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Research Article

Some Improvments of The Cordero-Torregrosa Method for The Solution of Nonlinear Equations

M. Mohamadi Zadeh *, T. Lotfi †‡, M. Amirfakhriyan §¶

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Abstract

In this paper, two adaptive methods with memory are improved based on the Cordero-Torregrosa method. The technique of adaptive method increases the efficiency index as high as possible. The new proposed derivative free methods have possessed the convergence order 7.46315 and 7.99315, and they use the information from the last two iterations. Finally, we provide convergence analysis and numerical examples to illustrate the efficiency and applicability of the proposed methods.

Keywords : Nonlinear equation; Iterative methods; The method without and with memory; Efficiency index; Adaptive method; Convergence order.

1 Introduction

Finding the solution of nonlinear equation $f(x) = 0$, where $f : D \subset R \longrightarrow R$ which is a scalar function at an open interval, is important and challenging a problem in the field of computational mathematics. To approximate the simple root of $f(x) = 0$, it is appropriate to use iterative methods where it starts with initial guess x_0 and conjectures by generating new iterations, for example 1, $x_{n+1} = \varphi(x_n)$. The first well- known iterative method was introduced by the Newton about 300 years ago and it is defined as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

and continued with new published methods [1, 9, 13, 14, 15, 16, 17, 18]. Kung and Traub in [5] proposed the following method with memory based on the Newton method:

$$\begin{cases} w_n = x_n + \gamma f(x_n), \\ \gamma = -\dfrac{1}{f\left[x_n, x_{n-1}\right]}, \\ x_{n+1} = x_n - \dfrac{f\left(x_n\right)}{f\left[x_n, w_n\right]}, \quad n = 0, 1, 2, \cdots \end{cases} \tag{1.1}$$

with the order of convergence 2.414, where $f[x_n, w_n] = \dfrac{f(x_n) - f(w_n)}{x_n - w_n}$ signies a divided difference. Kung and Traub proposed that each n-step method without memory is optimal if it uses $n + 1$ functional evaluations with conver-

*Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran.

†Corresponding author. lotfitaher@yahoo.com, Tel:+98(81)344163085.

‡Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran.

§Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.

¶Department of Computer Sciences, University of Calgary, Canada.

gence order 2^n [5]. According to this conjecture, there have been many efforts to construct such optimal methods without memory. Certain authors used the Newtons interpolation polynomial to construct the self-accelerating parameter and gave some effective iterative methods with memory [4, 12, 15]. The acceleration in convergence is based on the utilizing of a variation of at least one free non-zero parameter in each iterative step. This parameter is calculated using information from the current and previous iterations. Despite many successful and genuine attempts in developing methods with memory, a motivation for studying adaptive method with memory is needed. Previous scientists just use the information of the current and the previous iterations while it is possible to reuse all the information of the previous iterations to increase the efficiency as high as possible. In this paper, two adaptive methods have been introduced based on the Cordero-Torregrosa method with high-efficiency index.

This paper is organized as follows: In Section 2, based on the Cordero-Torregrosa method [1], an optimal fourth-order method is proposed for solving nonlinear equations, then two iterative methods with memory are constructed two adaptive methods. Using the new technique, the new iterative methods with memory reach the desired convergence order. Numerical examples are given in Section 3 to display the implementation of the new method, which confirms the theoretical results. Section 4 is tendered a short conclusion.

2 The method and analysis of convergence

In this section, we consider the following two-step method without memory that is kind of Cordero-Torregrosa method [1],

$$\begin{cases} w_n = x_n + \gamma f(x_n), \\ y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, \\ x_{n+1} = y_n - \dfrac{f(y_n) f[x_n, w_n]}{f[x_n, y_n] f[y_n, w_n]}, \\ \qquad n = 0, 1, 2, \ldots \end{cases} \tag{2.2}$$

when x_0 is given. This method is optimal with convergence order four that uses three functional evaluations.

Theorem 2.1. *Method (2.2) has convergence order four if x_0 is close enough to the zero. It's error equation is*

$$e_{n+1} = (1 + f'(\alpha)\gamma)^2 c_2 (2c_2^2 - c_3) e_n^4 + O(e_n^5).$$

Proof. Instead of involving reporting some algebraic expressions, we prefer to present the prove according to implementation in mathematica. The commands are self-explained and simple to understand, so we neglect to decipher them.

$$\texttt{f}[\texttt{e}\_] = f1a * \left(e + \sum_{i=2}^{4} c_i * e^i \right);$$

$$\texttt{ew} = \gamma * f[e] + e;$$

$$\texttt{f}[\texttt{x}\_, x\_] := f'[x];$$

$$\texttt{f}[\texttt{x}\_, y\_] := (f[x] - f[y])/(x - y);$$

$$\texttt{In}[1] := \mathrm{ey} = \mathrm{Series}\left[e - \frac{f[e]}{f[e, \mathrm{ew}]}, \{e, 0, 4\} \right]$$

$$//\mathrm{FullSimplify} \qquad (*First \quad step*)$$

$$\begin{aligned} \texttt{Out}[1] = {} & c_2(\gamma \mathrm{f1a} + 1)e^2 \\ & + \left(\texttt{c}_2^2(-(\texttt{'}\mathrm{f1a}(\texttt{'}\mathrm{f1a} + \texttt{2}) + \texttt{2})) \right. \\ & + c_3 + (\gamma \mathrm{f1a} + 1)(\gamma \mathrm{f1a} + 2)\mathrm{e}^3 \\ & + \left(\texttt{c}_2^3(\texttt{'}\mathrm{f1a}(\texttt{'}\mathrm{f1a}(\texttt{'}\mathrm{f1a} + \texttt{3}) + \texttt{5}) + \texttt{4}) \right. \\ & - c_2 c_3(\gamma \mathrm{f1a}(\gamma \mathrm{f1a}(2\gamma \mathrm{f1a} + 7) + 10) + 7) \\ & + \texttt{c}_4 + (\texttt{'}\mathrm{f1a} + \texttt{1})(\texttt{'}\mathrm{f1a}(\texttt{'}\mathrm{f1a} + \texttt{3}) + \texttt{3})\mathrm{e}^4 + O[e]^5 \end{aligned}$$

$$\texttt{In}[2] := \mathrm{ez} = \frac{f[\mathrm{ew}] f[\mathrm{ey}]}{(f[\mathrm{ew}] - f[\mathrm{ey})] f[e, \mathrm{ey}]}$$

$$//\mathrm{FullSimplify} \qquad (*Second \quad step*)$$

$$\texttt{Out}[2] = (1 + \gamma \mathrm{f1a})^2 c_2 \left(2c_2^2 - c_3 \right) e^4 + O[e]^5.$$

This completes our proof. □

To develop a method with memory, it is necessary to introduce a parameter that can be approximated as the procedure follows, so we modify Method (2.2) as follows:

$$\begin{cases} w_n = x_n + \gamma f(x_n) \\ y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n] + \beta f(w_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n) f[x_n, w_n]}{f[x_n, y_n] f[y_n, w_n]}, \\ \qquad n = 0, 1, 2, \ldots, \end{cases} \tag{2.3}$$

Table 1: Numerical results for $f_1(x)$ by the methods with memory

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	r_c	EI
Method (3.27) with (3.28)	0.9027(-3)	0.1827(-10)	0.7583(-41)	3.9489	1.5806
Method (3.27) with (3.29)	0.9027(-3)	0.1787(-10)	0.6924(-41)	4.1477	1.6067
Method (3.27) with (3.30)	0.9027(-3)	0.4948(-13)	0.4048(-51)	3.9116	1.5756
Method (3.27) with (3.31)	0.9027(-3)	0.3226(-10)	0.1607(-85)	3.9783	1.5845
Method (3.32)	0.2415	0.6840(-2)	0.7002(-3)	6.4360	1.8601
Method (2.4)	0.1501(-2)	0.3601(-19)	0.2462(-129)	7.0001	1.9129
Method (2.15)	0.1501(-2)	0.3781(-19)	0.7333(-129)	7.5234	1.9594

Table 2: Numerical results for $f_2(x)$ by the methods with memory

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	r_c	EI
Method (3.27) with (3.28)	0.1284(-1)	0.6392(-6)	0.2098(-22)	3.9306	1.5781
Method (3.27) with (3.29)	0.1288(-6)	0.6157(-6)	0.1592(-22)	4.1242	1.6036
Method (3.27) with (3.30)	0.1284(-1)	0.3521(-7)	0.1663(-27)	4.2464	1.6193
Method (3.27) with (3.31)	0.1616(-1)	0.1044(-7)	0.4112(-37)	4.4466	1.6444
Method (3.32)	0.2415	0.6840(-2)	0.7002(-3)	6.3360	1.8504
Method (2.4)	0.1181(-1)	0.8712(-64)	0.3771(-447)	7.3223	1.9418
Method (2.15)	0.1182(-1)	0.5791(-10)	0.1491(-63)	7.7432	1.9783

By adding the parameter β in the method (2.2) remains optimal with following error equation.

Theorem 2.2. *The error equation of method (2.3) is given by*

$$\begin{aligned} e_{n+1} =& (1+f'(\alpha)\gamma)^2(\beta+c_2) \\ & (c_2(\beta+2c_2)-c_3)e_n^4+O(e_n^5). \end{aligned}$$

To derive a method with memory, we suppose that $\gamma = -\dfrac{1}{f'(\alpha)} \approx \dfrac{-1}{N_3'(x_k)}$ and $\beta = -c_2 = -\dfrac{f''(\alpha)}{2f'(\alpha)} \approx -\dfrac{N_4''(w_k)}{2N_4'(w_k)}$, where $N_3(x_k)$ and $N_4(w_k)$ are Newton's interpolating polynomials of two and third degree on the nodes $(x_k, y_{k-1}, x_{k-1}, w_{k-1})$ and $(x_k, w_k, y_{k-1}, x_{k-1}, w_{k-1})$, respectively.

Here we affirmative that albeit this method is with memory but it is not adaptive yet. It consumes information on the last two iterations. To develop this method to an adaptive method with memory we update γ_n and β_n based on all the available information from the first iteration to the current iteration. So in each iteration, we use the following accelerators:

$$\begin{cases} \gamma_n = \dfrac{-1}{N_{3n}'(x_n)}, \\ \beta_n = -\dfrac{N_{3n+1}''(w_n)}{2N_{3n+1}'(w_n)}, \\ w_n = x_n + \gamma_n f(x_n), \\ y_n = x_n - \dfrac{f(x_n)}{f[e_n, w_n] + \beta_n f(w_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n) f[x_n, w_n]}{f[x_n, y_n] f[y_n, w_n]}, \\ \quad n = 0, 1, 2, \ldots, \end{cases} \tag{2.4}$$

where x_0 and γ_0 are given suitably.

Lemma 2.1. *If* $\gamma_n = \dfrac{1}{N_{3n}'(x_n)}$, *and* $\beta_n = \dfrac{-N_{3n+1}''(w_n)}{N_{3n+1}'(w_n)}$, *then*

$$1 + \gamma_n f'(\alpha) \sim \prod_{k=0}^{n-1} e_k e_{k,y} e_{k,w},$$

$$c_2 + \beta_n \sim \prod_{k=0}^{n-1} e_k e_{k,y} e_{k,w}, \qquad n = 1, 2, \ldots$$

where

$$e_k = x_k - \alpha, \quad e_{k,y} = y_k - \alpha, \quad e_{k,w} = w_k - \alpha.$$

Proof. The proof is very similar to Lemma 4, [14] and Lemma 1, [19]. □

The convergence order of the adaptive Method (2.4) is given in the following theorem:

Theorem 2.3. *If* $\gamma_n = \frac{1}{N'_{3n}(x_n)}$ *and,* $\beta_n = \frac{-N''_{3n+1}(w_n)}{N'_{3n+1}(w_n)}$ *then convergence order of method (2.3) is obtained from solution of the following nonlinear system of equations:*

$$\begin{cases} r^{n+1} - 3(1+q+p)(1+r+\ldots+r^{n-1}) \\ -4r^n = 0, \\ r^n q - 2(1+q+p)(1+r+\ldots+r^{n-1}) \\ -2r^n = 0, \\ r^n p - (1+q+p)(1+r+\ldots+r^{n-1}) \\ -r^n = 0, \qquad n = 1, 2, \ldots \end{cases} \tag{2.5}$$

Proof. Let x_n, y_n and w_n have convergent orders r, p and q respectively, i.e.,

$$e_{n+1} \sim e_n^r \sim e_{n-1}^{r^2} \sim \ldots \sim e_0^{r^{n+1}}, \tag{2.6}$$

$$e_{n,y} \sim e_n^q \sim e_{n-1}^{rq} \sim \ldots \sim e_0^{r^n q}. \tag{2.7}$$

$$e_{n,w} \sim e_n^p \sim e_{n-1}^{rp} \sim \ldots \sim e_0^{r^n p}. \tag{2.8}$$

Also, considering Lemma (2.1), we have

$$\begin{aligned} 1 + \gamma_n f'(\alpha) &\sim \prod_{k=0}^{n-1} e_k e_{k,y} e_{k,w} \\ &= (e_0 e_{0,y} e_{0,w})(e_1 e_{1,y} e_{1,w}) \ldots (e_{n-1} e_{n-1,y} e_{n-1,w}) \\ &= (e_0 e_0^q e_0^p)(e_0^r e_0^r q e_0^r p) \ldots (e_0^{r(n-1)} e_0^{rq(n-1)} e_0^{rp(n-1)}) \\ &= e_0^{(1+q+p)(1+r+r^2+\ldots+r^{n-1})}. \end{aligned} \tag{2.9}$$

Therefore,

$$\begin{aligned} e_{n,w} \sim& (1+\gamma_n f'(\alpha)) e_n \\ =& e_0^{(1+q+p)(1+r+r^2+\ldots+r^{n-1})} e_0^{r^n}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} e_{n,y} \sim& (1+\gamma_n f'(\alpha))(c_2+\beta) e_n^2 \\ =& e_0^{2(1+q+p)(1+r+r^2+\ldots+r^{n-1})} e_0^{2r^n}, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} e_{n+1} \sim& (1+\gamma_n f'(\alpha))^2 (c_2+\beta) \\ & (c_2(\beta+2c_2) - c_3) e_n^4 \\ \sim& e_0^{3(1+q+p)(1+r+r^2+\ldots+r^{n-1})} e_0^{4p^n}. \end{aligned} \tag{2.12}$$

Combining the right-hand-side (2.6)-(2.12), (2.7)-(2.11) and (2.8)-(2.10) we have the following system of nonlinear equations:

$$\begin{cases} r^{n+1} - 3(1+q+p)(1+r+\ldots+r^{n-1}) \\ -4r^n = 0, \\ r^n q - 2(1+q+p)(1+r+\ldots+r^{n-1}) \\ -2r^n = 0, \\ r^n p - (1+q+p)(1+r+\ldots+r^{n-1}) \\ -r^n = 0 \end{cases} \tag{2.13}$$

This complets the proof. □

Remark 2.1. *We have solved the nonlinear system of equations (2.13) by varying n. If $n = 1$, we use the information from the last iteration and the convergence order is* 7.0000*, and for $n = 2$ the convergence order is* 7.40515*, while if $n = 3$ the convergence order is* 7.45635*, we use the information from the last two iterations, and so on.*

Now we consider the following two-step method without memory by three parameters

$$\begin{cases} w_n = x_n + \gamma f(x_n), \\ y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n] + \beta f(w_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n) f[x_n, w_n]}{b}. \\ \quad n = 0, 1, 2, \ldots, \end{cases} \tag{2.14}$$

where
$b = f[x_n, y_n] f[y_n, w_n] + \lambda(y_n - x_n)(y_n - w_n)$
This method is optimal with convergence order 4.

Theorem 2.4. *The error equation of method (2.14) is given by*

$$e_{n+1} = \frac{d}{f'(\alpha)^2} + o(e_n^5).$$

where

$d = (1 + f'(\alpha)\gamma)^2(\beta + c_2)(\lambda + f'(\alpha)^2(c_2(\beta + 2c_2) - c_3))e_n^4$

Table 3: Numerical results for $f_3(x)$ by the methods with memory

Methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c	EI
Method (3.27) with (3.28)	0.1973(-1)	0.4185(-1)	0.7094(-3)	2.6296	1.3802
Method (3.27) with (3.29)	0.1973(-1)	0.2466(-1)	0.6695(-4)	2.8418	1.4164
Method (3.27) with (3.30)	0.1973(-1)	0.3578(-1)	0.5356(-5)	3.6013	1.5328
Method (3.27) with (3.31)	0.1973(-1)	0.7848(-1)	0.5077(-3)	4.3339	1.6303
Method (3.32)	0.1345	0.5325(-7)	0.1349(-51)	5.9941	1.8165
Method (2.4)	0.1921(-3)	0.5121(-22)	0.4932(-152)	7.0404	1.9166
Method (2.15)	0.1941(-3)	0.5593(-22)	0.8862(-168)	7.4334	1.9516

Table 4: Numerical results for $f_4(x)$ by the methods with memory

Methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c	EI
Method (3.27) with (3.28)	0.1814(-1)	0.2164(-2)	0.3004(-9)	3.7760	1.5571
Method (3.27) with (3.29)	0.1814(-1)	0.5435(-2)	0.9601(-8)	4.0711	1.5965
Method (3.27) with (3.30)	0.1814(-1)	0.5592(-2)	0.3220(-8)	4.4546	1.6453
Method (3.27) with (3.31)	0.1814(-1)	0.1762(-1)	0.7697(-7)	5.29261	1.7427
Method (3.32)	0.1991	0.6470(-6)	0.5272(-44)	6.1360	1.8307
Method (2.4)	0.1441(-4)	0.1341(-34)	0.6462(-240)	7.2688	1.9371
Method (2.15)	0.1612(-2)	0.7771(-12)	0.2281(-51)	7.9733	1.9977

To derive an adaptive method with memory, we suppose that $\gamma_n = -\dfrac{1}{f'(\alpha)} \approx -\dfrac{1}{N'_{3n}(x_n)}$, $\beta_n = -c_2 = -\dfrac{f''(\alpha)}{2f'(\alpha)} \approx -\dfrac{N''_{3n+1}(w_n)}{2N'_{3n+1}(w_n)}$ and $\lambda_n = f'(\alpha)c_3 = \dfrac{f'''(\alpha)}{6f'(\alpha)} \simeq \dfrac{N'''_{3n+2}(y_n)}{6N'_{3n+2}(y_n)}$, where $N'_{3n}(x_n)$, $N''_{3n+1}(w_n)$, and $N'''_{3n+2}(y_n)$ are Newton 's interpolation polynomials around the nodes $\{x_n, x_{n-1}, w_{n-1}, y_{n-1}\}$, $\{w_n, x_n, x_{n-1}, w_{n-1}, y_{n-1}\}$ and $\{y_n, w_n, x_n, x_{n-1}, w_{n-1}, y_{n-1}\}$, respectively.

So in each iteration, we use the following accelerators:

$$
\begin{cases}
\gamma_n = \dfrac{-1}{N'_{3n}(x_n)}, \\
\beta_n = -\dfrac{N''_{n+1}(w_n)}{2N'_{3n+1}(w_n)}, \\
\lambda_n = \dfrac{N'''_{3n+2}(y_n)}{6N'_{3n+2}(y_n)}, \\
w_n = x_n + \gamma_n f(x_n), \\
y_n = x_n - \dfrac{f(x_n)}{f[e_n, w_n] + \beta_n f(w_n)}, \\
x_{n+1} = y_n - \dfrac{f(y_n) f[x_n, w_n]}{g}. \quad n = 0, 1, 2, \ldots
\end{cases}
\tag{2.15}
$$

where $g = f[x_n, y_n] f[y_n, w_n] + \lambda_n (y_n - x_n)(y_n - w_n)$, x_0 and γ_0 are given suitably.

Lemma 2.2. *If* $\gamma_n = \dfrac{-1}{N'_{3n}(x_n)}$, $\beta_n = \dfrac{-N''_{3n+1}(w_n)}{2N'_{3n+1}(w_n)}$ *and* $\lambda_n = \dfrac{N'''_{3n+2}(y_n)}{6N'_{3n+2}(y_n)}$, *then*

$$1 + \gamma_n f'(\alpha) \sim \prod_{k=0}^{n-1} e_k e_{k,y} e_{k,w},$$

$$c_2 + \beta_n \sim \prod_{k=0}^{n-1} e_k e_{k,y} e_{k,w},$$

$$(\lambda_n + f'(\alpha)^2 (c_2(\beta_n + 2c_2) - c_3) \sim \prod_{k=0}^{n-1} e_k e_{k,y} e_{k,w}.$$

where

$$e_k = x_k - \alpha, \quad e_{k,y} = y_k - \alpha, \quad e_{k,w} = w_k - \alpha.$$

Proof. The proof is very similar to Lemma 4, [14] and Lemma 1, [19]. □

The convergence order of the adaptive method (2.15) is given in the following theorem:

Theorem 2.5. *If* $\gamma_n = \frac{-1}{N'_{3n}(x_n)}$, $\beta_n = \dfrac{-N''_{3n+1}(w_n)}{2N'_{3n+1}(w_n)}$ *and* $\lambda_n = \dfrac{N'''_{3n+2}(y_n)}{6N'_{3n+2}(y_n)}$,

then convergence order of method (2.15) is obtained from solution of the following nonlinear system of equations:

$$\begin{cases} r^{n+1} - 4(1+q+p)(1+r+\ldots+r^{n-1}) \\ -4r^n = 0, \\ r^n q - 2(1+q+p)(1+r+\ldots+r^{n-1}) \\ -2r^n = 0, \\ r^n p - (1+q+p)(1+r+\ldots+r^{n-1}) \\ -r^n = 0, \qquad n = 1, 2, \ldots \end{cases} \tag{2.16}$$

Proof. Let x_n, y_n and w_n have convergent orders r, p and q respectively, i.e.,

$$e_{n+1} \sim e_n^r \sim e_{n-1}^{r^2} \sim \ldots \sim e_0^{r^{n+1}}. \tag{2.17}$$

$$e_{n,y} \sim e_n^q \sim e_{n-1}^{rq} \sim \ldots \sim e_0^{r^n q}. \tag{2.18}$$

$$e_{n,w} \sim e_n^p \sim e_{n-1}^{rp} \sim \ldots \sim e_0^{r^n p}. \tag{2.19}$$

Also, considering Lemma (2.2), we have

$$\begin{aligned} 1 + \gamma_n f'(\alpha) &\sim \prod_{k=0}^{n-1} e_k e_{k,y} e_{k,w} \\ &= (e_0 e_{0,y} e_{0,w})(e_1 e_{1,y} e_{1,w}) \ldots (e_{n-1} e_{n-1,y} e_{n-1,w}) \\ &= (e_0 e_0^q e_0^p)(e_0^r e_0^r q e_0^r p) \ldots (e_0^{r(n-1)} e_0^{rq(n-1)} e_0^{rp(n-1)}) \\ &= e_0^{(1+q+p)(1+r+r^2+\ldots+r^{n-1})}, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} c_2 + \beta &\sim \prod_{k=0}^{n-1} e_k e_{k,y} e_{k,w} \\ &= e_0^{(1+q+p)(1+r+r^2+\ldots+r^{n-1})}, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \lambda + f'(\alpha)^2(c_2(\beta + 2c_2) - c_3) &\sim \prod_{k=0}^{n-1} e_k e_{k,y} e_{k,w} \\ &= e_0^{(1+q+p)(1+r+r^2+\ldots+r^{n-1})}. \end{aligned} \tag{2.22}$$

Therefore,

$$\begin{aligned} e_{n,w} &\sim (1 + \gamma_n f'(\alpha)) e_n \\ &= e_0^{(1+q+p)(1+r+r^2+\ldots+r^{n-1})} e_0^{r^n}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} e_{n,y} &\sim (1 + \gamma_n f'(\alpha))(c_2 + \beta_n) e_n^2 \\ &= e_0^{2(1+q+p)(1+r+r^2+\ldots+r^{n-1})} e_0^{2r^n}, \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} e_{n+1} &\sim (1 + \gamma_n f'(\alpha))^2 (c_2 + \beta_n) \\ &\quad (\lambda_n + f'(\alpha)^2 (c_2(\beta_n + 2c_2) - c_3)) e_n^4 \\ &\sim e_0^{4(1+q+p)(1+r+r^2+\ldots+r^{n-1})} e_0^{4p^n}. \end{aligned} \tag{2.25}$$

Incorporate the right-hand-side of the relations (2.17)-(2.25), (2.18)-(2.24) and (2.19)-(2.23) , we reach the following system of nonlinear equations

$$\begin{cases} r^{n+1} - 4(1+q+p)(1+r+\ldots+r^{n-1}) \\ -4r^n = 0, \\ r^n q - 2(1+q+p)(1+r+\ldots+r^{n-1}) \\ -2r^n = 0, \\ r^n p - (1+q+p)(1+r+\ldots+r^{n-1}) \\ -r^n = 0 \end{cases} \tag{2.26}$$

This complets the proof. □

Remark 2.2. *By solving the nonlinear system of equations (2.26) by varying n. If $n = 1$, we use the information from the last iteration and the convergence order is 7.530 and $n = 2$, we use the information from the last two iterations the convergence order is 7.94449., while if $n = 3$ the convergence order is 7.99315, and so on.*

3 Numerical results and comparisons

In this section, the family of methods with memory (2.13) and (2.25) are tested using four examples of nonlinear equations. The errors $\mid x_k - \alpha \mid$ of approximations to the sought zeros, are given in Tables where $a(-b)$ shows $a * 10^{-b}$. Computational order of convergence COC, in Tables $1-4$ computed by follow expresssion [12]

$$r_c = \frac{log|\frac{f(x_n)}{f(x_{n-1})}|}{log|\frac{f(x_{n-1})}{f(x_{n-2})}|}.$$

Moreover, the following test functions are used:

$$f_1(x) = (x-2)\left(x^6+x^3+1\right)e^{(-x^2)}, \quad x_0 = 1.8,\ \alpha = 2,$$
$$f_2(x) = e^{(x^2-3x)}\sin(x) + \log\left(x^2+1\right), \quad x_0 = 0.35,\ \alpha = 0,$$
$$f_3(x) = \sin(t)e^{(t^2+t\cos(t)-1)} + t\log(t+1), \quad x_0 = 0.6,\ \alpha = 0,$$
$$f_4(x) = e^{(t)}\sin(t) + \log\left(t^4-3t+1\right), \quad x_0 = 0.4,\ \alpha = 0.$$

Also, we compare our methods to some of the existing methods given as follows:

Example 3.1. *Xiaofeng Wang method method with order $2\sqrt{6} \approx 4.4495$ [16]:*

$$\begin{cases} w_n = x_n + f(x_n) \\ z_n = x_n - \dfrac{f(x_n)}{f[x_n,w_n]}, \\ y_n = z_n - \dfrac{\lambda_n(z_n-x_n)^2}{1+\lambda_n(z_n-x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)f[x_n,w_n]}{f[x_n,y_n]f[y_n,w_n]}, \end{cases} \tag{3.27}$$

where

$$\lambda_n = \frac{z_{n-1}-z_n}{(z_{n-1}-x_{n-1})^2}. \tag{3.28}$$

$$\lambda_n = \frac{z_{n-1}-x_n}{(z_n-x_{n-1})(x_n-x_{n-1})}. \tag{3.29}$$

$$\begin{aligned} \lambda_n =& \frac{(z_{n-1}-x_n)(y_{n-1}-x_{n-1})}{(x_n-x_{n-1})^3} \\ &\times \left\{ \frac{2y_{n-1}-x_n-w_{n-1}}{y_{n-1}-w_{n-1}} \right\} \\ &+ \frac{2z_n-x_n-w_n}{(y_{n-1}-x_n)(z_{n-1}-x_{n-1})} \\ &+ \frac{(z_n-z_{n-1})^2(w_{n-1}-x_{n-1})}{(z_{n-1}-x_{n-1})^2(z_n-w_{n-1})(w_{n-1}-z_{n-1})}, \end{aligned} \tag{3.30}$$

$$\begin{aligned} \lambda_n =& \frac{(x_n-z_{n-1})(2x_n-y_{n-1}-z_{n-1})}{(z_n-x_{n-1})(z_{n-1}-x_{n-1})^2} \\ &- \frac{2z_n-x_n-w_n}{(x_n-y_{n-1})(z_{n-1}-x_{n-1})} \\ &+ \frac{(w_{n-1}-y_{n-1})(x_n-z_{n-1})}{(x_n-x_{n-1})(z_{n-1}-w_{n-1})(z_n-x_{n-1})} \\ &- \frac{(z_n-z_{n-1})^2(w_{n-1}-x_{n-1})^2}{(z_{n-1}-w_{n-1})^2(z_{n-1}-x_{n-1})^3}. \end{aligned} \tag{3.31}$$

Example 3.2. *Alicia Cordero et al. method method with order 7 [2]:*

$$\begin{cases} x_0, \gamma_0, \lambda_0 \qquad w_0 = x_0 + \gamma_0 f(x_0), \\ \gamma_k = -\dfrac{1}{N_3'(x_k)}, \qquad \lambda_k = -\dfrac{N_4''(w_k)}{2N_4'(w_k)}, \\ w_k = x_k + \gamma_k f(x_k), \quad k \geqslant 1, \\ y_k = x_k - \dfrac{f(x_k)}{f[x_k,w_k]+\lambda_k f(w_k)}, \\ x_{k+1} = y_k - \dfrac{f(y_k)}{f[y_k,x_k]+f[y_k,x_k,w_k](y_k-x_k)}. \end{cases} \tag{3.32}$$

4 Conclusion

In this study, we introduced two adaptive methods with memory based on the variant of the Cordero-Torregrosa method. The technique of adaptive method uses all the available information from the first to current iterations. Consequently, it increases the efficiency index as high as possible. The new derivative free methods have possessed the swift convergence order 7.46315 and 7.99315, and they only use the information from the last two iterations. Observing the tables and examples, we found that the proposed new adaptive methods have higher convergence compared to the Wang method [16] and other methods. For the future work, similar to [6, 10], local and semi local analysis of the proposed methods can be carried out.

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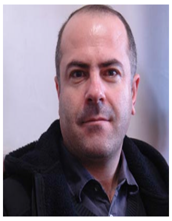
Maryam Mohamadi Zade got her PhD degree in applied mathematics from the department of applied mathematics, Hamedan Branch, in 2019. Her research interest in numerical analysis includes approximating numerical solutions using iterative methods for nonlinear systems of equations.



Taher Lotfi is an associate professor in applied mathematics and numerical analysis. His research interest includes iterative methods for approximating numerical solution of nonlinear systems of equations.

Majid Amirfakhrian is full professorship of Applied Mathematics at Azad University, Central Tehran Branch (IAUCTB). He is currently a Visiting Researcher at the University of Calgary, Canada. His main research interests lie in Numerical Analysis and Approximation. Some of his specific interests are solving the problems of Fuzzy Approximation, Data Sciences, Multivariate Approximation, Image Processing and Numerical Partial Differential Equations. He uses approximation techniques in some applied problems, but at the same time, he works on the theoretical aspects of the mathematical problems. He has side interests in Optimization and Statistics. He uses computational methods in his research.