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Research Article

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Weighted Differentiation Composition Operators from Weighted Bergman Spaces with Admissible Weights to Bloch-type Spaces

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Abstract

For an analytic self-map φ of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, a nonnegative integer n, and u analytic function on $\mathbb{D}$, weighted differentiation composition operator is defined by $(D^n_{\varphi,u}f)(z) = u(z)f^{(n)}(\varphi(z))$, where f is an analytic function on $\mathbb{D}$ and $z \in \mathbb{D}$. In this paper, we study the boundedness and compactness of $D^n_{\varphi,u}$, from weighted Bergman spaces with admissible weights to Bloch-type spaces.

Keywords : Weighted differentiation composition operator; Weighted Bergman space; Bloch-type space; Admissible weight; Boundedness; Compactness.

1 Introduction

Let $\mathcal{H}ol(\mathbb{D})$ be the class of all analytic functions in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. Let T denote the boundary of $\mathbb{D}$. A positive integrable function ω over $\mathbb{D}$, is called a weight function or simply a weight. It is radial, if $\omega(z) = \omega(|z|)$, for all $z \in \mathbb{D}$. For $0 < p < \infty$, and a weight ω, the weighted Bergman space A^p_ω consists of those $f \in \mathcal{H}ol(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\omega} := \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty,$$

where $dA(z) = \frac{dxdy}{\pi}$ stands for the normalized Lebesgue area measure on $\mathbb{D}$. As usual for standard radial weights $\omega(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, A^p_ω is the well known classical Bergman space $\mathcal{A}^p_\alpha(\mathbb{D})$ (see, e.g., [1, 2, 7, 9, 10]). It is also worth noting that the Bergman spaces with weights other than the classical weight are defined, (see, e.g., [19, 21, 22, 25, 32]).

Recall that a positive continuous function μ on $[0,1)$ is called normal, if there exist positive numbers a and b, $0 < a < b < \infty$, and $\delta \in [0,1)$ such that

- $\frac{\mu(r)}{(1-r)^a}$ is decreasing on $[\delta, 1)$ and $\lim_{r\to 1} \frac{\mu(r)}{(1-r)^a} = 0$,
- $\frac{\mu(r)}{(1-r)^b}$ is increasing on $[\delta, 1)$ and $\lim_{r\to 1} \frac{\mu(r)}{(1-r)^b} = \infty$,

were first studied by shields and williams in [30]. Note that a normal function $\mu : [0,1) \to [0,\infty)$ is decreasing in the neighborhood of 1 and satisfies $\lim_{r\to 1^-} \mu(r) = 0$. For example,

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- $\mu(r) = (1-r^2)^\alpha \quad \alpha > 0,$
- $\mu(r) = \frac{1}{\log\log \frac{e^2}{1-r^2}},$
- $\mu(r) = (1-r^2)^\alpha \log^\beta \frac{e^{\frac{\beta}{\alpha}}}{1-r^2}, \alpha > 0, \beta \geqslant 0,$

are such kind of normal functions.

For a normal function μ on $[0,1)$, the Bloch-type space $\mathcal{B}_\mu$ is the space of all $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \mu(|z|)|f'(z)| < \infty.$$

It is easy to see that $\mathcal{B}_\mu$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}_\mu} := |f(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f'(z)|.$$

For $\mu(r) = 1-r^2$, we obtain the classical Bloch space (see, e.g., [29]).

Let $u \in \mathcal{H}ol(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$, $f^{(n)}$ denote the n-th derivative of f and $f^{(0)} = f$. For $f \in \mathcal{H}ol(\mathbb{D})$, the weighted differentiation composition operator is defined by

$$(D^n_{\varphi,u} f)(z) := u(z) f^{(n)}(\varphi(z)), \quad z \in \mathbb{D}.$$

In fact, if $n = 0$ and $u(z) = 1$, then $D^n_{\varphi,u}$ is the composition operator C_φ. If $u(z) = 1$, then $D^n_{\varphi,u}$ is the operator $C_\varphi D^n$, which was studied, for example, in [11, 24, 33]. If $n = 0$, then $D^n_{\varphi,u}$ is just the weighted composition operator uC_φ. If $n = 1$ and $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$, which was studied in [11, 13, 15, 16, 17, 24, 26]. The operator $D^n_{\varphi,u}$ was introduced by Zhu in [36], and studied in [12, 14, 23, 26, 27, 28, 31, 34, 35, 36, 37].

This paper focuses on the boundedness and compactness of $D^n_{\varphi,u}$ from A^p_ω spaces with admissible weights to the Bloch-type spaces. Note that, throughout the remainder of this paper, C will stand for a positive constant and may differ from one occurrence to the other. As usual, for two positive real-valued functions f_1 and f_2 we write $f_1 \preceq f_2$, if there exists a positive constant C independent of the parameters such that $f_1 \leqslant C f_2$, and from $f_1 \asymp f_2$ is understood that both $f_1 \preceq f_2$ and $f_2 \preceq f_1$ hold.

2 Preliminaries

Let us first gather some auxiliary definitions and lemmas.

The distortion function of a radial weight $\omega : [0,1) \to (0,\infty)$ is defined by

$$\psi_\omega(r) = \frac{1}{\omega(r)} \int_r^1 \omega(s)ds, \quad 0 \leqslant r < 1,$$

and was introduced by siskakis in [20]. A radial weight ω is called regular, if ω is continuous and its distortion function satisfies

$$\psi_\omega(r) \asymp (1-r), \quad 0 \leqslant r < 1.$$

The class of all regular weights is denoted by $\mathcal{R}$. By [19, Section 1.4], if $\omega \in \mathcal{R}$, Then

$$\frac{\omega(r)}{\omega(t)} \asymp 1, \quad 0 \leqslant r \leqslant t \leqslant r + s(1-r) < 1, \quad (2.1)$$

where the constants of comparison depend on $s \in [0,1)$ and ω. This implies

$$\psi_\omega(r) \geqslant C(1-r), \quad 0 \leqslant r < 1,$$

for some constant $C > 0$. However does not imply the existence of $C > 0$ such that $\psi_\omega(r) \leqslant C(1-r)$, $0 \leqslant r < 1$. Note that a normal continuous weight ω is regular and for regular weight ω, the weighted Bergman space A^p_ω lies between two classical weighted Bergman spaces [19].

We will use the following lemma for constructing two family of test functions that allows us to show main results.

Lemma 2.1 *[19] If $\omega \in \mathcal{R}$, then there exists $\gamma_0 = \gamma_0(\omega)$ such that*

$$\int_D \frac{\omega(z)}{|1-\bar{\lambda}z|^{\gamma+1}} dA(z) \asymp \frac{\int_{|\lambda|}^1 \omega(r)dr}{(1-|\lambda|)^\gamma} \asymp \frac{\omega(\lambda)}{(1-|\lambda|)^{\gamma-1}}$$

for all $\gamma > \gamma_0$.

The proof of [19, Lemma 1.1] gives

$$\lim_{|\lambda|\to 1} \frac{(1-|\lambda|)^\gamma}{\int_{|\lambda|}^1 \omega(r)rdr} = 0, \quad (2.2)$$

if $\gamma > \gamma_0$ is enough large.

Bekollé and Bonami introduced the following set of weights in [3, 4].

Let $1 < p_0, p_0' < \infty$ such that $\frac{1}{p_0} + \frac{1}{p_0'} = 1$, and let $\eta > -1$. The class Bekollé-Bonami $\mathcal{B}_{p_0}(\eta)$

consists of the weights ω with the property that there exists a constant $C > 0$ such that

$$(\int_{S(I)} \omega(z)(1-|z|)^{\eta} dA(z)) \times (\int_{S(I)} \omega^{\frac{-p_0'}{p_0}}(z)(1-|z|)^{\eta} dA(z))^{\frac{p_0}{p_0'}} \leqslant C|I|^{(1+\eta)p_0},$$

for any Carleson square

$$S(I) = \{re^{i\theta} \in \mathbb{D} : e^{i\theta} \in I, 1-|I| \leqslant r < 1\},$$

where $|I|$ denote the Lebesgue measure of the measurable set $I \subset T$. Bekollé and Bonami showed that $\omega \in \mathcal{B}_{p_0}(\eta)$ if and only if the Bergman projection

$$P_{\eta}(f)(z) = (1+\eta)\int_D \frac{f(\xi)}{(1-\bar{\xi}z)^{\eta+2}}(1-|\xi|^2)^{\eta} dA(\xi),$$

is bounded from $L^{p_0}_{\omega}$ to $A^{p_0}_{\omega}$ [4]. Thus the dual space of $A^{p_0}_{\omega}$ is $A^{p_0'}_{\omega}$. Authors in [19, Lemma 1.4] showed that, if $\omega \in \mathcal{R}$, then for each $p_0 > 1$ there exists $\eta > -1$ depend only on p_0 and ω such that $\frac{\omega(z)}{(1-|z|)^{\eta}}$ belongs to $\mathcal{B}_{p_0}(\eta)$ and , if a countinous radial weight ω satisfies (2.1) and $\frac{\omega(z)}{(1-|z|)^{\eta}}$ belongs to $\mathcal{B}_{p_0}(\eta)$ for some $p_0 > 0$ and $\eta > -1$, then $\omega \in \mathcal{R}$.

In order to state our main results we introduce the following larger class of weights.

For $p_0 > 1$, a weight function ω belongs to $\mathcal{C}_{p_0}$, if there exists a constant $C > 0$ such that

$$(\int_{D_{\lambda,\alpha}} \omega(z) dA(z)) \times (\int_{D_{\lambda,\alpha}} \omega^{\frac{-p_0'}{p_0}}(z) dA(z))^{\frac{p_0}{p_0'}} \leqslant CA^{p_0}(D_{\lambda,\alpha}),$$

for every disc $D_{\lambda,\alpha} = \{z \in \mathbb{D} : |z-\lambda| < \alpha(1-|\lambda|)\}$, where $\frac{1}{p_0} + \frac{1}{p_0'} = 1$. Here $\alpha \in (0,1)$ is fixed, but the class $\mathcal{C}_{p_0}$ is actually independent of $\alpha \in (0,1)$. Moreover, $\mathcal{B}_{p_0}(\eta) \subset \mathcal{C}_{p_0}$ for every $\eta > -1$ and the inclusion is strict [5, 18].

We fix the basic notation and introduce the following classes of radial weights that are used in the remainder of this paper.

A weight function ω is called admissible weight if

(ω_1) $\frac{\omega(r)}{\omega(t)} \asymp 1$, where $0 \leqslant r \leqslant t \leqslant r + s(1-r) < 1$, $s \in [0,1)$.

(ω_2) $\frac{\omega(z)}{(1-|z|)^{\eta}}$ belongs to $\mathcal{B}_{p_0}(\eta)$ for some $p_0 > 0$ and $\eta > -1$.

(ω_3) $\frac{\omega(z)}{(1-|z|)^{\eta}} \in C^2[0,1)$ and $\Delta(\frac{\omega(z)}{(1-|z|)^{\eta}}) \geqslant 0$, where Δ stands for the Laplace operator.

Since $\omega \in C^2[0,1)$, (ω_1) and (ω_2) imply that $\omega \in \mathcal{R}$.

Let now γ_0 be the constant in Lemma 2.1 and let $a, p > 1$ be the constant large enough such that $ap + 1 > \gamma_0$. Then for each $\lambda \in \mathbb{D}$, define the functions

$$f_{\lambda,p}(z) = \frac{(1-|\lambda|)^a}{(\omega(\lambda))^{\frac{1}{p}}(1-\bar{\lambda}z)^{a+\frac{2}{p}}},$$

and

$$h_{\lambda,p}(z) = (\frac{1-|\lambda|}{1-\bar{\lambda}z}) f_{\lambda,p}(z).$$

Lemma 2.1 immediately yields the following lemma.

Lemma 2.2 *Let ω be a regular weight. Then we have $\|f_{\lambda,p}\|_{A^p_{\omega}} \asymp 1$, and $\|h_{\lambda,p}\|_{A^p_{\omega}} \asymp 1$.*

Proof. Using Lemma 2.1, we have

$$\int_D \frac{\omega(z) dA(z)}{|1-\bar{\lambda}z|^{ap+2}} \asymp \frac{\omega(\lambda)}{(1-|\lambda|)^{ap}}.$$

Then

$$\begin{aligned}\|f_{\lambda,p}\|^p_{A^p_{\omega}} &= \frac{(1-|\lambda|)^{ap}}{\omega(\lambda)}(\int_{\mathbb{D}} \frac{\omega(z) dA(z)}{|1-\bar{\lambda}z|^{ap+2}}) \\ &\asymp \frac{(1-|\lambda|)^{ap}}{\omega(\lambda)} \times \frac{\omega(\lambda)}{(1-|\lambda|)^{ap}} \\ &\asymp 1.\end{aligned}$$

Also

$$\begin{aligned}\|h_{\lambda,p}\|^p_{A^p_{\omega}} &= \int_D \frac{(1-|\lambda|)^p}{|1-\bar{\lambda}z|^p} |f_{\lambda,p}|^p \omega(z) dA(z) \\ &\asymp \frac{(1-|\lambda|)^{ap+p}}{\omega(\lambda)}(\int_{\mathbb{D}} \frac{\omega(z) dA(z)}{|1-\bar{\lambda}z|^{ap+p+2}}) \\ &\asymp 1.\end{aligned}$$

We will use of the following characterization for A^p_{ω} functions.

Lemma 2.3 *Let ω be an admissible weight. Then for $f \in A^p_\omega$, we have*

$$|f^{(n)}(z)| \leqslant C \frac{\|f\|_{A_{\omega^p}}}{(\omega(z))^{\frac{1}{p}}(1-|z|^2)^{n+\frac{2}{p}}}, \tag{2.3}$$

with constant independent of f.

Proof. For f analytic in $\mathbb{D}$, by the Cauchy formula together with subharmonicity, and, subsequently, by Holder's inequality we have

$$\begin{aligned}
|f^{(n)}(z)|^{\frac{p}{p_0}} &\leqslant \frac{C}{(1-|z|^2)^{2+\frac{np}{p_0}}} \int_{D_{z,\alpha}} |f|^{\frac{p}{p_0}} dA \\
&\leqslant \frac{C}{(1-|z|^2)^{2+\frac{np}{p_0}}} \\
&\times \Big(\int_{D_{z,\alpha}} |f(\xi)|^p \frac{\omega(\xi)}{(1-|\xi|)^\eta} dA(\xi)\Big)^{\frac{1}{p_0}}
\end{aligned}$$

1p˙0’. $\times \Big(\int_{D_{z,\alpha}} \big(\frac{\omega(\xi)}{(1-|\xi|)^\eta}\big)^{-\frac{p_0'}{p_0}} dA(\xi)\Big)$

Since $\mathcal{B}_{p_0}(\eta) \subset \mathcal{C}_{p_0}$ for every $\eta > -1$, therefore $\frac{\omega(\xi)}{(1-|\xi|)^\eta}$ belongs to $\mathcal{C}_{p_0}$ and we deduce

$$\begin{aligned}
|f^{(n)}(z)|^p &\preceq C \frac{A^{p_0}(D_{z,\alpha})}{(1-|z|^2)^{2p_0+np}} \\
&\times \Big(\frac{1}{(1-|z|)^\eta} \int_{D_{z,\alpha}} |f|^p \omega dA\Big) \\
&\times \Big(\int_{D_{z,\alpha}} \frac{\omega(\xi)}{(1-|\xi|)^\eta} dA(\xi)\Big)^{-1} \\
&\asymp C \frac{(\int_{D_{z,\alpha}} \frac{\omega(\xi)}{(1-|\xi|)^\eta} dA(\xi))^{-1}}{(1-|z|^2)^{np+\eta}} \\
&\times \int_{D_{z,\alpha}} |f|^p \omega dA,
\end{aligned} \tag{2.4}$$

here we have used the estimate $A(D_{z,\alpha}) \asymp (1-|z|^2)^2$. We also notice that by (ω_3), $\frac{\omega(\xi)}{(1-|\xi|)^\eta}$ is subharmonic on $\mathbb{D}$ (see [8]). Thus

$$\begin{aligned}
\frac{\omega(z)}{(1-|z|)^\eta} &\leqslant \frac{1}{\pi\alpha^2(1-|z|)^2} \\
&\times \int_{D_{z,\alpha}} \frac{\omega(\xi)}{(1-|\xi|)^\eta} dA(\xi).
\end{aligned} \tag{2.5}$$

Using estimates (2.4) and (2.5) we deduce (2.3). The proof lemma 2.4 is based on a well-known method that has been used in the existing literature, see, for example [6].

Lemma 2.4 *Let $p > 1$ and ω be an admissible weight. Assume that φ is an analytic self-map of $\mathbb{D}$, $u \in \mathcal{H}ol(\mathbb{D})$, μ is normal and n is a nonnegative integer. Then $D^n_{\varphi,u} : A^p_\omega \to \mathcal{B}_\mu$ is compact if and only if $D^n_{\varphi,u} : A^p_\omega \to \mathcal{B}_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in A^p_ω which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|D^n_{\varphi,u} f_k\|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$.*

3 Main Results

In this section we give our main results and proofs.

Theorem 3.1 *Assume that φ is an analytic self-map of $\mathbb{D}$, $u \in \mathcal{H}ol(\mathbb{D})$, μ is normal and n is a nonnegative integer. Let ω be an admissible weight. Then $D^n_{\varphi,u} : A^p_\omega \to \mathcal{B}_\mu$ is bounded if and only if*

$$\sup_{z\in D} \frac{\mu(|z|)|u'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+\frac{2}{p}}} < \infty, \tag{3.6}$$

$$\sup_{z\in D} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+1+\frac{2}{p}}} < \infty. \tag{3.7}$$

Proof. Suppose that $D^n_{\varphi,u} : A^p_\omega \to \mathcal{B}_\mu$ is bounded. Let us first show (3.6). By Lemma 2.2, we see that for each $\lambda \in \mathbb{D}$, $f_{\varphi(\lambda),p}, h_{\varphi(\lambda),p} \in A^p_\omega$. Moreover $\|f_{\varphi(\lambda),p}\|_{A^p_\omega}$ and $\|h_{\varphi(\lambda),p}\|_{A^p_\omega}$ are bounded by constants independent of λ. The boundedness of $D^n_{\varphi,u} : A^p_\omega \to \mathcal{B}_\mu$, implies that

$$\begin{aligned}
\sup_{\lambda\in\mathbb{D}} \|D^n_{\varphi,u} f_{\varphi(\lambda),p}\|_{\mathcal{B}_\mu} &\leqslant \|D^n_{\varphi,u}\| \sup_{\lambda\in\mathbb{D}} \|f_{\varphi(\lambda),p}\|_{A^p_\omega} \\
&\leqslant C\|D^n_{\varphi,u}\| < \infty,
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\lambda\in\mathbb{D}} \|D^n_{\varphi,u} h_{\varphi(\lambda),p}\|_{\mathcal{B}_\mu} &\leqslant \|D^n_{\varphi,u}\| \sup_{\lambda\in\mathbb{D}} \|h_{\varphi(\lambda),p}\|_{A^p_\omega} \\
&\leqslant C\|D^n_{\varphi,u}\| < \infty,
\end{aligned}$$

as desired. Also by elementary calculations, we see that

$$\begin{aligned}
f^{(n)}_{\varphi(\lambda),p}(z) &= \prod_{i=0}^{n-1} \Big(a + \frac{2}{p} + i\Big) \\
&\times \frac{(1-|\varphi(\lambda)|^2)^a \overline{\varphi(\lambda)}^n}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-\overline{\varphi(\lambda)}z)^{a+n+\frac{2}{p}}},
\end{aligned} \tag{3.8}$$

and

$$h^{(n)}_{\varphi(\lambda),p}(z) = \prod_{i=1}^{n}(a+\frac{2}{p}+i) \times \frac{(1-|\varphi(\lambda)|^2)^{a+1}\overline{\varphi(\lambda)}^n}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-\overline{\varphi(\lambda)}z)^{a+n+1+\frac{2}{p}}}. \tag{3.9}$$

We use the fact that

$$(D^n_{\varphi,u}f)'(z) = u(z)\varphi'(z)f^{(n+1)}(\varphi(z)) + u'(z)f^{(n)}(\varphi(z))$$

with (3.8), for any $\lambda \in \mathbb{D}$ we obtain

$$(D^n_{\varphi,u}f_{\varphi(\lambda),p})'(\lambda) = [\prod_{i=0}^{n}(a+\frac{2}{p}+i) \times \frac{\overline{\varphi(\lambda)}^{n+1}\varphi'(\lambda)u(\lambda)}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+1+\frac{2}{p}}}] + [\prod_{i=0}^{n-1}(a+\frac{2}{p}+i) \times \frac{\overline{\varphi(\lambda)}^{n}u'(\lambda)}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+\frac{2}{p}}}], \tag{3.10}$$

and

$$(D^n_{\varphi,u}h_{\varphi(\lambda),p})'(\lambda) = [\prod_{i=1}^{n+1}(a+\frac{2}{p}+i) \times \frac{\overline{\varphi(\lambda)}^{n+1}\varphi'(\lambda)u(\lambda)}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+1+\frac{2}{p}}}] + [\prod_{i=1}^{n}(a+\frac{2}{p}+i) \times \frac{\overline{\varphi(\lambda)}^{n}u'(\lambda)}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+\frac{2}{p}}}]. \tag{3.11}$$

Multiplying both sides of equality (3.10) by $\mu(|\lambda|)$ then using the triangle inequality to obtain

$$\frac{\mu(|\lambda|)|\varphi(\lambda)|^n|u'(\lambda)|}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+\frac{2}{p}}} \leqslant \frac{\|D^n_{\varphi,u}f_{\varphi(\lambda),p}\|_{\mathcal{B}_\mu}}{\prod_{i=0}^{n-1}(a+\frac{2}{p}+i)} + \frac{(a+\frac{2}{p}+n)\mu(|\lambda|)|\varphi(\lambda)|^{n+1}|\varphi'(\lambda)u(\lambda)|}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+1+\frac{2}{p}}}. \tag{3.12}$$

In addition, we first multiply (3.10) by $a+\frac{2}{p}+n$ and (3.11) by $a+\frac{2}{p}$, then subtract such obtained equalities and subsequently, we use the triangle inequality to obtain

$$\prod_{i=0}^{n}(a+\frac{2}{p}+i)\frac{|\varphi(\lambda)|^{n+1}|\varphi'(\lambda)u(\lambda)|}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+1+\frac{2}{p}}} \leqslant (a+\frac{2}{p}+n)|(D^n_{\varphi,u}f_{\varphi(\lambda),p})'(\lambda)| + (a+\frac{2}{p})|(D^n_{\varphi,u}h_{\varphi(\lambda),p})'(\lambda)|. \tag{3.13}$$

Hence, multiplying both sides of inequality (3.13) by $\mu(|\lambda|)$, we get

$$\frac{\mu(|\lambda|)|\varphi(\lambda)|^{n+1}|\varphi'(\lambda)u(\lambda)|}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+1+\frac{2}{p}}} \leqslant \frac{1}{\prod_{i=0}^{n-1}(a+\frac{2}{p}+i)}\|D^n_{\varphi,u}f_{\varphi(\lambda),p}\|_{\mathcal{B}_\mu} + \frac{1}{\prod_{i=1}^{n}(a+\frac{2}{p}+i)}\|D^n_{\varphi,u}h_{\varphi(\lambda),p}\|_{\mathcal{B}_\mu}. \tag{3.14}$$

Replace now inequality (3.14) in (3.12), we deduce

$$\frac{\mu(|\lambda|)|\varphi(\lambda)|^{n}|u'(\lambda)|}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+\frac{2}{p}}} \leqslant \frac{a+\frac{2p}{+}n+1}{\prod_{i=0}^{n-1}(a+\frac{2}{p}+i)}\|D^n_{\varphi,u}f_{\varphi(\lambda),p}\|_{\mathcal{B}_\mu} + \frac{1}{\prod_{i=1}^{n-1}(a+\frac{2p}{+}i)}\|D^n_{\varphi,u}h_{\varphi(\lambda),p}\|_{\mathcal{B}_\mu}. \tag{3.15}$$

Let $r \in (0,1)$ be fix. If $|\varphi(\lambda)| > r$, then from (3.15) we obtain

$$\frac{\mu(|\lambda|)|u'(\lambda)|}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+\frac{2}{p}}} \leqslant \frac{a+\frac{2p}{+}n+1}{r^n\prod_{i=0}^{n-1}(a+\frac{2}{p}+i)}\|D^n_{\varphi,u}f_{\varphi(\lambda),p}\|_{\mathcal{B}_\mu} + \frac{1}{r^n\prod_{i=1}^{n-1}(a+\frac{2}{p}+i)}\|D^n_{\varphi,u}h_{\varphi(\lambda),p}\|_{\mathcal{B}_\mu} < \infty. \tag{3.16}$$

Let us now turn to the case $|\varphi(\lambda)| \leqslant r$. For this we take the function $f(z) = z^n$, which belong to A^p_ω. So by the boundedness of $D^n_{\varphi,u}$ we have

$$L_1 := \sup_{z\in\mathbb{D}}\mu(|z|)|u'(z)| < \infty. \tag{3.17}$$

Hence, by the condition (ω_1) we get

$$\frac{\mu(|\lambda|)|u'(\lambda)|}{(\omega(\varphi(\lambda)))^{\frac{1}{p}}(1-|\varphi(\lambda)|^2)^{n+\frac{2}{p}}} \preceq \frac{1}{(\omega(r))^{\frac{1}{p}}(1-r^2)^{n+\frac{2}{p}}} \sup_{\lambda\in\mathbb{D}} \mu(|\lambda|)|u'(\lambda)| < \infty. \tag{3.18}$$

Take (3.16) and (3.18) we deduce (3.6). To show (3.7), taking the function $f(z) = z^{n+1}$. By the boundedness of $D^n_{\varphi,u}$, and (3.17) we get

$$L_2 := \sup_{z\in\mathbb{D}} \mu(|z|)|u(z)\varphi'(z)| < \infty. \tag{3.19}$$

Now using similar arguments, from (3.14) and (3.19) we can obtain (3.7).

It remains to prove the sufficiency of (3.6) and (3.7). For any $z \in \mathbb{D}$ and $f \in A^p_\omega$ we have

$$\begin{aligned}\mu(|z|)|(D^n_{\varphi,u}f)'(z)| &\leqslant \mu(|z|)|f^{(n+1)}(\varphi(z))\varphi'(z)u(z) + f^{(n)}(\varphi(z))u'(z)| \\ &\leqslant C\Big(\frac{\mu(|z|)|u(z)\varphi'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+1+\frac{2}{p}}} + \frac{\mu(|z|)|u'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+\frac{2}{p}}}\Big)\|f\|_{A^p_\omega},\end{aligned}$$

where the last step above follows by Lemma 2.3. Taking the supremum with respect to $z \in \mathbb{D}$, then applying (3.6) and (3.7), we deduce

$$\sup_{z\in\mathbb{D}} \mu(|z|)|(D^n_{\varphi,u}f)'(z)| < \infty.$$

Also

$$|f^{(n)}(\varphi(0))u(0)| \leqslant \frac{C|u(0)|\|f\|_{A^p_\omega}}{(\omega(\varphi(0)))^{\frac{1}{p}}(1-|\varphi(0)|^2)^{n+\frac{2}{p}}},$$

we get that $D^n_{\varphi,u}$ is bounded from A^p_ω to $\mathcal{B}_\mu$.

Theorem 3.2 *Assume that φ is an analytic self-map of $\mathbb{D}$, $u \in \mathcal{H}ol(\mathbb{D})$, μ is normal and n is a nonnegative integer. Let ω be an admissible weight. Then $D^n_{\varphi,u} : A^p_\omega \to \mathcal{B}_\mu$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|u'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+\frac{2}{p}}} = 0, \tag{3.20}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+1+\frac{2}{p}}} = 0. \tag{3.21}$$

Proof. Let us first show the necessity. Suppose that $D^n_{\varphi,u} : A^p_\omega \to \mathcal{B}_\mu$ is compact. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty}|\varphi(\lambda_k)| = 1$ (if such sequence does not exist then conditions (3.20) and (3.21) are vacuously satisfied). Clearly $(f_{\varphi(\lambda_k),p})_{k\in\mathbb{N}}$ and $(h_{\varphi(\lambda_k),p})_{k\in\mathbb{N}}$ are bounded in A^p_ω and converge to zero, uniformly on the compact subsets of $\mathbb{D}$, because by (ω_1) and (ω_2), ω is regular hence using (2.2), for $ap+1 > \gamma_0$ large enough, we deduce

$$\frac{(1-r)^{ap}}{\omega(r)} = \frac{(1-r)^{ap}}{\frac{1}{\psi_\omega(r)}\int_r^1 \omega(s)ds} \preceq \frac{(1-r)^{ap+1}}{\int_r^1 \omega(s)sds} \to 0,$$

as $r \to 1$. On the other hand, for z in compact subset K of $\mathbb{D}$ we have $|z| \leqslant t$ for some t, since

$$\begin{aligned}|f_{\varphi(\lambda_k),p}(z)| &\leqslant \frac{C(1-|\varphi(\lambda_k)|^2)^a}{(\omega(\varphi(\lambda_k)))^{\frac{1}{p}}(1-\overline{\varphi(\lambda_k)}z)^{a+\frac{2}{p}}} \\ &\leqslant \frac{C(1-|\varphi(\lambda_k)|^2)^a}{(\omega(\varphi(\lambda_k)))^{\frac{1}{p}}(1-t)^{a+\frac{2}{p}}}\end{aligned} \tag{3.22}$$

therefore, $|f_{\varphi(\lambda_k),p}| \to 0$ as $k \to \infty$, on the K. So the compactness of $D^n_{\varphi,u}$ implies

$$\|D^n_{\varphi,u}f_{\varphi(\lambda_k),p}\|_{\mathcal{B}_\mu} \to 0,$$

and

$$\|D^n_{\varphi,u}h_{\varphi(\lambda_k),p}\|_{\mathcal{B}_\mu} \to 0,$$

as $k \to \infty$, from lemma 2.4. Thus, using the inequality (3.14), we obtain

$$\begin{aligned}&\frac{\mu(|\lambda_k|)|\varphi'(\lambda_k)u(\lambda_k)|}{(\omega(\varphi(\lambda_k)))^{\frac{1}{p}}(1-|\varphi(\lambda_k)|^2)^{n+1+\frac{2}{p}}} \leqslant \\ &\Big(\frac{1}{|\varphi(\lambda_k)|^{n+1}\prod_{i=0}^{n-1}(a+\frac{2}{p}+i)} \times \|D^n_{\varphi,u}f_{\varphi(\lambda_k),p}\|_{\mathcal{B}_\mu}\Big) \\ &+\Big(\frac{1}{|\varphi(\lambda_k)|^{n+1}\prod_{i=1}^{n}(a+\frac{2}{p}+i)} \times \|D^n_{\varphi,u}h_{\varphi(\lambda_k),p}\|_{\mathcal{B}_\mu}\Big) \\ &\to 0,\end{aligned}$$

as $k \to \infty$. In addition, applying the inequality (3.15), we get

$$\frac{\mu(|\lambda_k|)|u'(\lambda_k)|}{(\omega(\varphi(\lambda_k)))^{\frac{1}{p}}(1-|\varphi(\lambda_k)|^2)^{n+\frac{2}{p}}} \leqslant \Big(\frac{a+\frac{2}{p}+n+1}{|\varphi(\lambda_k)|^n\prod_{i=0}^{n-1}(a+\frac{2}{p}+i)} \times \|D^n_{\varphi,u}f_{\varphi(\lambda_k),p}\|_{\mathcal{B}_\mu}\Big) + \Big(\frac{1}{|\varphi(\lambda_k)|^n\prod_{i=1}^{n-1}(a+\frac{2}{p}+i)} \times \|D^n_{\varphi,u}h_{\varphi(\lambda_k),p}\|_{\mathcal{B}_\mu}\Big) \to 0,$$

as $k \to \infty$. as $k \to \infty$.

Conversely, to prove that $D^n_{\varphi,u}$ is compact, it suffices to show that $D^n_{\varphi,u}f_k \to 0$ in $\mathcal{B}_\mu$ whenever $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A^p_ω which converges to zero uniformly on compact subsets of $\mathbb{D}$, by lemma 2.4. Suppose (3.20) and (3.21) hold and let $(f_k)_{k\in\mathbb{N}}$ be as above. Then for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that

$$\frac{\mu(|z|)|u'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+\frac{2}{p}}} < \varepsilon, \tag{3.23}$$

$$\frac{\mu(|z|)|u(z)\varphi'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+1+\frac{2}{p}}} < \varepsilon, \tag{3.24}$$

when $\delta < |\varphi(z)| < 1$. Since $D^n_{\varphi,u}$ is bounded, in a way similar to the proof of Theorem 3.1, by relations (3.17), (3.19), (3.23) and (3.24), also per Lemma 2.3, for $\Omega = \{z \in \mathbb{D} : |\varphi(z)| \leqslant \delta\}$, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\mu(|z|)|(D^n_{\varphi,u}f_k)'(z)| \leqslant \\
&\quad \sup_{z\in\Omega}\mu(|z|)|f_k^{(n+1)}(\varphi(z))\varphi'(z)u(z)| \\
&\quad + \sup_{z\in\Omega}\mu(|z|)|f_k^{(n)}(\varphi(z))u'(z)| \\
&\quad + C\Big(\sup_{z\in\mathbb{D}\setminus\Omega}\frac{\mu(|z|)|u'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+\frac{2}{p}}} \\
&\quad + \sup_{z\in\mathbb{D}\setminus\Omega}\frac{\mu(|z|)|u(z)\varphi'(z)|}{(\omega(\varphi(z)))^{\frac{1}{p}}(1-|\varphi(z)|^2)^{n+1+\frac{2}{p}}}\Big) \\
&\quad \times \|f_k\|_{A^p_\omega} \\
&\leqslant L_2\sup_{z\in\Omega}|f_k^{(n+1)}(\varphi(z))| + L_1\sup_{z\in\Omega}|f_k^{(n)}(\varphi(z))| \\
&\quad + C\varepsilon\|f_k\|_{A^p_\omega}.
\end{aligned} \tag{3.25}$$

Since $(f_k)_{k\in\mathbb{N}}$ converges uniformly to zero on the compact subsets $\{\varphi(0)\}$ and $|z| \leqslant \delta$, applying Cauchy's estimate we deduce $f_k^{(n)} \to 0$ as $k \to \infty$, on compact subsets $\{\varphi(0)\}$ and $|z| \leqslant \delta$. Using our last considerations in (3.25), we have $\lim_{k\to\infty}\|D^n_{\varphi,u}f_k\|_{\mathcal{B}_\mu} = 0$. As ε was arbitrary chosen, we obtain that $D^n_{\varphi,u}$ is compact.

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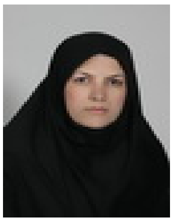

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