in (IM :_R M)$. Then $rM \otimes_R N \subseteq IM \otimes_R N$. This implies that $r(M \otimes_R N) \subseteq I(M \otimes_R N)$, so that $r \in (I(M \otimes_R N) :_R M \otimes_R N) = I$. Hence M and similarly N are multiplication-like R-modules. □

It is clear that, if M is a multiplication-like R-module and N is a free R-module, then the converse of above proposition is true.

Proposition 2.5. *Let M_1 and M_2 be two R-modules which M_1 or M_2 is multiplication-like R-module. Then $M_1 \oplus M_2$ is a multiplication-like R-module.*

Proof. Let I be an ideal of R such that $r(M_1 \oplus M_2) \subseteq I(M_1 \oplus M_2)$ and M_1 be a multiplication-like R-module. Then $I = (IM_1 :_R M_1)$ which implies that $r \in I$. Therefore, $M_1 \oplus M_2$ is a multiplication-like R-module. □

The converse of above lemma is not true in general.

Example 2.6. *Consider $M = \mathbb{Z}_6 = (\bar{2}) \oplus (\bar{3})$ and $R = \mathbb{Z}_6$. Then M is a multiplication-like R-module. But it is easy to see that $N = \bar{2}\mathbb{Z}_6$ and $L = \bar{3}\mathbb{Z}_6$ are not multiplication-like module.*

Corollary 2.8. *Let M_i $(i \in I)$ be R-modules such that for some i, M_i is a multiplication-like R-module. Then $\oplus_{i \in I} M_i$ is a multiplication-like R-module.*

Lemma 2.3. *Let R be a ring and M be an R-module such that $I \neq (IM :_R M)$, for some ideal I. Then there exists an ideal K and $r \notin K$ such that $I \subseteq K$ and $(K :_R r)$ is maximal ideal of ring R.*

Proof. By hypothesis there exists an element r in R such that $r \in (IM :_R M)$ but $r \notin I$. Let S denote the collection of ideals L of R such that $I \subseteq L$ but $r \notin L$. Clearly S is non-empty and so by Zorn's Lemma, S contains a maximal member like K.
Thus $I \subseteq K$ and $r \notin K$. Let s be an element of R such that $s \notin (K :_R r)$. It follows that K is a

proper subset of $K+Rsr$ and hence $K+Rsr \notin S$. Thus $r \in K+Rsr$. Therefore, there exists $b \in R$ and $u \in K$ such that $r = u+bsr$ and so $(1-bs)r \in K$. It follows that $(K :_R r)$ is a maximal ideal of R. □

Theorem 2.3. *Let R be a ring.Then the following statements are equivalent for R-module M.*

(i) M is a multiplication-like R-module.

(ii) $I = (IM :_R M)$, for every ideal I of R.

(iii) Given ideals I, J of R, $IM \subseteq JM$ implies that $I \subseteq J$.

(iv) Given any ideal I of R and $r \in R$, $rM \subseteq IM$ implies that $r \in I$.

(v) Given any ideal I of R and $r \in R$, $rM \subseteq IM$ implies that $(I :_R r)$ is not a maximal ideal.

Proof. $(i) \Longleftrightarrow (ii)$ By Lemma 2.1.

$(ii) \Longrightarrow (iii)$ Let $IM \subseteq JM$. Then $(IM :_R M) \subseteq (JM :_R M)$. By (ii), $I \subseteq J$.

$(iii) \Longrightarrow (ii)$ We know that always $IM = (IM :_R M)M$. By (iii), $I = (IM :_R M)$.

$(iii) \Longleftrightarrow (iv)$ It is obvious.

$(iv) \Longrightarrow (v)$ Let $rM \subseteq IM$. By (iv), $r \in I$, and hence $(I :_R r) = R$. Therefore, $(I :_R r)$ is not a maximal.

$(v) \Longrightarrow (iv)$ Let $rM \subseteq IM$ such that $r \notin I$. By Lemma 2.3, there exists an ideal K of R such that $I \subseteq K$, $r \notin K$ and $(K :_R r)$ is maximal ideal. But this is a contradiction. □

3 Properties of Multiplication-Like Modules

In this section we shall show that multiplication-like modules have some interesting properties.

Theorem 3.1. *Let M be a multiplication-like R-module. Then*

(i) M is a faithful module.

(ii) M is an r-multiplication module.

(iii) The set of all prime submodules of an R-module M is non-empty ($Spec_R(M) \neq \emptyset$).

(iv) For every ideal I of R, $Ann_R(I) = Ann_R(IM)$.

(v)
$Z(R) = \{a \in R : \exists\ non-zero\ submodule\ N\ s.t\ (N :_R M) \neq 0,\ a(N :_R M) = 0\}$
(here $Z(R)$ denotes the set of zero divisor of R).

Proof. (i) By Lemma 2.1, $0 = (0M :_R M) = Ann_R(M)$.

(ii) If there exists a proper ideal I of R such that $IM = M$, then $I = (IM :_R M) = R$. This is a contradiction and the proof is completed.

(iii) Let $m \in Max(R)$. By part (ii), $m = (mM :_R M)$. This shows that mM is a prime submodule of M.

(iv) It is enough to prove that $Ann_R(IM) \subseteq Ann_R(I)$. Now let $r \in Ann_R(IM)$, then $rIM = 0$. Now by using part (i), we have $Ir = 0$.

(v) Let $a \in Z(R)$. Then there exists $0 \neq b \in R$ such that $ab = 0$. It implies that $a(bM :_R M) = 0$, because M ia multiplication-like module. The converse is clear. □

The following examples show that Converse parts of the previous theorem do not hold in general.

Example 3.1. *Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$. It is clear that M satisfies in parts (i), (iii), (iv) and (v), but M is not a multiplication-like R-module.*

Example 3.2. *Let $R = \mathbb{Z}$ and $M = \bigoplus_{p \in max(R)} \mathbb{Z}_p$. Clearly M is an r-multiplication and $4\mathbb{Z} \neq (4\mathbb{Z}M :_R M) = 2\mathbb{Z}$. Therefore M is not a multiplication-like module.*

Lemma 3.1. *Let M be an R-module. Then M is multiplication-like and second module if and only if M is a vector space.*

Proof. It is sufficient to show that R is a field. For each non-unit such as $r \in R$, $rM \neq M$, as M is multiplication-like module. So $r = 0$, beacause M is second and faithful. The set of non-units is zero ideal. Therefore R is a field and M is a vector space. □

Lemma 3.2. *Let M be an r-multiplication module which every proper submodule of it is multiplication-like R-module. Then M is a multiplication-like R-module.*

Proof. Let I be an ideal of R. By Lemma 2.1, $I = (I^2M :_R IM)$. Let $rM \subseteq IM$. It follows that $IrM \subseteq I^2M$. So we have $r \in I$. Therefore, M is a multiplication-like R-module. □

Corollary 3.1. *Let M be a finitely generated R-module that every submodule of it is multiplication-like R-module. Then M is a multiplication-like module.*

Example 2.2, show that if R-module M is multiplication-like module, then every non-zero submodule of M need not necessarily be multiplication-like. By Theorem 3.1 (ii) and Proposition 2.11.24 [13], we get the following lemma.

Lemma 3.3. *Let M be a flat and multiplication-like R-module. Then M is a faithfully flat.*

If M is a multiplication (comultiplication) module, then it is not concluded that M is a multiplication-like and conversely.

Example 3.3.
(i) $\mathbb{Z}_n$ as $\mathbb{Z}$-module is a multiplication module, but it is not multiplication-like.

(ii) $\mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$-module is multiplication-like, but is not multiplication.

(iii) $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module is a comultiplication, but is not multiplication-like.

(iv) $\mathbb{Z}$ as a $\mathbb{Z}$-module is multiplication-like module, but is not a comultiplicatiom module, by Example 3.9 [3].

Remark 3.1. By Example 2.2, we can see that if R-module M is multiplication-like, then every submodule of M is not r-multiplication.

Proposition 3.1. *Let R be a Noetherian domain which is not a field and M be a multiplication-like R-module. Then every non-zero maximal submodule of M, is r-multiplication.*

Proof. Suppose that N is a non-zero maximal submodule of M. If N is not an r-multiplication, then there exists a proper ideal I of R such that $IN = N$.
Since N is a maximal submodule and M is multiplication-like, we must have $N = IM$ and $I = I^2 = (N :_R M)$. Hence there exists $a \in I$ such that $(1 - a)I = 0$. Since R is domain, we have $I = R$ or $I = 0$, which is a contradiction. □

Proposition 3.2. *Let R be a local Noetherian ring that is not a field and M be a multiplication-like R-module. Then every non-zero maximal submodule of M is r-multiplication.*

Proof. Suppose that N is a non-zero maximal submodule of M. If N is not an r-multiplication, then there exists a proper ideal I of R such that $IN = N$. Since N is a maximal submodule and M is multiplication-like, we have $N = IM$ and $I = I^2 = (N :_R M)$. By Nakayama lemma, $I = 0$, which is a contradiction. Hence N is an r-multiplication. □

Lemma 3.4. *Let M be a multiplication R-module. Then M is a multiplication-like if and only if M is finitely generated and faithful.*

Proof. Let M be a multiplication-like R-module. By Theorem 3.1, M is faithful and for each proper ideal I of R, $IM \neq M$. It follows that M is finitely generated. Conversely, let I be a proper ideal of R. Note that $IM = (IM :_R M)M$. Since M is multiplication, faithful and finitely generated, $I = (IM :_R M)$. Therefore, M is multiplication-like. □

Lemma 3.5. *Let M be a faithful multiplication R-module. Then M is an r-multiplication if and only if M is a multiplication-like.*

Proof. Let M be a multiplication-like R-module. By Theorem 3.1, M is r-multiplication. Conversely, let I be an ideal of R. Note that $IM =$

$(IM :_R M)M$. Since M is faithful multiplication and r-multiplication, so M is finitely generated. Now by Lemma 3.4, M is multiplication-like. □

Corollary 3.2. *If M is a multiplication and multiplication-like R-module, then $|Spec_R(M)| = |Spec(R)|$.*

Corollary 3.3. *Let M be a multiplication and multiplication-like R-module. Then for every I ideal of R, there exists an unique R-submodule K of M such that $I = (K :_R M)$.*

Corollary 3.4. *Let M be a multiplication-like R-module. Then M is multiplication if and only if for every I of R, there exists an unique submodule N of M such that $I = (N :_R M)$.*

Lemma 3.6. *Assume that M is a comultiplication and multiplication-like R-module. Then M is a strong comultiplication.*

Proof. Suppose N be a submodule of M. If there exist ideals I and J such that $N = (0 :_M I)$ and $N = (0 :_M J)$, then $IM = JM$, by Proposition 4.1 [12]. Now by Theorem 2.3, $I = J$. □

Proposition 3.3. *If M is a comultiplication and multiplication-like R-module, then for every submodule N of M, there exists an ideal I of R such that $(N :_R M) = Ann_R(I)$.*

Proof. Let N be a submodule of M. Since M is a comultiplication R-module, there exists an ideal I of R such that $N = (0 :_M I)$ and hence
$(N :_R M) = ((0 :_M I) :_R M) = Ann_R(IM) = Ann_R(I)$. □

Lemma 3.7. *Let M be a multiplication-like R-module. Then for every ideal I and J of R*

(i) $(IJM :_R M) = (IM :_R M)(JM :_R M)$.

(ii) $(IM + JM :_R M) = (IM :_R M) + (JM :_R M)$.

(iii) $((I \cap J)M :_R M) = (IM :_R M) \cap (JM :_R M)$.

Proof. This follows from Lemma 2.1. □

Remark 3.2. Lemma 3.7 shows properties which hold for multiplication-like modules (for ideals of ring), but part (ii) is not valid in general for submodules of module.

Consider $M = \mathbb{Z}[X] \oplus \mathbb{Z}[X]$ as $R = \mathbb{Z}[X]$-module. Then $((X) \oplus \mathbb{Z}[X] :_R M) + (\mathbb{Z}[X] \oplus (X) :_R M) \subset ((X) \oplus \mathbb{Z}[X] + \mathbb{Z}[X] \oplus (X) :_R M) = R$.

Proposition 3.4. *Let M be a Noetherian multiplication-like R-module. Then R is Noetherian.*

Proof. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ be an ascending chain of ideals of R. It follows that $I_1M \subseteq I_2M \subseteq I_3M \subseteq ...$ is an ascending chain of submodules of M. So there exists a positive integer k such that $I_kM = I_{k+1}M = ...$, and hence $I_k = I_{k+1} = ...$, as M is multiplication-like. □

Proposition 3.5. *Let M be an Artinian multiplication-like R-module. Then R is Artinian.*

The following example shows that if M is multiplication-like over a Noetherian (Artinian) ring, then it is not necessarily to be a Noetherin (Artinian) module.

Example 3.4. *Let V be a vector space over a field F. It follows that V is multiplication-like and F is Artinian and Noetherian. But if V has an infinite dimension, then V is not Artinian and Noetherian.*

Proposition 3.6. *Let M be a faithful module over a Noetherian ring R such that for every primary ideal q of R, $q = (qM :_R M)$. Then M is multiplication-like.*

Proof. Let I be an ideal of R and let $I = \bigcap_{i=1}^n q_i$ be a reduced primary decomposition of I in R, where q_i are primary. It follows that
$I \subseteq (IM :_R M) = ((\bigcap_{i=1}^n q_i)M :_R M) \subseteq \bigcap_{i=1}^n (q_iM :_R M) = \bigcap_{i=1}^n q_i = I$. □

Lemma 3.8. *If R-module M is a multiplication-like R-module and each submodule of M has a reduced primary decomposition, then every ideal of R has a reduced primary decomposition.*

Proof. Let I be an ideal of R. Since M is multiplication-like it follows that $I = (IM :_R M)$. By hypothesis, $IM = \bigcap_{i=1}^n q_i$, when q_i are P_i-primary. Hence
$I = (IM :_R M) = (\bigcap_{i=1}^n q_i :_R M) = \bigcap_{i=1}^n (q_i :_R$

M).
It follows that I has reduced primary decomposition in R. □

Recall that an integral domain R is a valution ring if and only if the ideals of R are totally ordered by inclusion.

Lemma 3.9. *Let M be a multiplication-like R-module and R be an integral domain. Then for any submodules N, L of M, $(N :_R M) \subseteq (L :_R M)$ or $(L :_R M) \subseteq (N :_R M)$ if and only if R is valuation ring.*

Proof. Obvious □

Proposition 3.7. *If for some $P \in Max(R)$, PM is a multiplication-like R-module, then M is a multiplication-like R-module.*

Proof. If $PM = M$, then the proof is complete. Now assume that $PM \neq M$ and let I be any ideal of R and $r \in (IM :_R M)$.
It implies that $rPM \subseteq PIM$. Hence $r \in I$. □

Remark 3.3. Example 2.6 shows that the converse of Proposition 3.7 is not true, in general.

Anderson and Fuller [1] called the submodule N a pure submodule, if $IN = N \cap IM$ for every ideal I of R.

Proposition 3.8. *Let N be a pure submodule of an R-module M. If N is multiplication-like, then M is a multiplication-like module.*

Proof. Let I be an ideal of R. Then $I = (IN :_R N)$. Assume that $rM \subseteq IM$. Since N is pure, we have $rN \subseteq IN$, and hence $r \in I$. Therefore, M is multiplication-like. □

Recall that a ring R is discrete valuation ring (DVR) if and only if it is valuation and Noetherian ring. If R is a DVR, then every non-zero ideal I of R is uniquely of the type $I = m^n$ (for some $n \in \mathbb{N}$), where m is the unique maximal ideal R.

Lemma 3.10. *Let M be a faithful finitely generated module over discrete valuation ring R. Then M is a multiplication-like.*

Proof. Let I be an ideal of R and m be the unique maximal ideal. Then there exists $n \in \mathbb{N}$ such that $I = m^n$. We have $m^n \subseteq (m^n M :_R M) \subseteq m^{n-1}$. Hence $m^n = (m^n M :_R M)$ or $(m^n M :_R M) = m^{n-1}$. If $(m^n M :_R M) = m^{n-1}$, then $m^n M = m^{n-1} M$. Hence by Nakayama lemma, $m = 0$ which is a contradiction. So $(m^n M :_R M) = m^n$. □

A Dedekind domain (D.d) is a Noetherian integrally closed domain in which every non-zero primes ideal is maximal.

Corollary 3.5. *Let M be a faithful finitely generated module over D.d R. Then for every non-zero prime ideal P of R, M_P is multiplication-like R_P-module.*

Proposition 3.9. *Let M be a faithful finitely generated R-module. Then for every radical ideal like I, $I = (IM :_R M)$.*

Proof. Let I be a radical ideal of R. Then $I = \sqrt{I} = \bigcap_{P \in V(I)} P$. For each $P \in V(I)$, $(PM :_R M) = P$, as M is a faithful finitely generated module. Thus

$I \subseteq (IM :_R M) = ((\bigcap_{P \in V(I)} P)M :_R M) \subseteq \bigcap_{P \in V(I)} (PM :_R M) = \bigcap_{P \in V(I)} P = I$. □

Lemma 3.11. *Let N be an R-submodule of M. If N is a multiplication-like such that for every ideal I of R, IN is primary-like submodule and $rad(IN) \subset N$, then M is a multiplication-like module.*

Proof. Let I be an ideal of R. Since N is a multiplication-like, $I = (IN :_R N)$. We show that $IM \subseteq IN$. It follows to show that $I \subseteq (IN :_R M)$. Let $r \in I$. Since $rad_R(IN) \subset N$, we can find an element $n \in N - rad_R(IN)$. Then $rn \in IN$. Hence $r \in (IN :_R M)$, as IN is primary-like. Therefore, M is multiplication-like. □

Lemma 3.12. *Let M be a distributive multiplication-like R-module and for any two submodules N and L of M, $(N :_R M) + (L :_R M) = (N + L :_R M)$. Then R is a distributive ring.*

Proof. Let A, B and C be ideals of R. Since M is multiplication-like, there exist submodules N, K and L of M such that $A = (N :_R M), B = (K :_R M)$ and $C = (L :_R M)$. Then

$(A+B) \cap C = ((N :_R M) + (K :_R M)) \cap (L :_R M) = (N + K :_R M) \cap (L :_R M)$
$= ((N+K) \cap L :_R M) = ((N \cap L) + (K \cap L) :_R M) = (N \cap L :_R M) + (K \cap L :_R M) = (N :_R M \cap (L :_R M) + (K :_R M) \cap (L :_R M) = A \cap C + B \cap C.$ □

The following example shows that in above theorem, the conditions, M is distributive and for any two submodule N and L of M, $(N :_R M) + (L :_R M) = (N + L :_R M)$ can not be omitted.

Example 3.5. *Let $M = \mathbb{Z}[X] \oplus \mathbb{Z}[X]$, $R = \mathbb{Z}[X]$, $N = (X) \oplus \mathbb{Z}[X]$ and $L = \mathbb{Z}[X] \oplus (X)$. It is clear that $((X) \oplus \mathbb{Z}[X] :_R M) + (\mathbb{Z}[X] \oplus (X) :_R M) \subset ((X) \oplus \mathbb{Z}[X] + \mathbb{Z}[X] \oplus (X) :_R M) = R$. Also R is not distributive, by Theorem 6.6 [8] and M is not distributive module, by [14].*

Proposition 3.10. *Let M be a multiplication-like R-module. If I is an ideal of R such that IM is a second submodule of M, then I is a second ideal of R.*

Proof. Let $\psi_a : I \longrightarrow I$ be the non-zero homomorphism defined by $r \longmapsto ar$. Thus $aIM \neq 0$, because M is faithful module. It follows that $aIM = IM$, since IM is a second submodule. Since M is multiplication-like

$$aI = (aIM :_R M) = (IM :_R M) = I.$$

□

Corollary 3.6. *Let M be a multiplication-like R-module. If I is an ideal of R such that IM is a second submodule of M, then for each non-zero $r \in R$, $r \in Z(R)$ or $I = Ir$.*

Lemma 3.13. *Let M be a multiplication-like and prime R-module. Then for any non-zero ideal I of R, $Ann_R(I) = 0$.*

Proof. Let I be any ideal of R. By Theorem 3.1 (i) and (iv), $Ann_R(I) = 0$. □

Corollary 3.7. *Let M be a multiplication-like and prime R-module. Then $Z(R) = 0$.*

References

[1] F. W. Anderson, K. R. Fuller, Rings and categories of modules,*Springer-verlag, New York, Berlin*, (1974).

[2] S. Annin, Associated and attached primes over noncommutative ring, *Phd. thesis, University of California at Berkeley,* (2002).

[3] H. Ansari-Toroghy, F. Farshadifar, The dual notion of multiplication modules, *Taiwanese J. Math.* 11 (2007) 1189-1201.

[4] H. Ansari-Toroghy, F. Farshadifar, Strong comultiplication modules, *CMU. J. Nat. Sci.* 8 (2009) 105-113.

[5] A. Barnard, Multiplication modules, *Journal of Algebra* 71 (1981) 174-178.

[6] Z. A. El-Bast, P. F. Smith, Multiplication modules, *Comm. Algebra* 16 (1988) 755-779.

[7] H. Fazaeli, F. Rashidi, Primary like-submodules satisfying the primeful property, *Transactions on Algebra and its Applications* 1 (2015) 43-54.

[8] M. D. Larsen, P. J. Maccarthy, Mulltiplicative Theory of ideals, *volume 43, Academic press, INC. London*, (1971).

[9] C. P. Lu, M-radicals of submodules in modules, *Mathematica Japonica* 34 (1989) 211-219.

[10] C. P. Lu, M-radicals of submodules in modules II, *Mathematica Japonica* 35 (1990) 991-1001.

[11] R. McCasland, M. Moore, Prime submodules, *Comm. Algebra* 20 (1992) 1803-1817.

[12] A. Nikseresht, H. Sharif, On comultiplication and r-multiplication modules, *Journal of Algebraic Systems* 2 (2014) 1-19.

[13] L. H. Rowen, Ring Theory, *Volume I, Academic pres, INC. San Diego*, (1988).

[14] A. A. Tuganbaev, Distributive and Related topics, *Gordon and Breach Science Publisher, amsterdam*, (1999).

[15] S. Yassemi, The dual notion of prime modules, *Arch. Math (Brno)* 37 (2001) 273-278.

Maryam Ahmadi received the Ph.D. degree in pure mathematics from Hormozgan University, Iran in 2018. Her research interest is module theory.

Dr. Javad Moghaderi is a university assistant professor at university about 12 years out of 15 years of his work experience in university. He received his PhD in pure mathematics from Shahid Bahonar University of Kerman. His research interests include in three areas: Ring theory, module theory, and many valued logic.