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Research Article

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Using New Operational Matrix for Solving Nonlinear Fractional Integral Equations

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Abstract

In this paper, a numerical method for solving nonlinear fractional integral equations (NFIE) is introduced. This method is based on the new basis functions (NFs) introduced in [*M. Paripour and et al., Numerical solution of nonlinear Volterra Fredholm integral equations by using new basis functions, Communications in Numerical Analysis, (2013)*]. Since the conventional operational matrices for fractional kernels are singular, the definition of these matrices is modified. In order to increase the accuracy of approximating integrals, the operational matrices are exactly calculated and parametrically presented. Then, the solution procedure is proposed and applied on NFIE. Furthermore, the error analysis is performed and rate of convergence is obtained. In addition, various numerical examples are provided for a wide range of fractional orders and nonlinearity of integral equations. Comparison of the results with the exact solutions and those reported in previous studies indicate the capability, salient accuracy, and superiority of the proposed method over similar ones.

Keywords : Integral equations; Nonlinear Fractional Integral Equations; Numerical Method; Operational Matrix; New Basis Function.

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1 Introduction

Singular integral equations have been involved in studying numerous areas of science and engineering including mathematical physics, heat conduction, electrochemistry, scattering theory, fluid flow, semi-conductors, and population dynamics [1, 10, 20]. As a result, the theory of fractional integral has attracted much interest in recent pieces of research, owing to its high accuracy in mathematical modeling of various phenomena [1]. These investigations have been mainly devoted to considering the existence of a solution, as well as approximating the answer of fractional integral equations. There can be found several methods suggested for solving singular integral equations. In [20] second Chebyshev wavelet (SCW) was employed to solve weakly singular Volterra integral equations, specifically for the Abel's equations. Besides, on the basis of NewtonCotes rules, a product integration method is introduced in [4]. It is claimed in [7] that the most frequently used methods are Walsh functions, Laguerre polynomials, Fourier series, Laplace trans-

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form method, Haar wavelets, Legendre wavelets, and Chebyshev wavelets. Besides, different innovative methods have been introduced in literature. For instance, extrapolation algorithm [12], Hermite-type collocation method [6], Nyström interpolate method [2], spline collocation and iterated collocation methods [5], spectral collocation [1], graded mesh method [13], optimal homotopy asymptotic method [8], Tau method [17], Taylor series [18], and approximating fractional integrals and Caputo derivatives in [9] concede that there is no limit for researchers to improve accuracy of the solution of fractional integral equations. In addition, operational matrices based methods have received much attention recently. For instance, SCW operational matrix of fractional integration, combined with the block pulse functions, is given in [20]. Moreover, based on rational Haar wavelet, nonlinear fractional integro-differential equations were studied in [7]. Additionally, [16] presented transformation of nonlinear Volterra-Fredholm integral equations into a system of nonlinear algebraic equations by introducing a new set of complementary pair of orthogonal basis functions (NFs), derived from the well-known block pulse functions. Furthermore, on the basis of Chebyshev spectral method, operational matrix for both systems of fractional integro-differential equations and Abel's integral equations was obtained [19]. Besides, a short review of operational matrix of fractional derivatives, provided in [19], states that these matrices have been obtained in previous papers for Chebyshev, Laguerre, Bernstein and Legendre polynomials. In another study, a new numerical approach is developed in [14] by Modified Hat Functions (MHFs) for solving linear and nonlinear Volterra Fredholm integral equations. Although NFs and MHFs were successfully applied on nonlinear integral equations [16, 14], they have not been employed for solving nonlinear fractional integral equations (NFIE). As a result, their consistent operational matrices have not been developed for NFIE. In previous studies, operational matrices were generalized to investigate fractional calculus problems including integro-differential equations [3, 11].
In this paper, the operational matrices, constructed in [16] based on NFs, extended to fit NFIE and the matrices components are parametrically presented. All integrations during calculating these matrices, were exactly carried out. Hence, more accurate method compared to what was presented in [16] is built up. In Section 2, NFs and their properties are summarized. Then, they are utilized to obtain symbolic representation of operational matrices in Section 3. Subsequently, in Section 4, the solution procedure is presented. In Section 5, error analysis is provided and rate of convergence is obtained. Finally, numerical examples are added along with tabulated data to compare the results with the exact solutions and those are reported in former studies and to demonstrate the accuracy of the method.

2 An Overview of NFs

In this section, the definition and properties of a set of new basis functions (NFs), introduced in [16], are presented. For a uniform distribution of grid points lying on the interval $[0, 1)$, the i-th left and right hand functions are defined as follows:

$$N1_i(x) = \begin{cases} \frac{(i+1)^2h^2 - x^2}{(2i+1)h^2}, & ih \leqslant x < (i+1)h \\ 0, & \text{otherwise} \end{cases}$$
$$N2_i(x) = \begin{cases} \frac{x^2 - i^2h^2}{(2i+1)h^2}, & ih \leqslant x < (i+1)h \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where, $i = 0, 1, 2, \ldots, m-1$ and $h = \dfrac{1}{m}$. These functions form vectors $\mathbf{N1}(x)$ and $\mathbf{N2}(x)$, called left and right hand vectors, respectively:

$$\mathbf{N1}(x) = (N1_0(x), N1_1(x), \ldots, N_{1m-1}(x))^T,$$
$$\mathbf{N2}(x) = (N2_0(x), N2_1(x), \ldots, N2_{m-1}(x))^T. \tag{2.2}$$

Then, the vector of (NFs)is written as

$$\mathbf{N}(x) = (\mathbf{N1}^T(x), \mathbf{N2}^T(x))^T, \tag{2.3}$$

where $\mathbf{N}(x)$ is a $2m \times 2m$ vector. In addition, the following properties of NFs, stated in [16], are utilized in this paper:

$$\mathbf{N}(x) \cdot \mathbf{N}^T(x) \cong \text{diag}(\mathbf{N}(x)) = \tilde{N}(x), \tag{2.4a}$$
$$\mathbf{N}(x) \cdot \mathbf{N}^T(x) \cdot \mathbf{V} \cong \tilde{V} \cdot \mathbf{N}(x), \tag{2.4b}$$
$$\mathbf{N}^T(x) \cdot B \cdot \mathbf{N}(x) \cong \widehat{\mathbf{B}}^T \cdot \mathbf{N}(x). \tag{2.4c}$$

In Eq. (2.4a), $\tilde{N}$ and $\tilde{V}$ are diagonal matrices of vector $\mathbf{N}$ and $\mathbf{V}$, respectively; while $\widehat{\mathbf{B}}$ is a vector including all diagonal components of the matrix B. Moreover, a continuous and bounded function $f(x)$, defined on the interval $[0,1)$, can be approximated by NFs as follows:

$$f(x) \cong \mathbf{C}^T \cdot \mathbf{N}(x) = \mathbf{N}^T(x) \cdot \mathbf{C}, \qquad (2.5)$$

where

$$\begin{cases} C = \{C1^T, C2^T\}^T, (6.a) \\ C1 = \left\{f(ih)|_{i=0,1,\ldots,m-1}\right\}^T \\ = \{f(0), f(h), f(2h), \ldots, f((m-1)h)\}^T, (6.b) \\ C2 = \left\{f\left((i+1)h\right)|_{i=0,1,\ldots,m-1}\right\}^T \\ = \{f(h), f(2h), f(3h), \ldots, f(mh)\}^T. (6.c) \end{cases} \qquad (2.6)$$

Furthermore, if p is a positive integer, then:

$$f(x)^p \cong \mathbf{C_p^T} \cdot \mathbf{N}(x) = N^T(x) \cdot \mathbf{C_p}, \qquad (2.7)$$

where

$$\boldsymbol{C_p} = \{f(0)^p, f(h)^p, f(2h)^p, \ldots, f(mh)^p\}^T.$$

Additionally, a continuous and bounded function of two variables, $k : [0,1) \times [0,1) \to \mathbb{R}$ can be expanded by double NFs, using a uniform square grid of $m \times m$ points as follows:

$$k(x,t) \cong \mathbf{N}^T(x) \cdot K \cdot \mathbf{N}(t). \qquad (2.8)$$

In Eq. (2.8), K is a $2m \times 2m$ matrix, whose components are listed as:

$$\begin{aligned} K &= \begin{pmatrix} \kappa 11 & \kappa 12 \\ \kappa 21 & \kappa 22 \end{pmatrix}, \\ \kappa 11_{ij} &= k(ih, jh), \\ \kappa 12_{ij} &= k(ih, (j+1)h), \\ \kappa 21_{ij} &= k((i+1)h, jh), \\ \kappa 22_{ij} &= k((i+1)h, (j+1)h). \end{aligned} \qquad (2.9)$$

3 Modification and Generalization of Operational Matrices

As mentioned in Section 1, NFs have not been applied for solving fractional integral equations so far. Hence, unlike conventional integral equations, solving NFIEs necessitates some modifications in operational matrices to fit the solution procedure. These modifications are carried out in two aspects. The first is to eliminate any approximations made in [16] for evaluating integrals. In other words, the only approximations used in this paper are those made while expanding the functions. Therefore, all integrals are exactly evaluated while obtaining operational matrices. Besides, since the integral equation includes the fractional term, $(x-t)^{(\alpha-1)}$, it appears in calculation as a part of integrands. Thus, as a modification, this term must be added to the integrands of operational matrices:

$$\begin{aligned} \int_0^x (x-t)^{\alpha-1}\mathbf{N}(t)dt &\cong P_0 \cdot \mathbf{N}(x), \\ \int_x^1 (t-x)^{\alpha-1}\mathbf{N}(t)dt &\cong P_1 \cdot \mathbf{N}(x). \end{aligned} \qquad (3.10)$$

In Eq. (3.10), P_0 and P_1 are $2m \times 2m$ operational matrices. To calculate these matrices components, it is needed to consider left-hand side and right-hand side of Eq. (3.10) as functions of x, and then to expand them by NFs using Eq. (2.5):

$$\begin{aligned} &\int_0^x (x-t)^{\alpha-1}\mathbf{N}(t)dt = (\int_0^x (x-t)^{\alpha-1}N1_0(t)dt, \ldots, \\ &\int_0^x (x-t)^{\alpha-1}N1_{m-1}(t)dt, \ldots, \\ &\int_0^x (x-t)^{\alpha-1}N2_0(t)dt, \ldots, \int_0^x (x-t)^{\alpha-1}N2_{m-1}(t)dt)^T. \end{aligned} \qquad (3.11)$$

According to Eqs. (2.5)-(2.6), i-th row in Eq.(3.11) for $i = 1, \ldots, m$ can be expanded as follows:

$$\begin{aligned} &\int_0^x (x-t)^{\alpha-1}N1_{i-1}(t)dt \\ &\cong (\mathbf{C1}^{(i)^T}, \mathbf{D1}^{(i)^T})^T \cdot N(x), \end{aligned} \qquad (3.12)$$

where vectors $\mathbf{C1}^{(i)}$ and $\mathbf{D1}^{(i)}$ respectively play the role of $\mathbf{C1}$ and $\mathbf{C2}$, that appeared in Eq. (2.6). In addition, the superscript (i) corresponds to the i-th row of the matrix P_0. By considering Eq. (2.6), it can be realized that the n-th component of the vector $\mathbf{C1}^{(i)}$ is obtained by evaluating left-hand side of Eq. (3.12) at $x = (n-1)h$:

$$C\mathbf{1}_n^{(i)} = \int_0^x (x-t)^{\alpha-1}N1_{i-1}(t)dt\bigg|_{x=(n-1)h}. \qquad (3.13)$$

Applying $N1_i(x)$ defined in Eq. (2.1) on Eq. (3.13) and then exactly evaluating the integral, we concluded that:

$$C\mathbf{1}_n^{(i)} = \begin{cases} 0 & i \geq n \\ \beta_{i,n-1} & i < n \end{cases} \quad n, i = 1, 2, \ldots, m, \tag{3.14}$$

where, $\beta_{i,n}$ is defined as

$$\beta_{i,n} = \int_{(i-1)h}^{ih} (nh - t)^{\alpha-1} \frac{i^2h^2 - t^2}{(2i-1)h^2} dt,$$
$$\text{for } i = 1, 2, \ldots, m \text{ and } n = 0, 1, \ldots m - 1. \tag{3.15}$$

Similarly, the n-th component of the vector $\mathbf{D1}^{(i)}$ is obtained by setting $x = nh$ as follows:

$$\mathbf{D1}_n^{(i)} = \int_0^x (x-t)^{\alpha-1} N1_{i-1}(t) dt \Big|_{x=nh}$$
$$= \begin{cases} 0 & i \geq n+1 \\ \beta_{i,n} & i < n+1 \end{cases} \quad n, i = 1, 2, \ldots, m, \tag{3.16}$$

Likewise, Eqs. (2.5)-(2.6) suggest that $(i+m)$-th row in Eq. (3.11) for $i = 1, \ldots, m$ might be approximated by NFs in the following way:

$$\int_0^x (x-t)^{\alpha-1} N2_{i-1}(t) dt$$
$$\cong (\mathbf{C2}^{(i)^T}, \mathbf{D2}^{(i)^T})^T \cdot \mathbf{N}(x), \tag{3.17}$$

where $\mathbf{C2}^{(i)}$ and $\mathbf{D2}^{(i)}$ are constant vectors, whose n-th components are respectively calculated by setting $x = (n-1)h$ and $x = nh$ in the left-hand side of Eq. (3.17):

$$\mathbf{C2}_n^{(i)} = \int_0^x (x-t)^{\alpha-1} N2_{i-1}(t) dt \Big|_{x=(n-1)h}$$
$$= \begin{cases} 0 & i \geq n \\ \gamma_{i,n-1} & i < n \end{cases} \quad n, i = 1, 2, \ldots, m, \tag{3.18}$$

$$\mathbf{D2}_n^{(i)} = \int_0^x (x-t)^{\alpha-1} N2_{i-1}(t) d \Big|_{x=nh}$$
$$= \begin{cases} 0 & i \geq n+1 \\ \gamma_{i,n} & i < n+1 \end{cases} \quad n, i = 1, 2, \ldots, m,$$

where

$$\gamma_{i,n} = \int_{(i-1)h}^{ih} (nh-t)^{\alpha-1} \frac{t^2 - (i-1)^2h^2}{(2i-1)h^2} dt,$$
$$i = 1, 2, \ldots, m \text{ and } n = 0, 1, \ldots, m-1. \tag{3.19}$$

Consequently, P_0 can be represented in the following matrix form:

$$P_0 = \begin{bmatrix} \overline{C1} \overline{D1} \\ \overline{C2} \overline{D2} \end{bmatrix}, \tag{3.20}$$

where $\overline{C1}$, $\overline{D1}$, $\overline{C2}$ and $\overline{D2}$ are $m \times m$ matrices whose i-th rows are $\mathbf{C1}^{(i)^T}$, $\mathbf{D1}^{(i)^T}$, $\mathbf{C2}^{(i)^T}$, and $\mathbf{D2}^{(i)^T}$, respectively.
Besides, i-th row of matrix P_1 for $i = 1, \ldots, m$ is expanded in the following way:

$$\int_x^1 (t-x)^{\alpha-1} \mathbf{N1}_{i-1}(t) dt$$
$$\cong (\mathbf{E1}^{(i)^T}, \mathbf{F1}^{(i)^T})^T . \mathbf{N}(x). \tag{3.21}$$

Similar to the procedure of deriving matrix P_0, components of $\mathbf{E1}^{(i)}$ and $\mathbf{F1}^{(i)}$ are calculated as written below:

$$\mathbf{E1}_n^{(i)} = \int_x^1 (t-x)^{\alpha-1} N1_{i-1}(t) dt \Big|_{x=(n-1)h}$$
$$= \begin{cases} \pi_{i,n-1} & i \geq n \\ 0 & i < n \end{cases} \quad n, i = 1, 2, \ldots, m, \tag{3.22}$$

$$\boldsymbol{F1}_n^{(i)} = \int_x^1 (t-x)^{\alpha-1} N\mathbf{1}_{i-1}(t) dt \Big|_{x=nh}$$
$$= \begin{cases} \pi_{i,n} & i \geq n+1 \\ 0 & i < n+1 \end{cases} \quad n, i = 1, 2, \ldots, m, \tag{3.23}$$

where

$$\pi_{i,n} = \int_{(i-1)h}^{ih} (t - nh)^{\alpha-1} \frac{i^2h^2 - t^2}{(2i-1)h^2} dt, i = 1, 2,$$
$$\ldots, m \text{ and } n = 0, 1, \ldots, m-1. \tag{3.24}$$

Furthermore, $(i+m)$-th row of matrix P_1 for $i = 1, \ldots, m$ can be approximated by NFs as:

$$\int_x^1 (t-x)^{\alpha-1} N2_{i-1}(t) dt$$
$$\cong (\mathbf{E2}^{(i)^T}, \mathbf{F2}^{(i)^T})^T . \mathbf{N}(x). \tag{3.25}$$

Components of $\mathbf{E2}^{(i)}$ and $\mathbf{F2}^{(i)}$ are obtained as follows:

$$\mathbf{E2}_n^{(i)} = \int_x^1 (t-x)^{\alpha-1} N2_{i-1}(t) dt \Big|_{x=(n-1)h}$$
$$= \begin{cases} \theta_{i,n-1} & i \geq n \\ 0 & i < n \end{cases} \quad n, i = 1, 2, \ldots, m, \tag{3.26}$$

$$\boldsymbol{F2}_n^{(i)} = \int_x^1 (t-x)^{\alpha-1} N2_{i-1}(t)dt\Big|_{x=nh} \tag{3.27}$$

$$= \begin{cases} \theta_{i,n} & i \geq n+1 \\ 0 & i < n+1 \end{cases} \quad n, i = 1, 2, \ldots, m$$

where

$$\theta_{i,n} = \int_{(i-1)h}^{ih} (t-nh)^{\alpha-1} \frac{t^2-(i-1)^2h^2}{(2i-1)h^2} dt,$$

$$i = 1, 2, \ldots, m \quad \text{and } n = 0, 1, \ldots, m-1. \tag{3.28}$$

The exact values of $\beta_{i,n}, \gamma_{i,n}, \pi_{i,n}$ and $\theta_{i,n}$ are symbolically reported in the appendix.
Finally, P_1 can be represented in the following form:

$$P_1 = \begin{bmatrix} \overline{E1}\overline{F1} \\ \overline{E2}\overline{F2} \end{bmatrix}, \tag{3.29}$$

where $\overline{E1}$, $\overline{F1}$, $\overline{E2}$ and $\overline{F2}$ are $m \times m$ matrices for which i-th rows are $\mathbf{E1}^{(i)^T}$, $\mathbf{F1}^{(i)^T}$, $\mathbf{E2}^{(i)^T}$, and $\mathbf{F2}^{(i)^T}$, respectively.

4 Solution Procedure

Consider the following nonlinear fractional integral equation:

$$u(x) = f(x) + \int_0^x (x-t)^{\alpha-1}\kappa_1(x,t)u^{p_1}(t)dt$$

$$+ \int_0^1 |x-t|^{\alpha-1}\kappa_2(x,t)u^{p_2}(t)dt, \quad x \in [0,1], \tag{4.30}$$

where $\alpha > 0$ is called a fractional order, while κ_1 and κ_2 are respectively Volterra and Fredholm kernels, p_1 and p_2 are positive integers. In this section, we present a new method to solve this equation, provided that this equation has at least one continuous and bounded solution.

Eq. (4.30) can be rearranged as follows.

$$\begin{aligned} u(x) &= f(x) + \int_0^x (x-t)^{\alpha-1}\kappa_1(x,t)u^{p_1}(t)dt \\ &+ \int_0^x (x-t)^{\alpha-1}\kappa_2(x,t)u^{p_2}(t)dt \\ &+ \int_x^1 (t-x)^{\alpha-1}\kappa_2(x,t)u^{p_2}(t)dt \\ &= f(x) + \int_0^x (x-t)^{\alpha-1}(\kappa_1(x,t)u^{p_1}(t) \\ &+ \kappa_2(x,t)u^{p_2}(t))dt \\ &+ \int_x^1 (t-x)^{\alpha-1}\kappa_2(x,t)u^{p_2}(t)dt. \end{aligned} \tag{4.31}$$

According to Eqs. ((2.5)-(2.7)-(2.8)), the functions $u(x)$, $u^{p_i}(t)$, $f(x)$ and $\kappa_i(x,t)$ are expanded:

$$\begin{aligned} & u(x) \cong \boldsymbol{U}^T.\boldsymbol{N}(x), \\ & u(t)^{p_i} \cong \boldsymbol{N}^T(t) \cdot \boldsymbol{U}_{p_i}, \quad i = 1, 2, \\ & f(x) \cong \boldsymbol{F}^T \cdot \boldsymbol{N}(x), \\ & k_i(x,t) \cong \boldsymbol{N}^T(x).K_i.\boldsymbol{N}(t), \quad i = 1, 2. \end{aligned} \tag{4.32}$$

By substituting Eq. (4.32) in Eq. (4.31) and then applying Eq. (2.4b), we get:

$$\begin{aligned} & \boldsymbol{U}^T \cdot \boldsymbol{N}(x) \cong \boldsymbol{F}^T \cdot \boldsymbol{N}(x) \\ & + \int_0^x (x-t)^{\alpha-1} \Big\{ \mathbf{N}^T(x).K_1 \cdot \mathbf{N}(t)\mathbf{N}^T(t).\mathbf{U}_{p_1} \\ & + \mathbf{N}^T(x) \cdot K_2 \cdot \mathbf{N}(t)\mathbf{N}^T(t) \cdot \mathbf{U}_{p_2} \Big\} dt \\ & + \int_x^1 (t-x)^{\alpha-1}\mathbf{N}^T(x).K_2.\mathbf{N}(t)\mathbf{N}^T(t).\mathbf{U}_{p_2}dt \\ & = \boldsymbol{F}^T \cdot \boldsymbol{N}(x) + \int_0^x (x-t)^{\alpha-1}\mathbf{N}^T(x) \cdot \Big\{ K_1 \cdot \widetilde{\mathbf{U}}_{p_1} \\ & + K_2 \cdot \widetilde{\mathbf{U}}_{p_2}.\mathbf{N}(t)dt \\ & + \int_x^1 (t-x)^{\alpha-1}\mathbf{N}^T(x) \cdot K_2 \cdot \widetilde{\mathbf{U}}_{p_2} \cdot \mathbf{N}(t)dt. \end{aligned} \tag{4.33}$$

Hence

$$\boldsymbol{U}^T.\boldsymbol{N}(x) \cong \boldsymbol{F}^T.\boldsymbol{N}(x) +$$

$$\mathbf{N}^T(x) \cdot \Big\{ K_1, \widetilde{\mathbf{U}}_{p_1} + K_2.\widetilde{\mathbf{U}}_{p_2} \Big\} \cdot \int_0^x (x-t)^{\alpha-1}\mathbf{N}(t)dt \tag{4.34}$$

$$+ \mathbf{N}^T(x) \cdot K_2.\widetilde{\mathbf{U}}_{p_2} \cdot \int_x^1 (t-x)^{\alpha-1}\mathbf{N}(t)dt.$$

By replacing $=$ instead of $\cong$ in the above equation and employing operational matrices presented in Eq. (3.10), Eq. (4.34) changes to:

$$\boldsymbol{U}^T.\boldsymbol{N}(x) = \boldsymbol{F}^T \cdot \boldsymbol{N}(x)$$

$$+\boldsymbol{N}^T(x)\cdot\left\{K_1\cdot\widetilde{\boldsymbol{U}}_{p_1}+K_2.\widetilde{\boldsymbol{U}}_{p_2}\right\}\cdot P_0\cdot\boldsymbol{N}(x)$$
$$+N^T(x)\cdot K_2\cdot\widetilde{\boldsymbol{U}}_{p_2}\cdot P_1\cdot N(x). \tag{4.35}$$

Assuming that $B_0 = \left\{K_1\cdot\widetilde{U}_{p_1}+K_2.\tilde{U}_{p_2}\right\}\cdot P_0$ and $B_1 = K_2\cdot\widetilde{\boldsymbol{U}}_{p_2}\cdot P_1$ leads to:

$$\boldsymbol{U}^T\cdot\boldsymbol{N}(x) =$$
$$\boldsymbol{F}^T.\boldsymbol{N}(x)+\boldsymbol{N}^T(x)\cdot B_0\cdot\boldsymbol{N}(x)+\boldsymbol{N}^T(x)\cdot B_1\cdot\boldsymbol{N}(x). \tag{4.36}$$

By considering Eq. (2.4c), Eq. (4.36) is simplified to:

$$\boldsymbol{U}^T\cdot\boldsymbol{N}(x) = \boldsymbol{F}^T\cdot\boldsymbol{N}(x)+\widehat{\boldsymbol{B}}_0^T\cdot\boldsymbol{N}(x)+\widehat{\boldsymbol{B}}_1^T\cdot\boldsymbol{N}(x). \tag{4.37}$$

Finally,

$$\boldsymbol{U} = \boldsymbol{F}+\widehat{\boldsymbol{B}}_0+\widehat{\boldsymbol{B}}_1. \tag{4.38}$$

Eq. (4.38) shows a nonlinear algebraic set of $2m$ equations and $2m$ unknowns $\mathbf{U}$. This set might be solved by an appropriate method to find $\mathbf{U}$ and approximate the solution as: $u(x)\cong\mathbf{U}^T.\mathbf{N}(x)$. It should be noted that solving the mentioned set is out of the scope of this paper. However, it is supposed that this set has at least one solution.

5 Error Analysis

In this section, convergence theorems are provided to bound the error of the proposed method.

Definition 5.1. *The norm of an arbitrary bounded function $v:\mathcal{D}\to\mathbb{R}$ is defined as:*

$$\|v\|_{\mathcal{D}} = \sup_{x\in\mathcal{D}}|v(x)| \tag{5.39}$$

Definition 5.2. *Suppose that continuous and bounded functions $v:[0,1]\to\mathbb{R}$ and $w:[0,1]\times[0,1]\to\mathbb{R}$ are approximated as $v_m(x)$ and $w_{m,m}(x,y)$ by NFs according to Eqs. (2.5)-(2.6) and Eqs. (2.8)-(2.9), using m and m^2 quadrature points, respectively. The errors of these approximations are defined as:*

$$\begin{aligned}
&e_{i,m}^{(x,v)}(x) = v(x)-v_m(x), \quad x\in[ih,(i+1)h],\\
&\text{for } i=0,1,2,\ldots,m-1,\\
&e_{m}^{(x,v)}(x) = v(x)-v_m(x), \quad x\in[0,1],\\
&e_{i,j,m}^{(x,y,w)}(x,y) = w(x,y)-w_{m,m}(x,y),\\
&\quad x\in[ih,(i+1)h], \quad y\in[jh,(j+1)h],\\
&\text{for } i,j=0,1,2,\ldots,m-1,\\
&e_{m}^{(x,y,w)}(x,y) = w(x,y)-w_m(x,y), \quad x,y\in[0,1],
\end{aligned} \tag{5.40}$$

where the superscripts x and y indicate the independent variables while v and w show the function approximated. In addition, the subscripts i and j respectively, correspond to $(i+1)-th$ and $(j+1)-th$ subdomains of x and y, while m is the number of intervals.

Theorem 5.1. *Suppose that the continuous and bounded function $v(x)\in C^2([0,1])$ is approximated by NFs by m points as $v_m(x)$ by using Eqs. (2.5)-(2.6). Then, for $i=0,1,2,\ldots,m-1$, and $x\in[ih,(i+1)h]$, there exist $\xi_i(x)\in[ih,x],\xi_{i,1}\in[ih,(i+1)h]$, and $\eta_i(x)$ in a way that:*

$$\text{(I)}\, e_{i,m}^{(x,\nu)}(x) = \alpha_i(x)v'\left(\xi_{i,1}\right)+\beta_i(x)v''\left(\eta_i(x)\right),$$

$$\text{(II)}\, \left\|e_m^{(x,v)}(x)\right\|_{[0,1]} \le C^{(x,v)}h, \tag{5.41}$$

where, $\alpha_i(x) = \frac{h(x-ih)-(x-ih)^2}{(2i+1)h}, \beta_i(x) = \left(\xi_i(x)-\xi_{i,1}\right)(x-ih)$ and

$$C^{(x,v)} =$$
$$\frac{1}{4}\max\left\|\frac{v'\left(\xi_{i+1}\right)}{2i+1}\right\|_{[ih,(i+1)h]}+h\left\|v''(x)\right\|_{[0,1]}$$

for $i=0,1,2,\ldots,m-1$.
Besides, $v_m(x)$ converges to $v(x)$ with $\mathcal{O}(h)$.

Proof. By using Eqs. (2.5)-(2.6), the function $v(x)$ is approximated by NFs in the interval $[ih,(i+1)h)$ as:

$$v_m(x) = \frac{v((i+1)h)-v(ih)}{h^2(2i+1)}x^2+$$
$$\frac{h^2(1+i)^2v(ih)-h^2i^2v((i+1)h)}{h^2(2i+1)}, x\in[ih,(i+1)h). \tag{5.42}$$

According to the mean value theorem, and this fact that $x \in [ih,(i+1)h]$, there exists $\xi_i(x) \in [ih, x]$ such that:

$$v'(\xi_i(x)) = \frac{v(x)-v(ih)}{x-ih}$$

$$\Rightarrow v(x) = v(ih) + v'(\xi_i(x))(x-ih), \quad (5.43)$$

setting $x=(i+1)h$ and $\xi_i((i+1)h)=\xi_{i,1}$ in Eq. (5.43) results in:

$$v((i+1)h) = v(ih) + v'(\xi_{i,1})\, h. \quad (5.44)$$

Similarly by considering mean value theorem for second derivative, there exists $\eta_i(x) \in [\min(\xi_{i,1},\xi_i(x)), \max(\xi_{i,1},\xi_i(x))]$ in such a way:

$$v''(\eta_i(x)) = \frac{v'(\xi_i(x)) - v'(\xi_{i,1})}{\xi_i(x)-\xi_{i,1}}$$

$$\Rightarrow v'(\xi_i(x)) = v'(\xi_{t_1}) + v''(\eta_i(x))(\xi_i(x)-\xi_{i,1}), \quad (5.45)$$

substituting Eq. (5.45) in Eq. (5.43) gives:

$$v(x) = v(ih) + [v'(\xi_{i,1}) + v''(\eta_i(x))(\xi_i(x)-\xi_{i,1})](x-ih), \quad (5.46)$$

subtracting Eq. (5.42) from Eq. (5.46) and using Eq.(5.44) proves the first part of the theorem:

$$e^{(x,v)}_{i,m}(x) = v(x) - v_m(x)$$

$$= \frac{h(x-ih)-(x-ih)^2}{(2i+1)h} v'(\xi_{i,1}) + (\xi_i(x)-\xi_{i,1})(x-ih)v''(\eta_i(x)).$$

On the other hand,

$$\left\|e^{(x,v)}_{i,m}(x)\right\|_{[ih,(i+1)h]} \le \frac{1}{(2i+1)h}\left\|h(x-ih)-(x-ih)^2\right\|_{[i,h,(i+1)h]} \times \left|v'(\xi_{i,1})\right| + \|(\xi_i(x)-\xi_{i,1})(x-ih)\|_{[ih,(i+1)h]} \times \left\|v''(\eta_i(x))\right\|_{[i,h,(i+1)h]}.$$

Clearly, the maximum of the $h(x-ih)-(x-ih)^2$ in the interval $[ih,(i+1)h]$ is $\frac{h^2}{4}$. Since $\xi_i(x), \xi_{i,1} \in [ih,(i+1)h]$, it can be inferred that $\|(\xi_i(x)-\xi_{i,1})(x-ih)\| \le h\|(x-ih)\| = h^2$. Hence:

$$\left\|e^{(x,v)}_{i,m}(x)\right\|_{[ih,(i+1)h\}} \le \frac{h}{4(2i+1)}\left|v'(\xi_{i,1})\right| + h^2\left\|v''(\eta_i(x))\right\|_{[ih,(i+1)h]'}$$

thus, total error of the approximation $\left(\left\|e^{(x,v)}_{m}(x)\right\|\right)$ is obtained by selecting the maximum value of the error in each interval:

$$\left\|e^{(x,v)}_{m}(x)\right\|_{[0,1]} \le \max\left\{\left\|e^{(x,v)}_{i,m}(x)\right\| \mid i=0,1,2,\ldots,m-1\right\}$$
$$= \frac{h}{4}\max\left\{\left\|\frac{v'(\xi_{i,1})}{2i+1}\right\|_{[ih,(i+1)h]} | i=0,1,2,\ldots,m-1\right\} + h^2\left\|v''(x)\right\|_{[0,1]}$$
$$= C^{(x,v)}h.$$

Furthermore, by considering the assumption that $v(x) \in C^2([0,1])$, it is obvious that when $h \to 0$, then $C^{(x,v)} \leqslant \frac{1}{4}\|v'(x)\|_{[0,1]}$. Hence the Eq. (5.41) reveals that $v_m(x)$ converges to $v(x)$ with $\mathcal{O}(h)$. This completes the proof. □

Definition 5.3. *The following operator $L^{(i)}_x$ is defined for approximating the continuous and bounded function $v:[0,1]\to\mathbb{R}$ with respect to x in the interval $[ih,(i+1)h]$ as:*

$$L^{(i)}_x\{v(x)\} = v_m(x), \quad i=0,1,2,\ldots,m-1, \quad (5.47)$$

using Eq.(5.40), Eq. (5.47) changes to:

$$L^{(i)}_x\{v(x)\} = v_m(x) = v(x) - e^{(x,v)}_{i,m}(x) \quad (5.48)$$

$i=0,1,2,\ldots,m-1$, it can be simply shown that the operator $\mathcal{L}$ is linear. Furthermore, a continuous and bounded function $w:[0,1]\times[0,1]\to\mathbb{R}$ can be expanded as:

$$w_{m,m}(x,y) = \mathcal{L}^{(j)}_y\left\{\mathcal{L}^{(i)}_x\{w(x,y)\}\right\}. \quad (5.49)$$

Theorem 5.2. *Suppose that the continuous and bounded function $w(x,y) \in C^2([0,1]\times[0,1] \to \mathbb{R})$ is approximated by NFs by m^2 points as $w_{m,m}(x,y)$. Then, for $i,j=0,1,2,\ldots,m-1$ and $x\in[ih,(i+1)h]$, $y\in[jh,(j+1)h]$ there exist $\xi_i(x)\in[ih,x], \xi_j(y)\in[jh,y], \xi_{i,1}\in[ih,(i+1)h]$, $\xi^*_{i,1}\in[jh,(j+1)h], \eta_i(x)$ and $\eta^*_j(y)$ such that:*

$$(\mathrm{I})e^{(x,y,w)}_{i,j,m}(x,y) = \alpha_j(y)\left.\frac{\partial w}{\partial y}\right|_{y=\xi^*_{j,1}} + \beta_j(y)\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=\eta^*_j(y)} + \alpha_i(x)\alpha_j(y)\left.\frac{\partial^2 w}{\partial x\partial y}\right|_{\substack{x=\xi_{i,1}\\ y=\xi^*_{j,1}}}$$

$$+\alpha_i(x)\beta_j(y)\left.\frac{\partial^3 w}{\partial x\partial y^2}\right|_{\substack{x=\xi_{i,1}\\ y=\eta_j^*(y)}}$$

$$+\alpha_j(y)\beta_i(x)\left.\frac{\partial^3 w}{\partial y\partial x^2}\right|_{\substack{x=\eta_i(x)\\ y=\xi_{j,1}^*}}+\beta_i(x)\beta_j(y)\left.\frac{\partial^4 w}{\partial x^2\partial y^2}\right|_{\substack{x=\eta_i^*(y)\\ y=\eta_j^*(y)}} \quad (5.50)$$

$$(\ II\ )\left\|e_m^{(x,y,w)}(x,y)\right\|\leq C^{(x,y,w)}h, \quad (5.51)$$

where,

$$C^{(x,y,w)}=$$

$$\max\left\{\frac{1}{4(2j+1)}\left|\left.\frac{\partial w}{\partial y}\right|_{y=\xi_{j,1}^*}\right|+\frac{h}{16(2i+1)(2j+1)}\left|\left.\frac{\partial^2 w}{\partial x\partial y}\right|_{\substack{x=\xi_{i,1}\\ y=\xi_{j,1}^*}}\right|\right.$$

$$+\frac{h^2}{4(2i+1)}\left\|\left.\frac{\partial^3 w}{\partial x\partial y^2}\right|_{\substack{x=\xi_{i,1}\\ y=\eta_j^*(y)}}\right\|$$

$$\left.+\frac{h^2}{4(2j+1)}\left\|\left.\frac{\partial^3 w}{\partial y\partial x^2}\right|_{\substack{x=\eta_i(x)\\ y=\xi_{j,1}^*}}\right\|\ \middle|\ i,j,=0,1,2,\dots,m-1\right\}$$

$$+h\left\|\frac{\partial^2 w}{\partial y^2}\right\|+h^3\left\|\frac{\partial^4 w}{\partial x^2\partial y^2}\right\|.$$

Besides, $w_{m,m}(x,y)$ converges to $w(x,y)$ with $\mathcal{O}(h)$.

Proof. Using Eqs. (5.48)- (5.49), results in:

$$w_{m,m}(x,y)=\mathcal{L}_y^{(j)}\left\{w(x,y)-e_{i,m}^{(x,w)}(x)\right\}$$

$$=\mathcal{L}_y^{(j)}\{w(x,y)\}-\mathcal{L}_y^{(j)}\left\{e_{i,m}^{(x,w)}(x,y)\right\}$$

$$=w(x,y)-e_{j,m}^{(y,w)}(x,y)-\mathcal{L}_y^{(j)}\left\{e_{i,m}^{(x,w)}(x,y)\right\}\Rightarrow$$

$$e_{i,j,m}^{(x,y,w)}(x,y)=w(x,y)-w_{m,m}(x,y)$$

$$=e_{j,m}^{(y,w)}(x,y)+\mathcal{L}_y^{(j)}\left\{e_{i,m}^{(x,w)}(x,y)\right\}. \quad (5.52)$$

On the basis of Definition 5.3, $\mathcal{L}_y^{(j)}\left\{e_{i,m}^{(x,w)}(x)\right\}$ is calculated as given:

$$\mathcal{L}_y^{(j)}\left\{e_{i,m}^{(x,w)}(x,y)\right\}$$

$$=\mathcal{L}_y^{(j)}\left\{\alpha_i(x)\left.\frac{\partial w}{\partial x}\right|_{x=\xi_{i,1}}+\beta_i(x)\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=\eta_i(x)}\right\}$$

$$=\alpha_i(x)\mathcal{L}_y^{(j)}\left\{\left.\frac{\partial w}{\partial x}\right|_{x=\xi_{i,1}}\right\}$$

$$+\beta_i(x)\mathcal{L}_y^{(j)}\left\{\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=\eta_i(x)}\right\},$$

considering the linearity of the operator $\mathcal{L}_y^{(i)}$, leads to:

$$\mathcal{L}_y^{(j)}\left\{e_{i,m}^{(x,w)}\right\}=\alpha_i(x)\left[\left.\left(\frac{\partial}{\partial x}\mathcal{L}_y^{(j)}\{w\}\right)\right|_{x=\xi_{i,1}}\right]$$

$$+\beta_i(x)\left[\left.\left(\frac{\partial^2}{\partial x^2}\mathcal{L}_y^{(j)}\{w\}\right)\right|_{x=\eta_i(x)}\right]$$

$$=\alpha_i(x)\left[\left.\left(\frac{\partial w}{\partial x}-\frac{\partial e_{j,m}^{(y,w)}(x,y)}{\partial x}\right)\right|_{x=\xi_{i,1}}\right]$$

$$+\beta_i(x)\left[\left.\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 e_{j,m}^{(y,w)}(x,y)}{\partial x^2}\right)\right|_{x=\eta_i(x)}\right], \quad (5.53)$$

by employing Theorem 5.1 in y direction for calculating $e_{j,m}^{(y,w)}(x,y)$ and then applying to Eq. (5.53), results in:

$$e_{j,m}^{(y,w)}(y)=\alpha_j(y)\left.\frac{\partial w}{\partial y}\right|_{y=\xi_{j,1}^*}+\beta_j(y)\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=\eta_j^*(y)}, \quad (5.54)$$

using Eqs. (5.52)- (5.54) confirms the first part of the theorem:

$$e_{i,j,m}^{(x,y,w)}(x,y)=\alpha_j(y)\left.\frac{\partial w}{\partial y}\right|_{y=\xi_{j,1}^*}$$

$$+\beta_j(y)\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=\eta_j^*(y)}+\alpha_i(x)\alpha_j(y)\left.\frac{\partial^2 w}{\partial x\partial y}\right|_{\substack{x=\xi_{i,1}\\ y=\xi_{j,1}^*}}$$

$$+\alpha_i(x)\beta_j(y)\left.\frac{\partial^3 w}{\partial x\partial y^2}\right|_{\substack{x=\xi_{i,1}\\ y=\eta_j^*(y)}}$$

$$+\alpha_j(y)\beta_i(x)\left.\frac{\partial^3 w}{\partial y\partial x^2}\right|_{\substack{x=\eta_i(x)\\ y=\xi_{j,1}^*}}$$

$$+\beta_i(x)\beta_j(y)\left.\frac{\partial^4 w}{\partial x^2\partial y^2}\right|_{\substack{x=\eta_i(x)\\ y=\eta_j^*(y)}}.$$

The second part of the theorem 5.2 can be simply proved similar to that in Theorem 5.1. Moreover, by considering the assumption that $w\in C^2([0,1]\times[0,1]\to\mathbb{R})$, it is clear that when $h\to 0$, then $C^{(x,y,v)}\leq\frac{1}{4}\left\|\frac{\partial w}{\partial y}\right\|_{[0,1]\times[0,1]}$. Changing the independent variables x and y in this theorem and its proof, results in: $C^{(x,v)}\leq\frac{1}{4}\left\|\frac{\partial w}{\partial x}\right\|_{[0,1]\times[0,1]}$. Consequently when $h\to 0$, $C^{(x,y,v)}\leqslant\frac{1}{4}\min\left\{\left\|\frac{\partial w}{\partial x}\right\|_{[0,1]\times[0,1]},\left\|\frac{\partial w}{\partial y}\right\|_{[0,1]\times[0,1]}\right\}$. Hence the Eq. (5.51) proves that $w_m(x,y)$ converges to $w(x,y)$ with $\mathcal{O}(h)$. □

Theorem 5.3. *Suppose that Eq. (4.31) has at least one continuous and bounded exact solution, $u(x)$ and the functions $\kappa_1(x,y)$ and $\kappa_2(x,y)$ are bounded for $x,y\in[0,1]$ and satisfy conditions of theorems (5.1-5.2). Then, the proposed numerical solution of Eq. (4.31), $u_m(x)$, converges with the upper error bound of:*

$$\left\|e_m^{(x,u)}\right\|_{[0,1]}\le \frac{\left\{\alpha C^{(x,f)}+C^{(x,y,\kappa_1)}\|u(x)\|_{[0,1]}^{p_1}+2C^{(x,y,\kappa_2)}\|u(x)\|_{[0,1]}^{p_2}\right\}h}{A+B},$$

where

$$A=\alpha-p_1\|u(x)\|_{[0,1]}^{p_1-1}(\|\kappa_1(x,y)\|_{[0,1]\times[0,1]}+C^{(x,y,\kappa_1)})h),$$

and

$$B=-2p_2\|u(x)\|_{[0,1]}^{p_2-1}\left(\|\kappa_2(x,y)\|_{[0,1]\times[0,1]}+C^{(x,y,\kappa_1)}h\right)$$

subject to the following condition for h(mesh generation) to be satisfied:

$$p_1\|u(x)\|_{[0,1]}p_1^{-1}\left(\|\kappa_1(x,y)\|_{[0,1]\times[0,1]}+C^{(x,y,\kappa_1)}h\right)+2p_2\|u(x)\|_{[0,1]}^{p_2-1}\left(\|\kappa_2(x,y)\|_{[0,1]\times[0,1]}+C^{(x,y,\kappa_1)}h\right).<\alpha.$$

Besides, $u_m(x)$ converges to $u(x)$ with $\mathcal{O}(h)$.

Proof. The upper bound of the mentioned functions in the theorem are supposed as:

$$\|u(x)\|_{[0,1]}=\mathcal{U},\quad \|\kappa_i(x,y)\|_{[0,1]\times[0,1]}=\mathcal{K}_i\quad i=1,2, \tag{5.55}$$

additionally, it can be simply shown that for a positive integer p_i, the function x^{p_i} satisfies Lipchitz condition, given as:

$$\|x^{p_i}-a^{p_i}\|_{[0,\infty)}\le p_i|x|^{p_i-1}|x-a|,$$

thus by using Eq. (5.55):

$$\|u(x)^{p_i}-u_m(x)^{p_i}\|_{[0,1]}\le p_i\mathcal{U}^{p_i-1}|u(x)-u_m(x)|=p_i\mathcal{U}^{p_i-1}e_m^{(x,u)}. \tag{5.56}$$

Now, considering that the exact solution satisfies Eq. (4.31), the error of the solution can be written as:

$$e_m^{(x,u)}=u(x)-u_m(x)=f(x)-f_m(x)+\int_0^x(x-t)^{\alpha-1}\{\kappa_1(x,t)u(t)^{p_1}-\kappa_{1_m}(x,t)u_m(t)^{p_1}+\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}\}dt+\int_x^1(t-x)^{\alpha-1}\{\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}\}\,dt, \tag{5.57}$$

subsequently, the norm of the error is as follows:

$$\left\|e_m^{(x,u)}\right\|_{[0,1]}\le\|f(x)-f_m(x)\|_{[0,1]}+\|\int_0^x(x-t)^{\alpha-1}\Big\{\kappa_1(x,t)u(t)^{p_1}-\kappa_{1_m}(x,t)u_m(t)^{p_1}+\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}\Big\}dt+\int_x^1(t-x)^{\alpha-1}\{\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}dt\|_{[0,1]}, \tag{5.58}$$

using Theorem 5.1 leads to:

$$\left\|e_m^{(x,u)}\right\|_{[0,1]}\le C^{(x,f)}h+\left\|\int_0^x(x-t)^{\alpha-1}dt\right\|_{[0,1]}\times\left\|\int_0^x|\kappa_1(x,t)u(t)^{p_1}-\kappa_{1_m}(x,t)u_m(t)^{p_1}+\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}|dt\right\|+\left\|\int_x^1(t-x)^{\alpha-1}dt\right\|_{[0,1]}\times\|\int_x^1|\kappa_2(x,t)u(t)^{p_2}-\kappa_{2m}(x,t)u_m(t)^{p_2}|\,dt_{[0,1]},$$

it can be easily demonstrated that:

$$I_1=\left\|\int_0^x(x-t)^{\alpha-1}dt\right\|_{[0,1]}=\frac{1}{\alpha},$$

$$I_2=\left\|\int_x^1(t-x)^{\alpha-1}dt\right\|_{[0,1]}=\frac{1}{\alpha},$$

hence:

$$\left\|e_m^{(x,u)}\right\|_{[0,1]} \leq C^{(x,f)}h$$
$$+\frac{1}{\alpha}\left\|\int_0^x \left|\kappa_1(x,t)u(t)^{p_1}-\kappa_{1_m}(x,t)u_m(t)^{p_1}\right.\right.$$
$$\left.\left.+\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}\right|dt\right\|_{[0,1]}+$$
$$\frac{1}{\alpha}\left\|\int_x^1 |\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}|\,dt\right\|_{[0,1]}$$
$$\leq C^{(x,f)}h+\frac{1}{\alpha}\Big\{\|\kappa_1(x,t)u(t)^{p_1}-\kappa_{1_m}(x,t)u_m(t)^{p_1}\|_{[0,1]}$$
$$+\|\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}\|_{[0,1]}\Big\}$$
$$+\frac{1}{\alpha}\|\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}\|_{[0,1]},$$

that is simplified to:

$$\left\|e_m^{(x,u)}\right\|_{[0,1]} \leq C^{(x,f)}h$$
$$+\frac{1}{\alpha}\|\kappa_1(x,t)u(t)^{p_1}-\kappa_{1_m}(x,t)u_m(t)^{p_1}\|_{[0,1]} \tag{5.59}$$
$$+\frac{2}{\alpha}\|\kappa_2(x,t)u(t)^{p_2}-\kappa_{2_m}(x,t)u_m(t)^{p_2}\|_{[0,1]}.$$

For calculating the right hand side of Eq. (5.59), the following term must be obtained:

$$\|\kappa_i(x,t)u(t)^{p_i}-\kappa_{i_m}(x,t)u_m(t)^{p_i}\|_{[0,1]}$$
$$=\kappa_i(x,t)u(t)^{p_i}$$
$$-\left(\kappa_i(x,t)-e_m^{(x,y,\kappa_i)}\right)u_m(t)^{p_i}\|_{[0,1]}$$
$$\leq \|\kappa_i(x,t)\|_{[0,1]}\,\|u(t)^{p_i}-u_m(t)^{p_i}\|_{[0,1]}$$
$$+\left\|e_m^{(x,y,\kappa_i)}\right\|_{[0,1]}$$
$$\times\left\{\|u(t)^{p_i}-u_m(t)^{p_i}\|_{[0,1]}+\|u(t)^{p_i}\|_{[0,1]}\right\},$$

considering Eqs. ((5.55)- (5.56)) results in:

$$\|\kappa_i(x,t)u(t)^{p_i}-\kappa_{i_m}(x,t)u_m(t)^{p_i}\|_{[0,1]}\leq$$
$$\mathcal{K}_ip_i\mathcal{U}^{p_i-1}e_m^{(x,u)}+C^{(x,y,\kappa_i)}h\left(p_i\mathcal{U}^{p_i-1}e_m^{(x,u)}+\mathcal{U}^{p_i}\right)$$
$$=p_i\mathcal{U}^{p_i-1}\left(\mathcal{K}_i+C^{(x,y,\kappa_i)}h\right)e_m^{(x,u)}+C^{(x,y,\kappa_i)}\mathcal{U}^{p_i}h$$
$$=\mathcal{R}_ie_m^{(x,u)}+\mathcal{S}_ih, \tag{5.60}$$

where $\mathcal{R}_i = p_i\mathcal{U}^{p_i-1}\left(\mathcal{K}_i+C^{(x,y,\kappa_i)}h\right)$ and $\mathcal{S}_i = C^{(x,y,\kappa_i)}\mathcal{U}^{p_i}$. Substituting Eq. (5.60) in Eq. (5.59) leads to:

$$\left\|e_m^{(x,u)}\right\|_{[0,1]} \leq C^{(x,f)}h+\frac{1}{\alpha}\left(\mathcal{R}_1e_m^{(x,u)}+\mathcal{S}_1h\right)$$
$$+\frac{2}{\alpha}\left(\mathcal{R}_2e_m^{(x,u)}+\mathcal{S}_2h\right),$$

by rearranging, the bound of the error is obtained:

$$\left\|e_m^{(x,u)}\right\|_{[0,1]} \leq \frac{\left\{C^{(x,f)}+\frac{1}{\alpha}\mathcal{S}_1+\frac{2}{\alpha}\mathcal{S}_2\right\}h}{\left(1-\frac{1}{\alpha}\mathcal{R}_1-\frac{2}{\alpha}\mathcal{R}_2\right)}$$
$$=\frac{\left\{\alpha C^{(x,f)}+\mathcal{S}_1+2\mathcal{S}_2\right\}h}{(\alpha-\mathcal{R}_1-2\mathcal{R}_2)}. \tag{5.61}$$

Since the numerator of the right hand side of Eq. (5.61) is positive, its denominator must be positive to make a positive bound for the error. Thus, $(\alpha-\mathcal{R}_1-2\mathcal{R}_2)>0$ which proves the condition of the theorem by considering that $I_1 = I_2 = \frac{1}{\alpha}$. In addition according to the theorems (I) and (II) when $h\to 0$, the constants $C^{(x,f)}, C^{(x,y,\kappa_1)}$ and $C^{(x,y,\kappa_2)}$ approach to the following upper bounds, independent of the mesh and h as follows:

$$C^{(x,f)} \leq \frac{1}{4}\left\|f'(x)\right\|_{[0,1]}$$
$$C^{(x,y,\kappa_1)} \leq$$
$$\frac{1}{4}\min\left\{\left\|\frac{\partial\kappa_1}{\partial x}\right\|_{[0,1]\times[0,1]},\left\|\frac{\partial\kappa_1}{\partial y}\right\|_{[0,1]\times[0,1]}\right\}$$
$$C^{(x,y,\kappa_2)} \leq$$
$$\frac{1}{4}\min\left\{\left\|\frac{\partial\kappa_2}{\partial x}\right\|_{[0,1]\times[0,1]},\left\|\frac{\partial\kappa_2}{\partial y}\right\|_{[0,1]\times[0,1]}\right\},$$

since it is supposed that the solution of Eq. (4.31) and the kernels κ_1 and κ_2 are continuous and bounded, the values of $\|u(x)\|_{[0,1]}, \|\kappa_1(x,y)\|_{[0,1]\times[0,1]}$ and $\|\kappa_2(x,y)\|_{[0,1]\times[0,1]}$ are bounded. Hence, $u_m(x)$ converges to $u(x)$ with $\mathcal{O}(h)$. □

6 Results and Discussion

In this section, the proposed method is examined by comparing the absolute errors ($E(x) = |u(x)-u_m(x)|$) with those reported in the literature. Furthermore, the effect of nonlinearity as well as fractional order are studied.

Example 6.1. *Consider the following conventional Volterra-Fredholm integral equation:*

$$u(x) = \frac{-1}{30}x^6+\frac{1}{3}x^4-x^2+\frac{5}{3}x-\frac{5}{4}$$

$$+\int_0^x (x-t)u^2(t)dt + \int_0^1 (x+t)u(t)dt,$$

the exact solution is $u(x) = x^2 - 2$. *Table 1 compares the results of the proposed method with those of [16]. It admits that the presented method is one order of magnitude accurate than that in [16]. In addition, Figure 1 illustrates the absolute error versus* x*, showing that the maximum error slightly exceeds* 0.004.

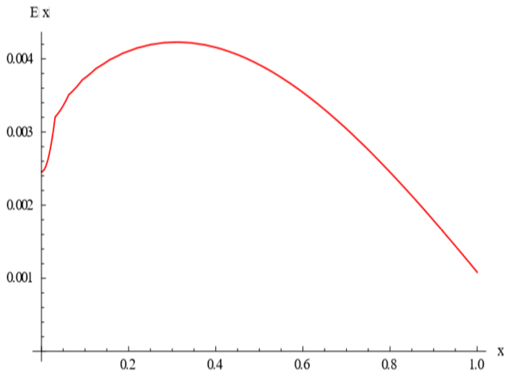

Figure 1: The Absolute error for Example 6.1 for $m = 32$.

Example 6.2. *The Abel integral equation with the following terms is assumed:*

$$u(x) = \frac{1}{\sqrt{1+x}} + \frac{\pi}{8} - \frac{1}{4}\arcsin\left(\frac{1-x}{1+x}\right)$$
$$-\frac{1}{4}\int_0^x (x-t)^{-\frac{1}{2}} u(t)dt,$$

where the exact solution is $u(x) = \dfrac{1}{\sqrt{1+x}}$ *and the results were compared with the solution obtained in [15] by block pulse function approximation. Table 2 reveals that the absolute errors of NFs method is remarkably smaller than those in [15]. Moreover, Figure 2 depicts that the absolute error of NFs method meets its maximum near* $x = 0$.

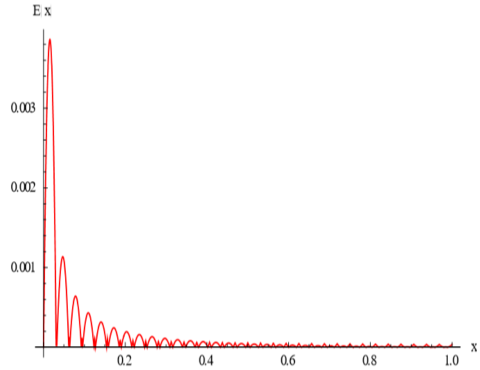

Figure 2: The Absolute error for Example 6.2 for $m = 32$.

Example 6.3. *For investigating the effects of the nonlinear term as well as the fractional order in Volterra integral equation, the subsequent equation is considered:*

$$u(x) = f(x) + \int_0^x xt(x-t)^{\alpha-1}u(t)^p dt,$$

with the exact solution of $u(x) = e^{-x}$. *Additionally,* $f(x)$ *can be accordingly calculated using the exact solution. The effects of* α *and* p *are demonstrated in Figures (3-4). Besides, the absolute errors for various values of* α *and* p *are tabulated in Table 3. It can be inferred that the absolute errors fluctuation rises, when the nonlinearity* (p) *increases.*

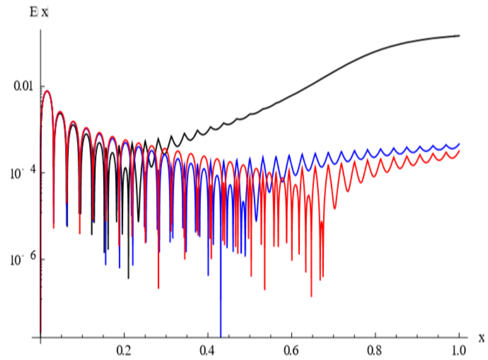

Figure 3: Absolute Error for Example 6.3 for $p = 3$. Black Line: $\alpha = 0.2$, Blue Line: $\alpha = 0.8$,Red Line:$\alpha = 1.8$.

Appendix

The functions $\beta_{i,n}$, $\gamma_{i,n}$, $\pi_{i,n}$ and $\theta_{i,n}$ are given as:

$$\beta_{i,n} =$$

$$h^{\alpha}\frac{2(-i+n)^{1+\alpha}(i+n+i\alpha)+(1-i+n)^{\alpha}H(n,i,\alpha)}{(-1+2i)\alpha(1+\alpha)(2+\alpha)},$$

Table 1: Comparison of the results of the proposed method with the method of [16] for $m = 32$

x_i	Proposed method	The method of [16]
0	0.00245	0.01455
0.1	0.00374	0.01744
0.2	0.00410	0.01967
0.3	0.00422	0.02118
0.4	0.00415	0.02195
0.5	0.00392	0.02351
0.6	0.00354	0.02269
0.7	0.00304	0.02122
0.8	0.00245	0.01199
0.9	0.00179	0.01671

Table 2: Comparison of the absolute errors of the NFs method with the method of [15] for Example 6.2 with $m = 32$.

x_i	The NFs method	The method of [15]
0	0.000231	0.000877
0.2	0.000104	0.000565
0.4	0.000022	0.000002
0.6	0.0000007	0.000041
0.8	0.000009	0.000005

Table 3: The absolute error of Example 6.3 for various values of α and p with $m = 32$.

	$p = 1$			$p = 2$			$p = 3$		
x_i	$\alpha = 0.2$	$\alpha = 0.8$	$\alpha = 1.8$	$\alpha = 0.2$	$\alpha = 0.8$	$\alpha = 1.8$	$\alpha = 0.2$	$\alpha = 0.8$	$\alpha = 1.8$
0.2	0.000112	0.000477	0.000553	0.000112	0.000477	0.000553	0.000112	0.000477	0.000553
0.4	0.000938	0.000014	0.000134	0.000938	0.000014	0.000134	0.000938	0.000014	0.000134
0.6	0.006140	0.000146	0.000006	0.006140	0.000146	0.000006	0.006140	0.000146	0.000006
0.8	0.064314	0.000235	0.064314	0.000009	0.000235	0.000088	0.064314	0.000235	0.000088

$\gamma_{i,n} =$

$$h^a \frac{2(1-i+n)^{1+\alpha}\varphi(n,i,\alpha) + (-i+n)^a}{G(n,i,\alpha)}\chi(i,\alpha),$$

$\pi_{i,n} =$

$$h^\alpha \frac{2(i-n)^{1+\alpha}(i+n+i\alpha) + (-1+i-n)^\alpha R(n,i,\alpha)}{(-1+2i)\alpha(1+\alpha)(2+\alpha)}$$

$\theta_{i,n} =$

$$h^\alpha \frac{2(-1+i-n)^{1+\alpha}\varphi(n,i,\alpha) + (i-n)^a}{Y(n,i,\alpha)}\chi(i,\alpha),$$

where $H(n,i,\alpha) =$

$$-2n^2+2n\alpha+2i^2(1+\alpha)-\alpha(1+\alpha)+2i\alpha(1-n+\alpha),$$

$G(n,i,\alpha) =$

$$-2n^2+2i^2(1+\alpha)+(1+\alpha)(2+\alpha)-2i(2+\alpha(3+n+\alpha)),$$

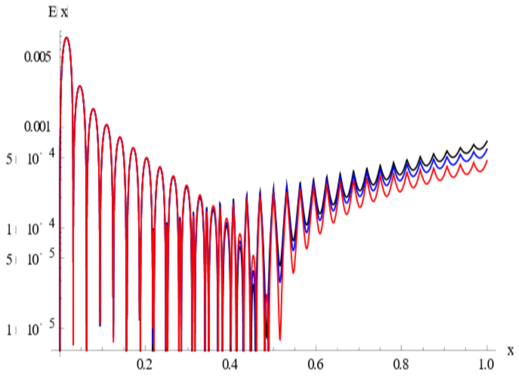

Figure 4: The Absolute Error for Example 6.3 for $\alpha = 0.8$. Black Line: $p = 1$, Blue Line: $p = 2$, Red Line: $p = 3$.

$$R(n,i,\alpha) = 2n^2 + \alpha - 2n\alpha + 2i(-1+n-\alpha)\alpha + \alpha^2 - 2i^2(1+\alpha),$$

$$Y(n,i,\alpha) = 2n^2 - 2i^2(1+\alpha) - (1+\alpha)(2+\alpha) + 2i(2+\alpha(3+n+\alpha)),$$

$$\chi(i,\alpha) = (-1+2i)\alpha(1+\alpha)(2+\alpha),$$

$$\varphi(n,i,\alpha) = (-1+i+n+(-1+i)\alpha).$$

7 Conclusion

This paper addressed a numerical solution to the nonlinear fractional integral equations via employing new basis functions firstly introduced in [16]. The proposed method suggested exact integration scheme for developing operational matrices in order to reduce absolute error compared to that obtained in [16]. Besides, the operational matrices were represented symbolically. Convergence and error analysis were provided to estimate an upper bound of the absolute error as well as order of convergence. The results were compared with previous studies, demonstrating the superiority of this method over some similar ones. The effect of nonlinearity and order of convergence were studied. It was concluded that the absolute error fluctuates more rapidly, as the nonlinearity increases.

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