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Research Article

A Numerical Algorithm for Solving Impulsive Fuzzy Initial Value Problem Based on Fuzzy Methods

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Abstract

In this paper, first the Newton's divided difference interpolation method based on the gH difference on fuzzy data is introduced. Then the numerical methods entitled fuzzy Euler and modified fuzzy Euler are used to solve fuzzy impulsive initial value problem. Moreover the algorithms for the fuzzy impulsive initial value problem are explained and their local truncation errors are obtained in details. Finally, for more illustration some numerical examples are solved.

Keywords : Generated of Hukuhara difference; Impulsive fuzzy differential equation; Fuzzy Modified Euler's method; Multline.

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1 Introduction

The purpose of this paper is to introduce a numerical algorithm for solving the great categories of impulsive fuzzy differential equations. Analytical solving of these equations is complex or impossible. Indeed, until now, was not introduced any numerical method for solving this equations. In recent years, with introducing generated Hukuhara difference, a useful way for solving fuzzy initial value problems by the fuzzy numerical methods is created. In most recently, [2] by using gH difference has obtained full fuzzy Euler's method for solving fuzzy IVP. In this paper, first with the idea taken from that paper [2] and using gH difference, we obtained fuzzy modified Euler's method. In the following, we presented a numerical algorithm for solving impulsive fuzzy initial value problems with obtaining full fuzzy method. The algorithm is discussed in detail and used for solving three numerical examples, and the numerical results were shown in the local truncation error tables and plotted graphs.

In this paper, we consider First-order impulsive differential equation [4, 5, 7, 8, 11] by fuzzy initial value [9], So we have First-order impulsive fuzzy differential equation as the following

$$y'_{gH}(t) = f(t, y(t)),\ t \in J = [0, T],\quad t \neq t_k,\ k = 1, 2, \dots, N, \tag{1.1}$$

$$\triangle y|_{t=t_k} = I_k(y(t_k)), \quad (t = t_k), \tag{1.2}$$

$$y(t_0) = y_0. \tag{1.3}$$

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Where $y_0 \in \mathbb{R}_F$ and $\mathbb{R}_F$ is the set of fuzzy numbers, by considering $J = [0, T]$, $f : J \times \mathbb{R}_F \to \mathbb{R}_F$ and $I_k : \mathbb{R}_F \to \mathbb{R}_F$, $k = 1, \cdots, N$ are given functions. We have $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = T$, and $\triangle y|_{t=t_k} = y(t_k^+) \ominus_{gH} y(t_k^-)$, $k = 1, \cdots, N$, $y(t_k^+)$ and $y(t_k^-)$ represent the right and left limits of $y(t)$ at $t = t_k$, where $\ominus_{gH}$ is generated Hukuhara difference.

2 Basic preliminaries

In this section, we recall the basic definitions and theorems that we need in the paper. The follows results are well known:

Definition 2.1. *[3] The generalized Hukuhara difference of two fuzzy numbers is defined by*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} & (i)\, u = v \oplus w; \\ or & (ii)\, u = v \oplus (-1)w. \end{cases}$$

The conditions for the existence of $u \ominus_{gH} v \in \mathbb{R}_F$ are given in [3].

Definition 2.2. *[3] Let be two fuzzy numbers; then*

(i) *If the gH-difference exists, it is unique;*

(ii) *$u \ominus_{gH} v = u \ominus_H v$ or $u \ominus_{gH} v = -(v \ominus_H u)$ whenever the expression on the right exist; in particular, $u \ominus_{gH} u = u \ominus_H u = 0$;*

(iii) *If $u \ominus_{gH} v$ exists in the sense (i), then $v \ominus_{gH} u$ exist in the sense (ii) and vice versa;*

(iv) *$(u + v) \ominus_{gH} v = u$;*

(v) *$0 \ominus_{gH} (u \ominus_{gH} v) = v \ominus_{gH} u$;*

(vi) *$u \ominus_{gH} v = v \ominus_{gH} u = w$ if and only if $w = -w$ furthermore $w = 0$ if and only if $u = v$.*

Definition 2.3. *[3] The Hausdorff distance between fuzzy numbers is given by $d : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \times \{0\}$ as $D(u, v) = \sup\max_{\alpha \in [0,1]}\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\}$. The metric space $(\mathbb{R}_F, d)$ is complete, separable and following properties of the metric are valid:*

1. $d(u \oplus w, v \oplus w) = d(u, v), \forall u, v, w \in \mathbb{R}_F,$
2. $d(u \oplus w, w \oplus z) \le d(u, v) + d(w, z),$
   $\forall u, v, w, z \in \mathbb{R}_F,$
3. $d(\lambda u, \lambda v) = |\lambda| d(u, v), \forall \lambda \in \mathbb{R}, u, v \in \mathbb{R}_F.$

Definition 2.4. *[6] For any decision maker, whether pessimistic ($\alpha = 0$), optimistic ($\alpha = 1$), or neutral ($\alpha = 0.5$), the ranking function of the trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of real numbers is defined as $R(\tilde{A}) = \sqrt{x_0^2 + y_0^2}$ which is the Euclidean distance from the Circumcenter of the Centroids and the original point. Using the above definitions we define ranking between fuzzy numbers as follows Let $\tilde{A}$ and $\tilde{A}_j$ two fuzzy numbers, then*

(i) *If $R(\tilde{A}_i) > R(\tilde{A}_j)$, then $\tilde{A}_j > \tilde{A}_j$;*

(ii) *If $R(\tilde{A}_i) < R(\tilde{A}_j)$, then $\tilde{A}_j < \tilde{A}_j$;*

(iii) *If $R(\tilde{A}_i) = R(\tilde{A}_j)$, then in this case the discrimination of fuzzy numbers is not possible. For a more detailed explanation, see [6].*

Definition 2.5. *[2] Consider $f : [a, b] \to \mathbb{R}_F$ is gH-differentiable such that type of differentiability f in $[a, b]$ dont change. Then for $a \le s \le b$*

(i) *If is $[i - gH]$-differentiable then $f'_{i,gH}(t)$ is (FR)-integrable over $[a, b]$*

$$f(s) = f(a) \oplus \int_a^s f'_{i,gH}(t)dt.$$

(ii) *If $f(t)$ is $[ii-gH]$-differentiable then $f'_{ii,gH}(t)$ is (FR)-integrable over $[a, b]$*

$$f(a) = f(s) \oplus (-1) \int_a^s f'_{i,gH}(t)dt.$$

The conditions for $t_0 \in C^k_{gH}([0, T], \mathbb{R}_F)$ is given in [2].

Definition 2.6. *[1] Let $T = [a, b]$ and $S = [c, d]$ be two closed intervals of real numbers, then $[a, b]/[c, d] = \Big[\min(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}), \max(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d})\Big]$, provided $0 \notin [a, b]$. The extremes of the resultant interval are given in the following table.*

Definition 2.7. *[3] Let $f : [a, b] \to \mathbb{R}_F$ and $x_0 \in [a, b]$. We say that f is strongly generalized Hukuhara differentiable at x_0 (gH-differentiable for short) if there exists an element $f'_G \in \mathbb{R}_F$, such that, for all $h > 0$ sufficiently small,*

Table 1: Table for computing the extremes of the division of two closed intervals.

$T=[a,b]$	$S=[c,d]$	T/S
$0 \leq a \leq b$	$0 < c \leq d$	$\left[\frac{a}{d}, \frac{b}{c}\right]$
	$c \leq d < 0$	$\left[\frac{b}{d}, \frac{a}{c}\right]$
$a \leq 0 \leq b$	$0 < c \leq d$	$\left[\frac{a}{c}, \frac{b}{c}\right]$
	$c \leq d < 0$	$\left[\frac{b}{d}, \frac{a}{d}\right]$
$a \leq b \leq 0$	$0 < c \leq d$	$\left[\frac{a}{d}, \frac{b}{c}\right]$
	$c \leq d < 0$	$\left[\frac{a}{c}, \frac{b}{d}\right]$

(i) $\exists f(x_0+h)\ominus_H f(x_0), f(x_0)\ominus_H f(x_0-h)$ *and*

$$\lim_{h\to 0}\frac{f(x_0+h)\ominus_H f(x_0)}{h} = \lim_{h\to 0}\frac{f(x_0)\ominus_H f(x_0-h)}{h} = f'_G(x_0).$$

Or

(ii) $\exists f(x_0)\ominus_H f(x_0+h), f(x_0-h)\ominus_H f(x_0)$ *and*

$$\lim_{h\to 0}\frac{f(x_0)\ominus_H f(x_0+h)}{-h} = \lim_{h\to 0}\frac{f(x_0-h)\ominus_H f(x_0)}{-h} = f'_G(x_0).$$

Or

(iii) $\exists f(x_0+h)\ominus_H f(x_0), f(x_0-h)\ominus_H f(x_0)$ *and*

$$\lim_{h\to 0}\frac{f(x_0+h)\ominus_H f(x_0)}{h} = \lim_{h\to 0}\frac{f(x_0-h)\ominus_H f(x_0)}{-h} = f'_G(x_0).$$

Or

(iv) $\exists f(x_0)\ominus_H f(x_0+h), f(x_0)\ominus_H f(x_0-h)$ *and*

$$\lim_{h\to 0}\frac{f(x_0)\ominus_H f(x_0+h)}{(-h)} = \lim_{h\to 0}\frac{f(x_0)\ominus_H f(x_0-h)}{h} = f'_G(x_0).$$

Definition 2.8. *[2] Let $f:(a,b)\to \mathbb{R}_F$. We say that $f(t)$ is gH-differentiable of the n^{th}-order at whenever the function $f(t)$ is gH-differentiable of the order i, $i=1,\cdots,n-1$ at t_0 with no switching point on $[a,b]$. Then there exist $f^{(n)}_{gH}(t_0)\in\mathbb{R}_F$ such that*

$$f^{(n)}_{gH}(t_0) = \lim_{h\to 0}\frac{f^{(n-1)}(t_0+h)\ominus_{gH} f^{(n-1)}(t_0)}{h}$$

In section 3, we will obtain full fuzzy modified Euler's method based on generating Hukuhara difference. In section 4, we use this method in the numerical algorithm for solving impulsive fuzzy initial value problem. In the following, in section 5, we present error and consistent theorem and prove these theorems for introducing full fuzzy modified Euler's method. And finally, in section 6, we solve three numerical examples and show results with local truncation error and plotted graphs.

3 Fuzzy Modified Euler's method

In this section we introduce and prove necessary fuzzy definitions and fuzzy theorems to achieve formulation of full fuzzy modified Euler's method. In all of the theorems, we have assumed that the generated Hukuhara difference is an existence and we limit the discussion to the case that all of the used fuzzy numbers are the trapezoidal fuzzy numbers.

3.1 Fuzzy Newton's Divided Difference

Theorem 3.1. *The fuzzy n-th Newton's divided difference between the crisp points $t_0, t_1, \ldots, t_n$ as follows form*

$$f[(t_0, y(t_0)), (t_1, y(t_1)), \ldots, (t_n, y(t_n))] = \frac{f[(t_0, y(t_0)), (t_1, y(t_1)), \ldots, (t_{n-1}, y(t_{n-1}))]}{t_0 - t_n} \ominus_{gH} \frac{f[(t_1, y(t_1)), (t_2, y(t_2)), \ldots, (t_n, y(t_n))]}{t_0 - t_n}. \quad (3.4)$$

Proof. Let $n+1$ distinct fuzzy data $\Big(t_i, f(t_i, y(t_i))\Big)$; $i = 0, 1, \cdots, n$; as interpolation nodes. Let the following set as the crisp polynomial basic functions

$$\{1, (t-t_0), (t-t_0)(t-t_1), \cdots, (t-t_0)(t-t_1)\cdots(t-t_n)\}.$$

Let $p(t)$ was fuzzy interpolation polynomial of fuzzy function $f(t, y(t))$ in points $t_0, t_1, \ldots, t_n$

$$p(t) = a_0 \oplus [a_1 \odot (t-t_0)] \oplus [a_2 \odot (t-t_0)(t-t_1)] \oplus \cdots \oplus [a_n \odot (t-t_0)(t-t_1)\cdots(t-t_n)].$$

According to interpolation condition $p(t_i) = f(t_i, y(t_i))$; $i = 0, 1, \cdots, n$; we can obtain interpolation coefficients a_i; $(i = 0, 1, \cdots, n)$. By satisfying interpolation condition, for $t = t_0$ and $t = t_1$, respectively we obtain

$$a_0 = f(t_0, y(t_0)),$$

and

$$a_1 = \frac{f(t_1, y(t_1)) \ominus_{gH} f(t_0, y(t_0))}{t_1 - t_0}.$$

In this way, Fuzzy Newton's Divided Difference for another fuzzy data $i = 2, \cdots, n$; is calculated. And we get iterative formula for Fuzzy Newtons Divided Difference. So $f[(t_i, y(t_i)), (t_{i+1}, y(t_{i+1}))]$ and $f[(t_i, y(t_i)), (t_{i+1}, y(t_{i+1})), (t_{i+2}, y(t_{i+2}))]$, respectively obtained as following for

$$f[(t_i, y(t_i)), (t_{i+1}, y(t_{i+1}))] = \frac{f(t_i, y(t_i)) \ominus_{gH} f(t_{i+1}, y(t_{i+1}))}{t_i - t_{i+1}},$$

$$f[(t_i, y(t_i)), (t_{i+1}, y(t_{i+1})), (t_{i+2}, y(t_{i+2}))] = \frac{f[(t_i, y(t_i)), (t_{i+1}, y(t_{i+1}))]}{t_i - t_{i+2}} \ominus_{gH} \frac{f[(t_i, y(t_i)), (t_{i+1}, y(t_{i+1})), (t_{i+2}, y(t_{i+2}))]}{t_i - t_{i+2}}.$$

By continuing this process fuzzy Newtons divided difference for $m+2$ fuzzy data denoted by $f[(t, y(t)), (t_0, y(t_0)), \cdots, (t_m, y(t_m))]$ will obtain as following form

$$f[(t, y(t)), (t_0, y(t_0)), \cdots, (t_m, y(t_m))] = \frac{f[(t, y(t)), (t_0, y(t_0)), \cdots, (t_{m-1}, y(t_{m-1}))]}{t - t_m} \ominus_{gH} \frac{f[(t_0, y(t_0)), \cdots, (t_m, y(t_m))]}{t - t_m}. \quad (3.5)$$

□

Remark 3.1. *Now we can introduce fuzzy Newton's divided difference for $f[(t_n, y(t_n)), (t_{n+1}, y(t_{n+1}))]$, as following form*

$$f[(t_n, y(t_n)), (t_{n+1}, y(t_{n+1}))] = \frac{f(t_{n+1}, y(t_{n+1})) \ominus_{gH} f(t_n, y(t_n))}{t_{n+1} - t_n}. \quad (3.6)$$

3.2 Fuzzy Newton's Divided Difference Interpolation

Theorem 3.2. *Fuzzy Newton's divided difference interpolation polynomial on points $t_0, t_1, \ldots, t_n$ defines as following form*

$$p(t) = f[((t_0, y(t_0))] \oplus \Big((t-t_0)\odot f[(t_0, y(t_0)), (t_1, y(t_1))]\Big) \oplus \cdots \oplus \Big((t-t_0)\cdots(t-t_{n-1})\odot f[(t_0, y(t_0)), \cdots, (t_n, y(t_n))]\Big).$$

Proof. By considering expression (3.5), setting $i+1$ instead of and simplify, we obtain

$$f[(t, y(t)), (t_0, y(t_0)), \cdots, (t_1, y(t_1))] = f[(t_0, y(t_0)), \cdots, (t_{i+1}, y(t_{i+1}))] \oplus \Big((t - t_{i+1}) \odot f[(t, y(t)), (t_0, y(t_0)), \cdots, (t_{i+1}, y(t_{i+1}))]\Big). \quad (3.7)$$

By setting $0, 1, \cdots, n-1$ instead of i respectively, inequality (3.7), calculation and simplify we obtain

$$f(t,y(t)) = f[(t_0,y(t_0))] \oplus \Big((x-x_0) \odot f[(t_0,y(t_0)),\cdots,(t_1,y(t_1))]\Big) \oplus \cdots \oplus \Big((t-t_0)(t-t_1)\cdots(t-t_{n-1}) \odot f[(t_0,y(t_0)),\cdots,(t_n,y(t_n))]\Big) \oplus \Big((t-t_0)(t-t_1)\cdots(t-t_n) \odot f[(t,y(t)),(t_0,y(t_0)),\cdots,f(t_n,y(t_n))]\Big).$$

By considering

$$p(t) = f[(t_0,y(t_0))] \oplus \Big((t-t_0)\odot f[(t_0,y(t_0)),(t_1,y(t_1))]\Big) \oplus \cdots \oplus ((t-t_0)(t-t_1)\cdots(t-t_{n-1}) \odot f[(t_0,y(t_0)),\cdots,f(t_n,y(t_n))]), \tag{3.8}$$

$$R(t) = ((t-t_0)(t-t_1)\cdots(t-t_n)\odot f[(t,y(t)),(t_0,y(t_0)),\cdots,f(t_n,y(t_n))]). \tag{3.9}$$

We have

$$f(t,y(t)) = p(t) \oplus R(t).$$

According to this point that the error of the interpolation polynomial at the each point is zero, we conclude that for

$$f(t,y(t)) = p(t_i)\ (i=0,1,\cdots,n).$$

According to that point, interpolation polynomial is unique, so the expression (3.8) is Fuzzy Newton's divided difference interpolation polynomial. Expression (3.9) is called error of interpolation polynomial. □

Remark 3.2. *So Newton's divided Difference linear interpolation polynomial for fuzzy data* $\Big(t_n, f(t_n,y(t_n))\Big), \Big(t_{n+1}, f(t_{n+1},y(t_{n+1}))\Big)$, *obtains as following form*

$$p_1(t) = f\Big(t_n,y(t_n)\Big) \oplus \Big((t-t_n) \odot f[(t_n,y(t_n)),(t_{n+1},y(t_{n+1}))]\Big). \tag{3.10}$$

According to expression (3.6), Indeed

$$p_1(t) = f\Big(t_n,y(t_n)\Big) \oplus \Big((t-t_n) \odot \frac{f(t_{n+1},y(t_{n+1})) \ominus_{gH} f(t_n,y(t_n))}{t_{n+1}-t_n}. \tag{3.11}$$

3.3 Fuzzy Trapezoidal Numerical Integration Method

Theorem 3.3. *By applying fuzzy Newton's divided Difference linear interpolation polynomial (3.11), the Fuzzy Trapezoidal Numerical Integration Method is as following form*

$$\int_{t_n}^{t_{n+1}} f(t,y(t))dt \cong \frac{h}{2}\odot \Big(f(t_n,y(t_n)) \oplus f(t_{n+1},y(t_{n+1}))\Big). \tag{3.12}$$

Proof. By applying an expression (3.11), and by changing the variable $t = t_n + \theta h$, $h = t_{n+1} - t_n$, $dt = hd\theta$, by considering $\delta f(t_n,y(t_n)) = f(t_{n+1},y(t_{n+1})) \ominus_{gH} f(t_n,y(t_n))$, we obtain fuzzy Newton's forward divided Difference interpolation polynomial formula as following form

$$p_1(t) = f(t_n,y(t_n)) \oplus (\theta \odot \delta f(t_n,y(t_n))). \tag{3.13}$$

We apply (FR)-integrable in definition 2.5 over interval on expression (3.13)

$$\int_{t_n}^{t_{n+1}} p_1(t)dt = \int_0^1 f(t_n,y(t_n))hd\theta \oplus (\theta\odot \int_0^1 \Big(f(t_{n+1},y(t_{n+1})) \ominus_{gH} f(t_n,y(t_n))\Big)hd\theta)$$
$$= \Big[h\odot f(t_n,y(t_n)) \int_0^1 d\theta\Big]\oplus \Big[h\odot f(t_{n+1},y(t_{n+1})) \int_0^1 \theta d\theta\Big]\ominus_{gH} \Big[h\odot f(t_n,y(t_n)) \int_0^1 \theta d\theta\Big].$$

Finally, we obtain

$$\int_{t_n}^{t_{n+1}} p_1(t)dt = \frac{h}{2}\odot\Big(f(t_n,y(t_n))\oplus f(t_{n+1},y(t_{n+1}))\Big)$$
$$\int_{t_n}^{t_{n+1}} f(t,y(t))dt \cong \frac{h}{2}\odot\Big(f(t_n,y(t_n))\oplus f(t_{n+1},y(t_{n+1}))\Big).$$

□

3.4 Error of linear interpolation polynomial

Theorem 3.4. *Consider fuzzy interpolation nodes* $\Big(t_i, f(t_i,y(t_i))\Big)$, $0 \le i \le n = 1$; *for fuzzy function* $f(t,y(t))$, *and* $p(t)$ *is fuzzy linear* *interpolation polynomial, derivative of function $f(t, y(t))$ exist until nth order; then for each $\tilde{t}$, exists unique δ from interval $I[t_0, \cdots, t_n, \tilde{t}]$, such that*

$$f(\tilde{t}, y(\tilde{t})) \ominus_{gH} p(\tilde{t}) = \frac{w(\tilde{t}) \odot f_{gH}^{(n+1)}(\delta)}{(n+1)!}. \quad (3.14)$$

Where $w(t) = (t - t_0)(t - t_1)$, and $f_{gH}^{(n+1)}$ is $(n+1)$th order of gH-differentiability of definition (2.7).

Proof. For $0 \leq i \leq n = 1$ we have $\tilde{t} \neq t_i$ because $p(t_i) = f(t_i, y(t_i))$, $0 \leq i \leq n = 1$. We can find the fuzzy constant $k = [\underline{k}, \overline{k}]$, such that the function $F(t)$ will zero, in the point $\tilde{t}$ as follows form.

$$\underline{F}(t) = \underline{f}(t, y(t)) - \underline{p}(t) - \underline{k}w(t), \quad (3.15)$$

$$\overline{F}(t) = \overline{f}(t, y(t)) - \overline{p}(t) - \overline{k}w(t). \quad (3.16)$$

On interval $[t_n, t_{n+1}]$ we have $t - t_{n+1} \leq 0, t - t_n \geq 0$, Thus in generality $w(t) = (t - t_n)(t - t_{n+1}) \leq 0$ i.e. $w(t)$ on $[t_n, t_{n+1}]$ does not change sign, Indeed (3.15) and (3.16) obtained as following form

$$\underline{F}(t) = \underline{f}(t, y(t)) - \underline{p}(t) - \overline{k}w(t),$$
$$\overline{F}(t) = \overline{f}(t, y(t)) - \overline{p}(t) - \underline{k}w(t).$$

We suppose that $\underline{F}(t) = \overline{F}(t)$. According to expressions (3.15) and (3.16) are crisp, as well as the error of interpolation polynomial theorem proof [11] we obtain, $\underline{k} = \frac{\underline{f}^{(n+1)}(\delta)}{(n+1)!}$ and $\overline{k} = \frac{\overline{f}^{(n+1)}(\delta)}{(n+1)!}$. By

$$\underline{f}(\tilde{t}, y(\tilde{t})) - \underline{p}(\tilde{t}) = \frac{w(\tilde{t})\underline{f}^{(n+1)}(\delta)}{(n+1)!},$$

and

$$\overline{f}(\tilde{t}, y(\tilde{t})) - \overline{p}(\tilde{t}) = \frac{w(\tilde{t})\overline{f}^{(n+1)}(\delta)}{(n+1)!},$$

we conclude that

$$\Big[\underline{f}(\tilde{t}, y(\tilde{t})) - \underline{p}(\tilde{t}), \overline{f}(\tilde{t}, y(\tilde{t})) - \overline{p}(\tilde{t})\Big]$$
$$\Big[\frac{w(\tilde{t})\underline{f}^{(n+1)}(\delta)}{(n+1)!}, \frac{w(\tilde{t})\overline{f}^{(n+1)}(\delta)}{(n+1)!}\Big] \in \mathbb{R}_F.$$

Or

$$\Big[\overline{f}(\tilde{t}, y(\tilde{t})) - \overline{p}(\tilde{t}), \underline{f}(\tilde{t}, y(\tilde{t})) - \underline{p}(\tilde{t})\Big]$$
$$\Big[\frac{w(\tilde{t})\overline{f}^{(n+1)}(\delta)}{(n+1)!}, \frac{w(\tilde{t})\underline{f}^{(n+1)}(\delta)}{(n+1)!}\Big] \in \mathbb{R}_F.$$

It means that finally we obtain

$$f(\tilde{t}, y(\tilde{t})) \ominus_{gH} p(\tilde{t}) = \frac{w(\tilde{t}) \odot f^{(n+1)}(\delta)}{(n+1)!}.$$

□

3.5 Error of Fuzzy Trapezoidal Numerical Integration Method

Theorem 3.5. *By considering, η_n between crisp points t_n and t_{n+1}, and $f''^{gH}(t, y(t))$ is continuous. The error of Fuzzy Trapezoidal Numerical Integration Method is as following form*

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt \ominus_{gH} \Big[\frac{h}{2}\Big(f(t_n, y(t_n)) \oplus f(t_{n+1}, y(t_{n+1}))\Big)\Big] = -\frac{h^3}{12} f''_{ii,gH}\Big(\eta_n, y(\eta_n)\Big).$$

Proof. By considering fuzzy linear interpolation polynomial error of fuzzy function $f(t, y(t))$ (3.14), we have

$$f(t, y(t))dt \ominus_{gH} p_1(t) = \frac{(t - t_n)(t - t_{n+1})}{2!} \odot f''_{ii,gH}(\eta_n, y(\eta_n)),$$
$$\eta_n \in [t_n, t_{n+1}]. \quad (3.17)$$

On interval $[t_n, t_{n+1}]$ we have $t - t_{n+1} \leq 0, t - t_n \geq 0$. Thus, in generality $g(t) = (t - t_n)(t - t_{n+1}) \leq 0$ i.e. $g(t)$ on $[t_n, t_{n+1}]$ does not change sign, So $f''_{ii,gH}(\eta_n, y(\eta_n))$ is (ii)-gH differentiable without any switching point on the interval $[t_n, t_{n+1}]$. By applying (FR)-integral in definition 2.5 over interval $[t_n, t_{n+1}]$ from (3.17), we obtain

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt \ominus_{gH} \int_{t_n}^{t_{n+1}} p_1(t)dt = \int_{t_n}^{t_{n+1}} \frac{(t - t_n)(t - t_{n+1})}{2!} \odot f''_{ii,gH}(\eta_n, y(\eta_n))dt,$$

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt \ominus_{gH} \int_{t_n}^{t_{n+1}} p_1(t)dt = \frac{f''_{ii,gH}(\eta_n, y(\eta_n))}{2} \odot \int_{t_n}^{t_{n+1}} (t - t_n)(t - t_{n+1})dt. \quad (3.18)$$

Where η_n is a number $t_n < \eta_n < t_{n+1}$. The right hand side of integral by change of the variable $t = t_n + \theta h$, $h = t_{n+1} - t_n$, $dt = hd\theta$ became as following form

$$\int_{t_n}^{t_{n+1}} (t - t_n)(t - t_{n+1})dt$$
$$= \int_0^1 h\theta \times h(\theta - 1) \times hd\theta$$
$$= h^3 \int_0^1 (\theta^2 - \theta)d\theta = -\frac{h^3}{6}.$$

With an expression (3.16) we conclude

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt \ominus_{gH} \int_{t_n}^{t_{n+1}} p_1(t)dt$$
$$= -\frac{h^3}{12} f''_{ii,gH}(\eta_n, y(\eta_n)).$$

Indeed, for expression (3.12) we obtain

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt \ominus_{gH} \Big[\frac{h}{2} \odot \Big($$
$$f(t_n, y(t_n)) \oplus f(t_{n+1}, y(t_{n+1}))\Big)\Big]$$
$$= -\frac{h^3}{12} f''_{ii,gH}\Big(\eta_n, y(\eta_n)\Big).$$

□

3.6 Modified Euler's Method

Consider the following fuzzy initial value problem

$$y'_{gH}(t) = f(t, y(t)),\ t \in [0, T], \tag{3.19}$$
$$y(t_0) = y_0 \in \mathbb{R}_F.$$

Where $y(t)$ is an unknown fuzzy function of crisp variable t and $f : [0, T] \times \mathbb{R}_F \to \mathbb{R}_F$ is continuous, also $y'_{gH}(t)$ is the generalized Hukuhara derivative of $y(t)$ such that the set of switching point is finite. Now we introduce Modified Eulers Method for solving fuzzy initial value problem.
To derive Modified Euler's Method, let partition $I_N = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of the interval $[0, T]$. Where $t_k = kh$, $k = 0, 1, \cdots, N$.

Case 1.

Let us suppose that the unique solution of the problem $y(t)$ is $[(i) - gH]$-differentiable and belongs $t_0 \in C^3_{gH}([0, T], \mathbb{R}_F)$ such that the type of differentiability dose not change in $[0, T]$.
We apply (FR)-integrable on the fuzzy differential equation (3.19) over interval $[t_n, t_{n+1}]$ as follows form

$$\int_{t_n}^{t_{n+1}} y'_{i,gH}(s)ds = \int_{t_n}^{t_{n+1}} f(s, y(s))ds. \tag{3.20}$$

According to (FR)-integrable in definition 2.5 and fuzzy Trapezoidal numerical integration method in theorem 3.3 on the left and right side of fuzzy differential equation (3.18) over interval $[t_n, t_{n+1}]$ respectively we obtain

$$\int_{t_n}^{t_{n+1}} y'_{i,gH}(s)ds = y(t_{n+1}) \ominus_H y(t_n), \tag{3.21}$$

$$\int_{t_n}^{t_{n+1}} f(s, y(s))ds \cong \frac{h}{2} \odot \Big(f(t_n, y(t_n)) \oplus f(t_{n+1}, y(t_{n+1}))\Big). \tag{3.22}$$

Finally, we have modified Euler's method for **Case1** as following form

$$y(t_{n+1}) \ominus_H y(t_n) \cong \frac{h}{2} \odot \Big(f(t_n, y(t_n)) \oplus f(t_{n+1}, y(t_{n+1}))\Big).$$

By setting k instead n and simplify

$$y_{k+1} \cong y_k \oplus \Big[\frac{h}{2} \odot \Big(f(t_k, y(t_k)) \oplus f(t_{k+1}, y(t_{k+1}))\Big)\Big].$$

According to the Error of Fuzzy Trapezoidal Numerical Integration Method (3.17), we have

$$y_{k+1} = y_k \oplus \Big[\frac{h}{2} \odot \Big(f(t_k, y(t_k)) \oplus f(t_{k+1}, y(t_{k+1}))\Big)\Big] - \frac{h^3}{12} f''_{ii,gH}\Big(\eta_n, y(\eta_n)\Big).$$

We introduce numerical fuzzy modified Euler's method as

$$y_{k+1} = y_k \oplus \Big[\frac{h}{2} \odot \Big(f(t_k, y(t_k)) \oplus f(t_{k+1}, y(t_{k+1}))\Big)\Big]. \tag{3.23}$$

Case 2.

Now, consider $y(t)$ is $[(ii)-gH]$-differentiable and belongs to $t_0 \in C^3_{gH}([0,T],\mathbb{R}_F)$ such that the type of differentiability dose not change in $[0,T]$.
We apply (FR)-integrable over $[0,T]$ and fuzzy integration, the differential equation (3.19) over $[t_n,t_{n+1}]$ to obtain

$$\int_{t_n}^{t_{n+1}} y'_{ii,gH}(s)ds = \int_{t_n}^{t_{n+1}} f(s,y(s))ds. \tag{3.24}$$

By (FR)-integrable in definition 2.5 and fuzzy Trapezoidal numerical integration method in theorem 3.3 on the fuzzy differential equation (3.24) over interval $[t_n,t_{n+1}]$ we obtain respectively

$$\int_{t_n}^{t_{n+1}} y'_{ii,gH}(s)ds = -(y(t_n)\ominus_H y(t_{n+1})), \tag{3.25}$$

$$\int_{t_n}^{t_{n+1}} f(t,y(t))dt \cong \frac{h}{2}\odot\Big(f(t_n,y(t_n)) \oplus f(t_{n+1},y(t_{n+1}))\Big). \tag{3.26}$$

Finally, we have modified Euler's method for **Case2** as following form

Case 3.

$$y(t_{n+1}) \cong y(t_n)\ominus_{gH}\Big[-\frac{h}{2}\odot\Big(f(t_n,y(t_n)) \oplus f(t_{n+1},y(t_{n+1}))\Big)\Big].$$

By setting k instead n, and by according to the Error of Fuzzy Trapezoidal Numerical Integration Method (3.12), we have

$$y_{k+1} \cong y_k\ominus_{gH}\Big[-\frac{h}{2}\odot\Big(f(t_k,y(t_k)) \oplus f(t_{k+1},y(t_{k+1}))\Big)\Big]\oplus -\frac{h^3}{12}f''_{ii,gH}\Big(\eta_n,y(\eta_n)\Big).$$

We introduce numerical fuzzy modified Euler's method as

$$y_{k+1} = y_k\ominus_{gH}\Big[-\frac{h}{2}\odot\Big(f(t_k,y(t_k)) \oplus f(t_{k+1},y(t_{k+1}))\Big)\Big]. \tag{3.27}$$

4 Numerical Algorithm for Solving IFIVP

We introduce a numerical algorithm for solving first order impulsive fuzzy differential equation with initial value, (Algorithm for Solving Impulsive Fuzzy Differential Equations). We consider impulsive fuzzy differential equations (1.1), (1.2), (1.3) it is possible to obtain approximation solution for fixed value t_z of parameter t by Algorithm for Solving Impulsive Fuzzy Differential Equations. Where $t_z > t_0$, at the moment $t = t_z$. The key idea of our numerical algorithm based on full fuzzy method is taken from that paper [10].

4.1 Numerical Algorithm By Fuzzy Euler's Method

Step One:

Using fuzzy Euler's method *FEM*. We solve the function of argument t, by taking it from half-segment $(t_k,t_{k+1}]$, i.e. $y^{[j+1]} = FEM(y^{[j]})$.

Case 1.

Let us suppose that the unique solution of the problem $y(t)$ is $[(i)-gH]$-differentiable and belongs to $t_0 \in C^2_{gH}([0,T],\mathbb{R}_F)$ such that the type of differentiability dose not change on $[0,T]$:

$$\begin{cases} y_0 = y(t_0), \\ y_{k+1} = y_k\oplus[h\odot f(t_k,y_k)], k=0,1,\cdots,N-1. \end{cases} \tag{4.28}$$

Case 2.

Now, consider $y(t)$ is $(ii)-gH$-differentiable and belongs to $t_0 \in C^2_{gH}([0,T],\mathbb{R}_F)$ such that the type of differentiability dose not change on $[0,T]$:

$$\begin{cases} y_0 = y(t_0), \\ y_{k+1} = y_k\ominus[(-1)h\odot f(t_k,y_k)], k=0,1,\cdots,N-1. \end{cases} \tag{4.29}$$

Remark 4.1. *In the half-segment $(t_0,t_1]$ we consider $y_0 = y(t_0)$, that this value is given by problem initial condition, But in the other half-segment $(t_k,t_{k+1}]$ we take $y(t_k^+)$ for y_0, i.e. $y_0 = y(t_k^+)$.*

Step Two:

At the moment $t = t_k$ acts impulsive fuzzy operator I_k and brings rapidly changes of fuzzy functions y that amounts

$$J(t_k) = I_k\Big(y(t_k)\oplus\sum_p^{\star} J(t_N)\Big), \tag{4.30}$$

$$\Delta y|_{t=t_k} = y(t_k^+) \ominus_{gH} y(t_k^-) = I_k\Big(y(t_k^-)\Big),$$

$$y(t_k^+) \ominus_{gH} y(t_k^-) = I_k\Big(y(t_k^-)\Big)$$
$$\leftrightarrow \begin{cases} (i)\, y(t_k^+) = y(t_k^-) + I_k\Big(y(t_k^-)\Big), \\ or \\ (ii)\, y(t_k^-) = y(t_k^+) - I_k\Big(y(t_k^-)\Big), \end{cases}$$

where $p = \{t_i \,|\, i < k\}$, $\sum_p^\star$ denotes the fuzzy summation.

Step Three:

The steps one and two are repeated while $t_{k+1} \leq t_z$.

Step Four:

We add a sum of all jumps to the function y

$$Y = y \oplus \sum_q^\star J(t_i). \tag{4.31}$$

Where $p = \{t_i \,|\, 0 < i < z\}$, $\sum_q^\star$ denotes the fuzzy summation.

4.2 Numerical Algorithm By Fuzzy Modified Euler's Method

Step One:

Using fuzzy Modified Euler's method *(FEM)*, we solve the functions of argument t by taking it from half-segment $(t_k, t_{k+1}]$, i.e. $y^{[j+1]} = FEM(y^{[j]})$.

Case 1.

Let us suppose that the unique solution of the problem $y(t)$ is $[(i) - gH]$-differentiable, and belongs to $t_0 \in C^2_{gH}([0,T], \mathbb{R}_F)$ such that the type of differentiability dose not change in $[0,T]$:

$$\begin{cases} y_0 = y(t_0), \\ y_{k+1} = y_k \oplus \Big[\frac{h}{2} \odot \Big(f(t_k, y_k) \oplus f(t_{k+1}, y_{k+1})\Big)\Big]. \end{cases} \tag{4.32}$$

Case 2.

Now, consider $y(t)$ is $[(ii) - gH]$-differentiable, and belongs to $t_0 \in C^2_{gH}([0,T], \mathbb{R}_F)$ such that the type of differentiability dose not change in $[0,T]$:

$$\begin{cases} y_0 = y(t_0), \\ y_{k+1} = y_k \ominus_{gH} \Big[\frac{-h}{2} \odot \Big(f(t_k, y_k) \oplus f(t_{k+1}, y_{k+1})\Big)\Big]. \end{cases} \tag{4.33}$$

Remark 4.2. *In the half-segment $(t_0, t_1]$ we consider $y_0 = y(t_0)$, that is given by problem initial condition, But in the other half-segment $(t_k, t_{k+1}]$ we take $y(t_k^+)$ for y_0, i.e. $y_0 = y(t_k^+)$.*

Step Two:

At the moment $t = t_k$ acts impulsive fuzzy operator I_k, and brings rapidly changes of fuzzy functions that amounts

$$J(t_k) = I_k\Big(y(t_k) \oplus \sum_p^\star J(t_N)\Big), \tag{4.34}$$

$$\Delta y|_{t=t_k} = y(t_k^+) \ominus_{gH} y(t_k^-) = I_k\Big(y(t_k^-)\Big),$$

$$y(t_k^+) \ominus_{gH} y(t_k^-) = I_k\Big(y(t_k^-)\Big)$$
$$\leftrightarrow \begin{cases} (i)\, y(t_k^+) = y(t_k^-) + I_k\Big(y(t_k^-)\Big), \\ or \\ (ii)\, y(t_k^-) = y(t_k^+) - I_k\Big(y(t_k^-)\Big), \end{cases}$$

where $p = \{t_i \,|\, i < k\}$, $\sum_p^\star$ denotes the fuzzy summation.

Step Three:

The steps one and two are repeated while $t_{k+1} \leq t_z$.

Step Four:

We add a sum of all jumps to the function y

$$Y = y \oplus \sum_q^\star J(t_i). \tag{4.35}$$

Where $p = \{t_i \,|\, 0 < i < z\}$, $\sum_q^\star$ denotes the fuzzy summation.

5 Error Analysis

5.1 Local Truncation Error of fuzzy Modified Euler's Method

For numerical method written as following form, we define the residual as $\mathcal{R}_k$, given by

Case 1.

$$y_{k+1} = y_k \oplus \Big[\frac{h}{2} \odot \Big(f(t_k, y_k) \oplus f(t_{k+1}, y_{k+1})\Big)\Big],$$

$$\mathcal{R}_k = y_{k+1} \ominus_{gH} \Big[y_k \oplus \frac{h}{2} \odot \Big(f(t_k, y_k) \oplus f(t_{k+1}, y_{k+1})\Big)\Big].$$

Case 2.

$$y_{k+1} = y_k \ominus_{gH} \Big[-\frac{h}{2} \odot \Big(f(t_k, y_k) \oplus f(t_{k+1}, y_{k+1})\Big)\Big],$$

$$\mathcal{R}_k = y_{k+1} \ominus_{gH} \Big[y_k \ominus_{gH} -\frac{h}{2} \odot \Big(f(t_k, y_k) \oplus f(t_{k+1}, y_{k+1})\Big)\Big].$$

Hence, the coefficient of fuzzy function is negative, is [(ii)-gH]-differentiable, so we have

$$\mathcal{R}_k = -\frac{h^3}{12} \odot f''_{ii,gH}\Big(\eta_n, y(\eta_n)\Big),$$

$$\tau_k = -\frac{h^2}{12} \odot f''_{ii,gH}\Big(\eta_n, y(\eta_n)\Big).$$

5.2 Consistent of Modified Euler's Method

Definition 5.1. *[2] The fuzzy Euler's method is said to be consistent if*

$$\lim_{h \to 0} \max_{t_k \le b} \mathcal{D}(\tau_k, 0) = 0.$$

So consistency of fuzzy modified Euler's method analysis as following
Consider $D\Big(f''_{ii,gH}\Big(\eta_n, y(\eta_n)\Big), 0\Big) \le M$, *we conclude*

$$\begin{aligned}
&\lim_{h \to 0} \max_{t_k \le b} \mathcal{D}(\tau_k, 0) \\
&= \lim_{h \to 0} \max_{t_k \le b} \mathcal{D}\Big(-\frac{h^2}{12} \odot f''_{ii,gH}(\eta_n, y(\eta_n)), 0\Big) \\
&= \lim_{h \to 0} -\frac{h^2}{12} \max_{t_k \le b} \mathcal{D}\Big(f''_{ii,gH}(\eta_n, y(\eta_n)), 0\Big) \\
&\le \lim_{h \to 0} -\frac{h^2}{12} M = 0.
\end{aligned}$$

It means the fuzzy modified Euler's method is consistent.

5.3 Global Truncation Error of Modified Euler's Method

Definition 5.2. *[2] The numerical method is convergent if the global truncation error goes to zero as the step size goes to zero; in other words, the numerical solution converges to the exact solution:*

$$\begin{aligned}
&\lim_{h \to 0} \max_{k} \mathcal{D}(e_{k+1}, 0) = 0 \\
&\Rightarrow \lim_{h \to 0} \max_{k} \mathcal{D}(y(t_{k+1}), y_{k+1}) = 0.
\end{aligned}$$

Definition 5.3. *The global truncation error is the agglomeration of the local truncation error over all the iterations, assuming perfect knowledge of the true solution at the initial time step. For fuzzy modified Euler's method case 1 (3.23) and case 2 (3.27), the global truncation error,* e_{k+1}, *at* t_{k+1} *is respectively defined by*

$$\begin{aligned}
&e_{k+1} = \mathcal{D}(y(t_{k+1}), y_{k+1}) = \mathcal{D}\Big(y(t_{k+1}), \\
&\Big[y_0 \oplus [\frac{h}{2} \odot f(t_0, y(t_0))] \oplus \Big[h \odot (f(t_1, y_1) \oplus \\
&f(t_2, y_2) \oplus \cdots \oplus f(t_k, y_k))\Big] \oplus \\
&[\frac{h}{2} \odot f(t_{k+1}, y_{k+1})]\Big]\Big).
\end{aligned}$$

And

$$\begin{aligned}
&e_{k+1} = \mathcal{D}(y(t_{k+1}), y_{k+1}) = \mathcal{D}\Big(y(t_{k+1}), \Big[y_0 \ominus_{gH} \\
&[-\frac{h}{2} \odot f(t_0, y_0)] \ominus_{gH} [-h \odot f(t_1, y_1) \ominus_{gH} \\
&[-h \odot f(t_2, y_2)] \ominus_{gH} \cdots \ominus_{gH} [-h \odot f(t_k, y_k)] \\
&\ominus_{gH} [-\frac{h}{2} \odot f(t_{k+1}, y_{k+1})]\Big]\Big).
\end{aligned}$$

Theorem 5.1. *Suppose that* $y''_{gH}(t)$ *exists and* $f(t, y(t))$ *is continuous and satisfies in Lipschitz condition on the* $\{t, y(t) | [0, p], y \in \overline{B}(y_0, q), p, q > 0\}$. *Then Modified Euler's method converges to the solution of fuzzy the initial value problem.*

Proof. Consider $y(t)$ is (i)-gH differentiable, by assumption $d_k = -\frac{h^3}{12} \odot f''_{ii,gH}\Big(\eta_n, y(\eta_n)\Big)$ and (3.23), the exact solution of the initial value problem satisfies

$$y_{k+1} = y_k \oplus \Big[\frac{h}{2} \odot \Big(f(t_k, y_k) \oplus f(t_{k+1}, y_{k+1})\Big)\Big] + d_k.$$

By use the distance it is concluded:

$$\mathcal{D}(y(t_{k+1}), y_{k+1}) = \mathcal{D}(y(t_k), y_k) \oplus \frac{h}{2}\Big[\mathcal{D}\Big(f(t_k, y(t_k)), f(t_k, y_k)\Big) + \mathcal{D}\Big(f(t_{k+1}, y(t_{k+1})), f(t_{k+1}, y_{k+1})\Big)\Big] + \mathcal{D}(d_k, 0).$$

Since $f(t, y(t))$ satisfies in Lipschitz condition

$$\mathcal{D}\Big(f(t_k, y(t_k)), f(t_k, y_k)\Big) \leq L_k \mathcal{D}(y(t_k), y_k),$$

we have

$$\mathcal{D}\Big(y(t_{k+1}), y_{k+1}\Big) \leq \Big(L_k + \frac{1}{2}L_k^2 h\Big)\mathcal{D}(y(t_k), y_k) + \mathcal{D}(d_k, 0).$$

Suppose that $L = \max_{0\leq k\leq N} L_k$ and $d = \max_{0\leq k\leq N} \mathcal{D}(d_k, 0)$, We obtain

$$\mathcal{D}(y(t_{k+1}), y_{k+1}) \leq \Big(L + \frac{1}{2}L^2 h\Big)\mathcal{D}(y(t_k), y_k) + d.$$

Therefore, the fuzzy function satisfies a Lipschitz condition. According to that point $f(t, y(t))$ is continuous, moreover $d = \max_{0\leq k\leq N-1} \mathcal{D}(d_k, 0) = -\frac{h^2}{12} \max_{0\leq t\leq T} \mathcal{D}\Big(f''_{ii,gH}(\eta_k, y(\eta_k)), 0\Big)$. And $\mathcal{D}\Big(f''_{ii,gH}(\eta_k, y(\eta_k)), 0\Big) = 0$. So the fuzzy modified Euler's method is Stable. Since the method is consistent, we conclude the method in convergent. Thus, $\lim_{h\to 0} \mathcal{D}(y(t_{k+1}), y_{k+1}) \to 0$, and (3.23) is converges. □

5.4 Error of Fuzzy Numerical Algorithm

By considering First-order impulsive fuzzy differential equations (1.1), (1.2), (1.3)

$$\begin{aligned} &y'_{gH}(t) = f(t, y(t)), \\ &t \in J = [0, T],\ t \neq t_k, k = 1, \cdots, N, \\ &\Delta y|_{t=t_k} = I_k(y(t_k))\ (t = t_k), \\ &y(t_0) = y_0. \end{aligned}$$

We get the analysis error of numerical algorithm in section 4 based on fuzzy modified Euler's method. We import small perturbations δ_3 on the right hand side of expression (1.3), then it creates perturbation δ_1, δ_2 on the right hand side of expressions (1.1) and (1.2), and we will have the solution as following form

$$Y = y \oplus \delta \odot y. \tag{5.36}$$

We got analysis with this perturbation, whether we will have a stable numerical algorithm, and we will have converged? By perturbation, we achieve to first order perturbation impulsive fuzzy differential equation as following form

$$\begin{aligned} &Y'_{gH}(t) = f(t, y(t)) \oplus [\delta_1 \odot f(t, y(t))], \\ &t \in J = [0, T],\ t \neq t_k, k = 1, \cdots, N, \end{aligned} \tag{5.37}$$

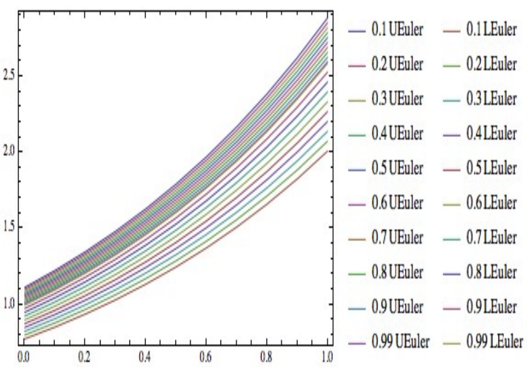

Figure 1: Fuzzy Euler's Method for example (6.1)

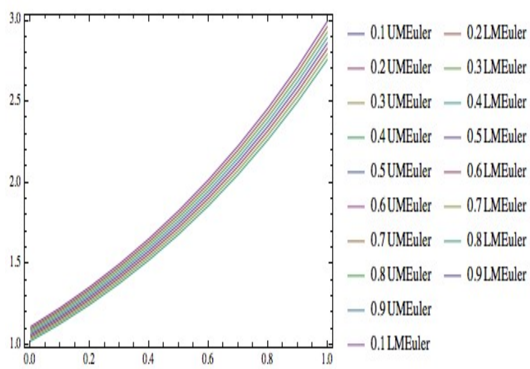

Figure 2: Fuzzy Modified Euler's Method for example (6.1)

$$\begin{aligned} &\Delta Y|_{t=t_k} = Y(t_k^+) \ominus_{gH} Y(t_k^-) \\ &= I_k(y(t_k^-)) \oplus [\delta_2 \odot I_k(y(t_k^-))],\ (t = t_k), \end{aligned} \tag{5.38}$$

Table 2: The local truncation error for first interval of example (6.1).

Time	F Euler	F M Euler
0.1	1.33×10^{-3}	5×10^{-5}
0.2	2.93×10^{-3}	2.4×10^{-4}
0.3	4.85×10^{-3}	5.9×10^{-4}
0.4	7.14×10^{-3}	1.15×10^{-3}
0.5	9.84×10^{-3}	1.94×10^{-3}
0.6	1.304×10^{-2}	3.03×10^{-3}
0.7	1.677×10^{-2}	4.45×10^{-3}
0.8	2.114×10^{-2}	6.25×10^{-3}
0.9	2.624×10^{-2}	8.53×10^{-3}

Table 3: The local truncation error for second interval of example (6.1)

Time	F Euler	F M Euler
1.1	1.6979×10^{-3}	1.5033×10^{-3}
1.2	2.9094×10^{-3}	1.6632×10^{-3}
1.3	2.1465×10^{-3}	1.8435×10^{-3}
1.4	2.4123×10^{-3}	2.0463×10^{-3}
1.5	2.7099×10^{-3}	2.2748×10^{-3}
1.6	3.0434×10^{-3}	2.5319×10^{-3}
1.7	3.4168×10^{-3}	2.8212×10^{-3}
1.8	3.8349×10^{-3}	3.1464×10^{-3}
1.9	4.3028×10^{-3}	3.512×10^{-3}

Table 4: The local truncation error for last interval of example (6.1)

Time	F Euler	F M Euler
2.1	1.00423×10^{-4}	9.01×10^{-5}
2.2	1.1205×10^{-4}	9.963×10^{-5}
2.3	1.25×10^{-4}	1.1028×10^{-4}
2.4	1.3944×10^{-4}	1.2217×10^{-4}
2.5	1.5553×10^{-4}	1.3546×10^{-4}
2.6	1.7344×10^{-4}	1.5029×10^{-4}
2.7	1.934×10^{-4}	1.6684×10^{-4}
2.8	2.1564×10^{-4}	1.8531×10^{-4}
2.9	2.404×10^{-4}	2.0594×10^{-4}
3	2.6799×10^{-4}	2.2896×10^{-4}

$$Y(t_0) = \delta_3 y(t_0). \tag{5.39}$$

By considering expression (5.38) and using expression (5.36) we conclude

$$Y(t_k^+) \ominus_{gH} Y(t_k^-) = [y(t_k^+) \ominus_{gH} y(t_k^-)] \oplus \delta_2 \odot I_k(y(t_k^-)),$$

$$\Big([y(t_k^+) \oplus \delta \odot (y(t_k^+)] \ominus_{gH} [y(t_k^-) \oplus \delta \odot (y(t_k^-)]\Big) \ominus_{gH} [y(t_k^+) \ominus_{gH} (y(t_k^-)] = \delta_2 \odot I_k(y(t_k)),$$

$$[\delta \odot (y(t_k^+)] \ominus_{gH} [\delta \odot (y(t_k^-)] = \delta_2 \odot I_k(y(t_k)),$$

$$\delta \odot [y(t_k^+) \ominus_{gH} y(t_k^-)] = \delta_2 \odot I_k(y(t_k)).$$

Table 5: The local truncation error for first interval of example (6.2)

Time	F Euler	F M Euler
0.1	1.203×10^{-3}	4.7×10^{-5}
0.2	2.172×10^{-3}	2.09×10^{-4}
1.3	2.942×10^{-3}	4.63×10^{-4}
0.489	3.544×10^{-3}	7.9×10^{-4}
0.5	4.00×10^{-3}	1.175×10^{-3}
0.6	4.335×10^{-3}	1.602×10^{-3}
0.7	4.567×10^{-3}	2.063×10^{-3}
0.8	4.714×10^{-3}	2.545×10^{-3}
0.9	4.79×10^{-3}	3.041×10^{-3}

Table 6: The local truncation error for first interval of example (6.2)

Time	F M Euler
1.1	3.252×10^{-5}
1.2	3.561×10^{-5}
1.3	3.848×10^{-5}
1.4	4.135×10^{-5}
1.5	4.314×10^{-5}
1.6	4.473×10^{-5}
1.7	4.565×10^{-5}
1.8	4.575×10^{-5}
1.9	6.67978×10^{-4}

Table 7: The local truncation error for last interval of example (6.2)

Time	F M Euler
2.1	1.3976×10^{-6}
2.2	1.5398×10^{-6}
2.3	1.689×10^{-6}
2.4	1.8449×10^{-6}
2.5	2.0074×10^{-6}
2.6	2.2958×10^{-6}
2.7	2.35×10^{-6}
2.8	2.531×10^{-6}
2.9	2.714×10^{-6}
3	2.901×10^{-6}

By using the ranking of fuzzy numbers, in definition 2.3

$$\delta \odot [y(t_k^+) \ominus_{gH} y(t_k^-)] \leq \delta_2 \odot I_k(y(t_k)).$$

Finally by considering definition 2.6 we have

$$\delta \leq \frac{I_k(y(t_k))}{[y(t_k^+) \ominus_{gH} y(t_k^-)]} \odot \delta_2. \tag{5.40}$$

Without reduce of generality, since for expression (4.33) is the same, we only consider fuzzy modified Euler's method 3.23, for expressions (5.36)

Table 8: The local truncation error for first interval of example (6.3)

Time	F Euler	F M Euler
3.1	6.46597×10^{-3}	3.63614×10^{-4}
3.2	2.28493×10^{-2}	2.85363×10^{-3}
3.3	4.14864×10^{-2}	8.13481×10^{-3}
3.4	5.888584×10^{-2}	1.59316×10^{-2}
3.5	7.33986×10^{-2}	2.56672×10^{-2}

Table 9: The local truncation error for second interval of example (6.3)

Time	F Euler	F M Euler
4.1	4.32516×10^{-3}	1.17608×10^{-5}
4.2	4.39935×10^{-3}	1.26349×10^{-4}
4.3	1.39573×10^{-2}	2.13926×10^{-4}
4.4	3.13065×10^{-2}	3.21227×10^{-4}
4.5	5.38253×10^{-2}	3.7512×10^{-4}

Table 10: The local truncation error for last interval of example (6.3)

Time	F Euler	F M Euler
5.1	4.99669×10^{-5}	7.58055×10^{-6}
5.2	4.9965×10^{-4}	7.58172×10^{-6}
5.3	4.9932×10^{-3}	7.67938×10^{-5}
5.4	4.99614×10^{-3}	7.5764×10^{-4}
5.5	4.99596×10^{-2}	7.57674×10^{-4}

and (5.38), we obtain

$$\begin{cases} y_0 = \delta_3 y(t_0), \\ y_{k+1} = y_k \oplus \Big[\frac{h}{2} \odot \delta_1(f(t_k, y_k) \\ \qquad \oplus f(t_{k+1}, y_{k+1}))\Big]. \end{cases} \tag{5.41}$$

By setting $k = 1$, in (5.41), we obtain

$$Y_1 = \delta_3 y_0 \oplus \Big[\frac{h}{2} \odot \delta_1(f(t_0, y_0) \oplus f(t_1, y_1))\Big].$$

According to the expression (5.36), we obtain

$$y_1 \oplus \delta \odot y_1 = \delta_3 y_0 \oplus \Big[\frac{h}{2} \odot \delta_1(f(t_0, y_0) \oplus .f(t_1, y_1))\Big].$$

By using the ranking of fuzzy numbers, in definition (2.3)

$$\delta \leq \frac{\delta_3 y_0 \oplus \frac{h}{2} \odot \delta_1(f(t_0, y_0) \oplus f(t_1, y_1))}{y_1}. \tag{5.42}$$

By $\delta_3 \to 0$ we obtain $\delta_1 \to 0$ and $\delta \to 0$, it means the numerical algorithm is stable and converge to the exact solution.
Without reduce of generality, since for $k = 3, \cdots, z$ is the same as for $k = 2$, we only consider for $k = 2$ and according to the expression (5.36), we obtain

$$y_2 \oplus \delta \odot y_2 = (y_1(t_k^+) \oplus \delta \odot y_1(t_k^+) \\ \oplus \Big[\frac{h}{2} \odot (\delta_1(f(t_1, y_1) \oplus f(t_2, y_2)))\Big].$$

By using the ranking of fuzzy numbers, in definition (2.3)

$$y_2 \oplus \delta \odot y_2 \leq (y_1(t_k^+) \oplus \delta \odot y_1(t_k^+) \\ \oplus \Big[\frac{h}{2} \odot (\delta_1(f(t_1, y_1) \oplus f(t_2, y_2)))\Big].$$

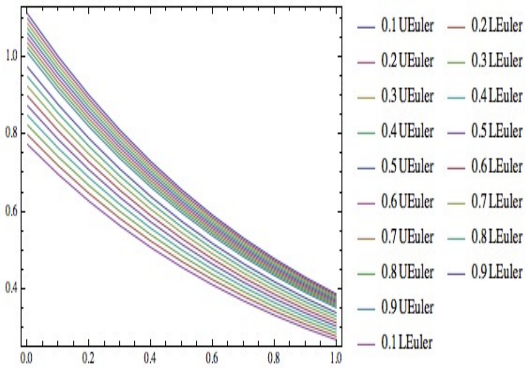

Figure 3: Fuzzy Euler's Method for example (6.2)

Finally, we obtain

$$\delta \leq \frac{(y_1(t_k^+) \oplus \delta \odot y_1(t_k^+))}{y_2} \oplus \frac{\left[\frac{h}{2} \odot (\delta_1(f(t_1, y_1) \oplus f(t_2, y_2)))\right]}{y_2}.$$

According to the expression (4.34) we obtain

$$\delta \leq \frac{\delta_3 y_0 \oplus [\frac{h}{2} \odot (\delta_1(f(t_0, y_0) \oplus f(t_1, y_1)))}{y_1} \odot \frac{I_k(y(t_k)) \odot \delta_2}{[y(t_k^+) \ominus_{gH} y(t_k^-)]}. \quad (5.43)$$

Upper bound obtained (5.43). Shows us for having the stable numerical algorithm and converges to exact solution, except need to $\delta_3 \to 0$, moreover, we need imported perturbation in right and side of the expression (5.38) near to zero too, it means $\delta_2 \to 0$.
Under this condition we can say for $k = 2, \cdots, z$, the numerical algorithm introduced is stable and converges to the exact solution

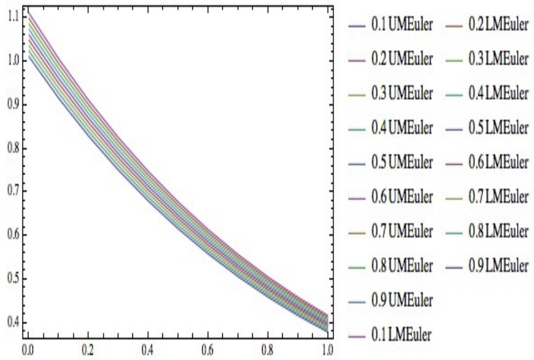

Figure 4: Fuzzy Modified Euler's Method for example (6.2)

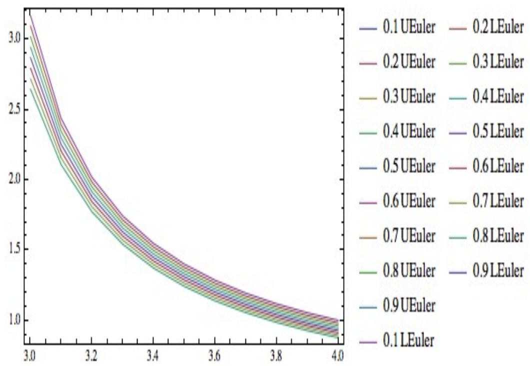

Figure 5: Fuzzy Modified Euler's Method for example (6.3)

6 Numerical Result

Example 6.1. *Consider the first order impulsive fuzzy initial value problem*

$$y'(t) = y(t), \quad t \in [0, 3], \quad (6.44)$$

$$y(t_k^+) = 0.01 y(t_k^-), \quad (6.45)$$

$$y(0) = (0.75 + 0.25r, 1.125 - 0.125r). \quad (6.46)$$

Where $0 \leq r \leq 1$.
And by considering $t_z = 3$.
The solution is calculated by using a numerical algorithm in section 4, and by considering $r = 0.9, h = 0.05, h = 0.025$.
The global truncation error calculated by the Hausdorff distance between fuzzy numbers is given in definition 2.3.
The local truncation error, for example (6.1).

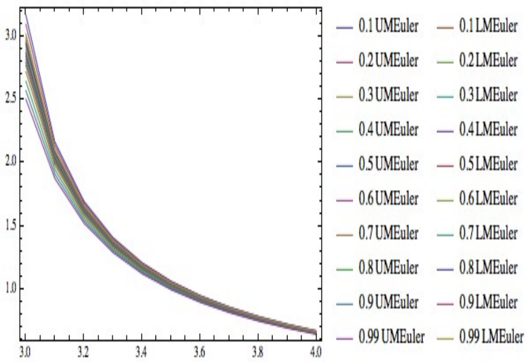

Figure 6: Fuzzy Modified Euler's Method for example (6.3)

Example 6.2. *Consider the first order impulsive fuzzy initial value problem*

$$y'(t) = -y(t), \tag{6.47}$$

$$y(t_k^+) = 0.01y(t_k^-), \tag{6.48}$$

$$y(0) = (0.75 + 0.25r, 1.125 - 0.75r). \tag{6.49}$$

Where $0 \leq r \leq 1$.
And by considering $t_z = 3$.
The solution is calculated by using a numerical algorithm in section 4, and by considering $r = 0.9, h = 0.05, h = 0.025$.
The global truncation error calculated by the Hausdorff distance between fuzzy numbers is given in definition 2.3.

Tables (6), (7), (8) show us at all intervals among pulse of the domain, the error of fuzzy modified Euler's method and error of Euler's method are stable. It shows the numerical algorithm for solving linear impulsive differential equation with the fuzzy initial equation by using fuzzy modified Euler's method, and Euler's method is stable.

Example 6.3. *Consider the first order impulsive fuzzy initial value problem*

$$y'(t) = -y^2(t), \tag{6.50}$$

$$y(t_k^+) = 0.01y(t_k^-), \tag{6.51}$$

$$y(0) = (2.75 + 0.25r, 3.125 - 0.125r). \tag{6.52}$$

Where $0 \leq r \leq 1$.
And by considering $t_z = 5.5$.
The solution is calculated by using a numerical algorithm in section 4, and by considering $r = 0.9, h = 0.05, h = 0.025$.
The global truncation error calculated by the Hausdorff distance between fuzzy numbers is given in definition 2.3.

Tables (8), (9), (10) show us at all intervals among pulse of domain, the error of fuzzy modified Euler's method is less than the error of Euler's method. In the final point the error of fuzzy modified Euler's method is little, and it shows the numerical algorithm for solving non-linear impulsive differential equation with the fuzzy initial equation by using fuzzy modified Euler's method, not only is acceptable but also is very satisfied. The algorithm is converging to the exact solution of the problem.

7 Conclusion

The plotted graphs and obtained global truncation error tables show the presented numerical algorithm for solving impulsive fuzzy initial value problem by using fuzzy modified Euler's method and fuzzy Euler's method is stable, and presentations are acceptable solution for this problem. This algorithm can run with each fuzzy repetitious method. As we now, solving this equation is complex or impossible, but our paper can give good ideas for solving impulsive fuzzy equations.

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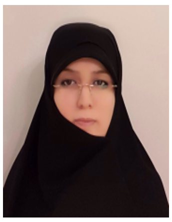

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