Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 13, No. 1, 2021 Article ID IJIM-1265, 14 pages
DOR: http://dorl.net/dor/20.1001.1.20085621.2021.13.1.2.5
Research Article

Science and Research Branch (IAU)

Numerical Solution of Interval Volterra-Fredholm-Hammerstein Integral Equations via Interval Legendre Wavelets Method

N. Khorrami *, A. Salimi Shamloo †‡, B. Parsa Moghaddam §

Received Date: 2018-12-23 Revised Date: 2019-02-06 Accepted Date: 2019-05-01

Abstract

In this paper, interval Legendre wavelet method is investigated to approximated the solution of the interval Volterra-Fredholm-Hammerstein integral equation. The shifted interval Legendre polynomials are introduced and based on interval Legendre wavelet method is defined. The existence and uniqueness theorem for the interval Volterra-Fredholm-Hammerstein integral equations is proved. Some examples show the effectiveness and efficiency of the approach.

Keywords : GInterval Volterra-Fredholm-Hammerstein integral equation; Interval Legendre Polynomial; Interval Shifted Legendre Polynomial; Interval Legendre wavelet method; Interval System of Equation.

1 Introduction

INterval analysis is a powerful tool to compute with intervals of real numbers in place of real numbers. While floating point arithmetic is affected by rounding errors, and can produce inaccurate results, interval arithmetic has the advantage of giving rigorous bounds for the exact solution. The concept of interval analysis was first introduced by Moore [14]. Since then, thousands of articles have appeared and numerous books published on the subject.

Interval algorithms may be used in most areas of numerical analysis, and are used in many applications such as engineering problems and computer aided design. Another application is in computer assisted proofs. Several conjectures have recently been proven using interval analysis, perhaps most famously Kepler's conjecture [6], which remained unsolved for nearly 400 years.

The starting point of the topic in the set valued differential equation and also fuzzy differential equation is Hukuhara's paper [7]. Since then, reliable computing, validated numerics and interval problems with differential equations are discussed in several monographs and research papers [1, 2, 9, 16, 17, 18, 19, 26, 27].

Integral equation arise in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, commu-

*Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran.
†Corresponding author. ali-salimi@iaushab.ac.ir, Tel:+98(914)4456325.
‡Department of Mathematics, Shabestar Branch, Islamic Azad University, Shabestar, Iran.
§Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran.

nication theory, mathematical economics, population genetics, radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics, etc [22, 23, 24, 25]. But the concept of interval integral equations have not been studied in many articles. In paper [29] the interval-valued Volterra integral equations are introduced and existence and uniqueness of solutions interval-valued Volterra integral equations is studied.

The aim of the present paper is two topics. One of our intentions is to prove the existence and uniqueness of a solution for interval Volterra-Fredholm-Hammerstein integral equation. The other aim is to obtain the solution of a interval Volterra-Fredholm-Hammerstein integral equation using interval Legendre wavelets method. To find this solution, we first introduce interval shifted Legendre polynomials and based on this definition, we define interval Legendre wavelets method. Using this method, the interval Volterra-Fredholm-Hammerstein integral equation is converted into a interval system of algebraic equations that can be solved by any numerical method.

The paper is organized as follows. Section 2 collects some definitions of basic notions concerning interval calculus, and then in Section 3 we describe the basic definitions and preliminaries about interval function, interval polynomials, interval Legendre polynomials and shifted interval Legendre polynomial. The interval Volterra-Fredholm-Hammerstein integrals equation and the existence and uniqueness of the solutions of this set of equations are studied in Section 4. To determine the approximate solution for the interval Volterra-Fredholm-Hammerstein integrals equation, interval Legendre wavelet method has been introduced in Section 5. In Section 6, some examples are given to show the efficiency of the proposed method and conclusions are drawn in Section 7

2 Preliminaries

In what follows, we briefly recall the basic definitions and properties of the interval calculus. Let $\mathbb{R}$ be the set of all real numbers. Interval X is a closed bounded compact subset of $\mathbb{R}$ that

$$X = \{x : x \in \mathbb{R}, a \leq x \leq b, a, b \in \mathbb{R}, \\ \infty < a \leq b < +\infty\}.$$

The set of all intervals will be showed by $I(\mathbb{R})$. If X is an interval, we will show that its lower (left) endpoint by $\underline{x}$ and its upper (right) endpoint by $\overline{x}$, so $X = [\underline{x}, \overline{x}]$. The set interval $[\varnothing, \varnothing]$ is a singleton which contains a single element : $\varnothing \in [\varnothing, \varnothing]$, $\varnothing = \{0\} = [0, 0]$.

We call two interval $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$ equal if and only if $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$. The width of an interval X is defined and denoted by

$$w(X) = \overline{x} - \underline{x}.$$

The absolute value of X, denoted $|X|$, is scalar values and defined by

$$|X| = \max\{|\underline{x}|, |\overline{x}|\}.$$

We call the dual of an interval X the value

$$dual(X) = [\overline{x}, \underline{x}].$$

The midpoint of an interval is scalar values and defined by

$$m(X) = \frac{(\underline{x} + \overline{x})}{2}.$$

We define the radius of an interval X by

$$rad(X) = \frac{(\overline{x} - \underline{x})}{2}.$$

Definition 2.1 *(See [11]) The Hausdorff distance between two intervals $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$ is:*

$$D(Y, X) = max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\},$$

and for all $A, B, C \in I(\mathbb{R})$ satisfies in the following properties

1. $D(A + C, B + C) = D(A, B)$,

2. $D(A, B) = D(B, A)$,

3. $D(\lambda A, \lambda B) = |\lambda| D(A, B), \qquad \forall \lambda \in \mathbb{R}$,

4. $D(A, B) \leq D(A, C) + D(C, B)$.

2.1 Interval Operations:

The four classical operations of real arithmetic, namely addition (+), subtraction (−), multiplication (∗) and division (/) can be extended to intervals. For any such binary operator, denoted by $\lozenge$, performing the operation associated with $\lozenge$ on the intervals $X \in I(\mathbb{R})$ and $Y \in I(\mathbb{R})$ means computing [15, 10]

$$X \lozenge Y = \{x \lozenge y \in \mathbb{R} | x \in X, y \in Y\}.$$

If X and Y are real intervals then specific equations for interval operations are:

$$X + Y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}].$$

$$X - Y = [\underline{x} - \overline{y}, \overline{x} - \underline{y}].$$

$$X * Y = [\underline{x}, \overline{y}] * [\underline{y}, \overline{y}] = \Big[min(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}), max(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y})\Big].$$

2.2 Interval Matrices and Interval System of Equation

Definition 2.2 *(See [15]) An interval matrix is a matrix whose elements are interval numbers.*

Definition 2.3 *(See [15]) The ij-th element C_{ij} of the product $C = AB$ of a m by p interval matrix A and a p by n interval matrix B gives sharp bounds on the range of $C_{ij} = \{M_{ij} = \sum_{k=1}^{p} P_{ik}Q_{kj} : P_{ik} \in A_{ik}$ and $Q_{kj} \in B_{kj}$ for $1 \le k \le p\}$ for each i, $1 \le i \le m$, and each j , $1 \le j \le n$.*

Most of the properties of determinants of classical matrices are held for the determinants of interval matrices under the modified interval arithmetic.

Definition 2.4 *(See [8]) A square interval matrix A is said to be non-singular or regular if $\det(A)$ is invertible (i.e. $0 \notin \det(A)$). Alternatively, a square interval matrix A is said to be invertible if $\det(A)$ is invertible (i.e. $0 \notin \det(A)$).*

Definition 2.5 *(See [8]) An interval matrix A is regular if every point matrix $A \in A$ is nonsingular.*

Definition 2.6 *(See [8]) Let A be a square interval matrix. The adjoint matrix A^* of A is the transpose of the matrix of cofactors of the elements of A. That is $A^* = adj(A) = (b_{ij})$, where $b_{ij} = \det(A_{ji})$, for all $i, j = 1, 2, 3, \dots, n$.*

Definition 2.7 *(See [8]) For any $A \in I(\mathbb{R})^{n\times n}$, if $\det(A)$ is invertible, then the common solution of equations $AX = I$ and $XA = I$ is called the inverse of A and is denoted by $A^{-1} = \frac{adj(A)}{\det(A)} = \frac{A^*}{\det(A)}$.*

3 Interval Function and Interval Polynomial

Let us consider $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$ are interval. We say that Y is an interval function of X, $Y = F(X)$, if to every X is a certain domain $D \subseteq I(\mathbb{R})$ there corresponds one interval Y. Symbolically $F : D \subseteq I(\mathbb{R}) \to I(\mathbb{R})$.

Definition 3.1 *(See [12]) We say the interval valued mapping $F : D \to I(\mathbb{R})$ is continuous at the point $t \in D$ if, for every $\varepsilon > 0$ there exists $\delta = \delta(t, \varepsilon) > 0$ such that for all $s \in D$ such that $|t - s| < \delta$ one has $D\Big(F(t), F(s)\Big) \le \varepsilon$.*

Let us to consider, we introduce in the space of interval continuous functions defined in $[a, b]$ which we denote by $\mathbb{C}[a, b]$.

Definition 3.2 *(See [14]) A real interval polynomial with degree n is defined by*

$$P_n(t) = \sum_{j=0}^{n} A_j x^{n-j}, \tag{3.1}$$

with $A_0 = 1$, $A_j = [\underline{a}_j, \overline{a}_j] \subset I(\mathbb{R})$, $j = 1, ..., n$.

By Eq.(3.1) it is easy to see that $P_n(t)$ is a family of polynomials

$$p_n(t) = \sum_{j=0}^{n} a_j x^{n-j}, \tag{3.2}$$

where $a_0 = 1$, $a_j \in A_j$, $j = 1, ..., n$. According the definition of a real function, the graph of a real interval polynomial is introduced as follows.

Definition 3.3 *(See [20]) Let $P_n(t)$ be a real interval polynomial. the graph of $P_n(t)$ is denoted by $G(P_n)$ and is given by*

$$G(P_n) = \{(\tilde{t}, \tilde{y}) \in \mathbb{R}^2 : \exists p_n \in P_n, \tilde{y} = p_n(\tilde{x})\}.$$

Let $\overline{q}(t)$, $\overline{r}(t)$, $\underline{q}(t)$ and $\underline{r}(t)$ be the following real polynomials

$$\begin{aligned}\overline{q}(t) &= \sum_{j=0}^{n} \overline{q}_j t^{n-j}, \quad \overline{r}(t) = \sum_{j=0}^{n} \overline{r}_j t^{n-j} \quad (3.3)\\ \underline{q}(t) &= \sum_{j=0}^{n} \underline{q}_j t^{n-j}, \quad \underline{r}(t) = \sum_{j=0}^{n} \underline{r}_j t^{n-j},\end{aligned}$$

where

$$\begin{aligned}&\overline{q}_0 = \overline{r}_0 = \underline{q}_0 = \underline{r}_0 = 1,\\ &\overline{q}_j = \overline{a}_j, \quad \overline{r}_j = \underline{a}_j, \quad j = 1, ..., n.\end{aligned}$$

and

$$\begin{aligned}\underline{q}_j &= \begin{cases} \overline{a}_j, & \text{If } n-j \text{ is even;}\\ \underline{a}_j, & \text{If } n-j \text{ is odd .}\end{cases},\\ \underline{r}_j &= \begin{cases} \underline{a}_j, & \text{If } n-j \text{ is even;}\\ \overline{a}_j, & \text{If } n-j \text{ is odd .}\end{cases}\end{aligned}$$

Lemma 3.1 *(See [20]) Let $P_n(t)$ be the real interval polynomial given by (3.1). The graph of P_n is given by*

$$G(P_n) = \Big\{(x,y) \in \mathbb{R}^2 : \Big(\overline{r}(t) \le y \le \overline{q}(t) \; if \; t \ge 0\Big) or \Big(\underline{r}(t) \le y \le \underline{q}(t) \; if \; t < 0\Big)\Big\}.$$

with $\overline{q}(t)$, $\overline{r}(t)$, $\underline{q}(t)$ and $\underline{r}(t)$ are given by (3.3).

Definition 3.4 *(See [20]) Let $F(t)$ be a real interval function continuous in $[a,b]$ and $F(t) = [\underline{F}(t), \overline{F}(t)]$ for all $t \in [a,b]$. Then*

$$\int_a^b F(t)dt = \Big[\int_a^b \underline{F}(t)dt, \; \int_a^b \overline{F}(t)dt\Big]. \quad (3.4)$$

Remark 3.1 *(See [20]) If $F(t)$ is a real interval polynomial, that is $F(t) = P_n(t)$ so*

$$\int_a^b P_n(t)dt = \Big[\int_a^b \overline{r}(t)dt, \; \int_a^b \overline{q}(t)dt\Big], if [a,b] \ge 0$$

$$\int_a^b P_n(t)dt = \Big[\int_a^b \underline{r}(t)dt, \; \int_a^b \underline{q}(t)dt\Big], if [a,b] \ge 0$$

$$\int_a^b P_n(t)dt = \Big[\int_a^b \underline{r}(t)dt, \; \int_a^b \underline{q}(t)dt\Big] + \Big[\int_a^b \overline{r}(t)dt, \; \int_a^b \overline{q}(t)dt\Big], if [a,b] \cup [0,b]$$

The following inner product has values in set of all real intervals $\mathbb{R}$.

Definition 3.5 *Let $< .,. >: \mathbb{C}[a,b] \times \mathbb{C} \to \mathbb{R}$ be defined by*

$$< F, G >= \int_a^b F(x)G(x)dx, F, G \in \mathbb{C}[a,b]. \quad (3.5)$$

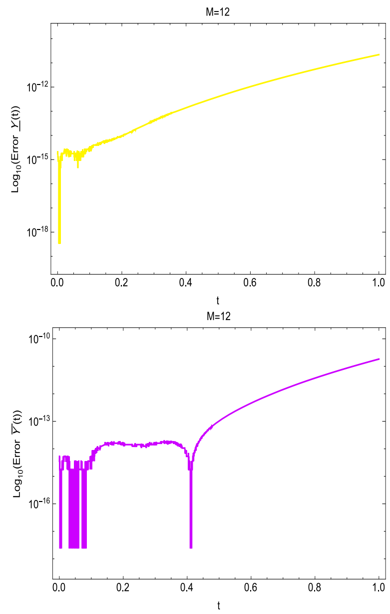

Figure 1: The error function graph for Example 6.2.

3.1 Interval Legendre Polynomial

Definition 3.6 *(See [20]) Let us consider, for each natural number ℓ, the family of interval polynomials defined by the following recursive formula*

1. $\mathbb{L}_{0,\ell}(t) = [1 - \frac{1}{\ell}, 1 + \frac{1}{\ell}]$,

2. $\mathbb{L}_{1,\ell}(t) = [1 - \frac{1}{\ell}, 1 + \frac{1}{\ell}]t$,

3. *for $m \in \mathbb{N}$,*

$$\mathbb{L}_{m+1,\ell}(t) = \frac{2m+1}{m+1} t\mathbb{L}_{m,\ell}(t) - \frac{m}{m+1}\mathbb{L}_{m-1,\ell}(t). \quad (3.6)$$

For each $\ell \in \mathbb{N}$ and $m \in \mathbb{N}$, we call $\mathbb{L}_{m,\ell}(t)$ interval Legendre polynomial.

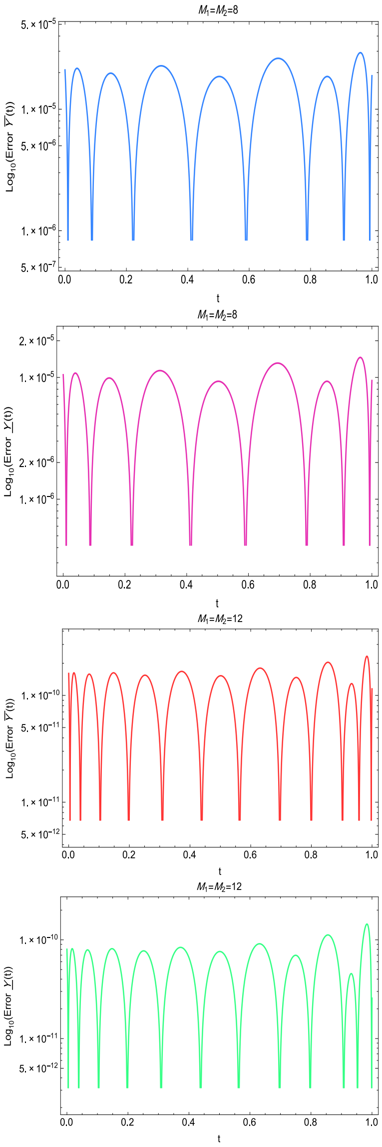

Figure 2: Graph of the Wavelet approximation error for Example 6.3.

Theorem 3.1 *(See [20]) The interval Legendre polynomial $\mathbb{L}_{m,\ell}(t)$ is equal to the interval polynomial obtained from the real Legendre polynomial $L_m(t)$ considering their coefficients multiplied by $[1-\frac{1}{\ell}, 1+\frac{1}{\ell}]$.*

Theorem 3.2 *(See [20]) The interval Legendre polynomials $\mathbb{L}_{m,\ell}$, $m \in \mathbb{N}$, satisfy*

1. *If m is even, then*

$$\begin{aligned}
\underline{r}(t) &= \overline{q}(t) \\
&= \sum_{j=0}^{\frac{m}{2}} a_j\Big(1+\frac{(-1)^j}{\ell}\Big)(-1)^j t^{m-2j}, \\
\overline{r}(t) &= \underline{q}(t) \\
&= \sum_{j=0}^{\frac{m}{2}} a_j\Big(1+\frac{(-1)^{j+1}}{\ell}\Big)(-1)^j t^{m-2j}.
\end{aligned}$$

2. *If m is odd, then*

$$\begin{aligned}
\underline{r}(t) &= \overline{q}(t) \\
&= \sum_{j=0}^{\frac{m-1}{2}} a_j\Big(1+\frac{(-1)^j}{\ell}\Big)(-1)^j t^{m-2j}, \\
\overline{r}(t) &= \underline{q}(t) \\
&= \sum_{j=0}^{\frac{m-1}{2}} a_j\Big(1+\frac{(-1)^{j+1}}{\ell}\Big)(-1)^j t^{m-2j},
\end{aligned}$$

where

$$a_j = \frac{(2m-2j)!}{2^m j!\,(m-j)!\,(m-2j)!}.$$

Definition 3.7 *The interval shifted Legendre polynomials are defined on $[0,1]$ as*

$$\tilde{\mathbb{L}}_{m,\ell}(t) = \mathbb{L}_{m,\ell}(2t-1),$$

and an explicit expression for the interval shifted Legendre polynomials is given by

$$\begin{aligned}
&\tilde{\mathbb{L}}_{m,\ell}(t) \\
&= \sum_{j=0}^{m} \binom{m}{j}\binom{m+j}{j}(-1)^{j+m}\Big[1-\frac{1}{\ell}, 1+\frac{1}{\ell}\Big]t^j.
\end{aligned}$$

Theorem 3.3 *The interval shifted Legendre polynomials $\tilde{\mathbb{L}}_{m,\ell}$, $m \in \mathbb{N}$, satisfies*

1. *If m is even, then*

$$\overline{r}(t) = \underline{r}(t) =$$

$$\sum_{j=0}^{m}\binom{m}{j}\binom{m+j}{j}\Big(1+\frac{(-1)^{j+1}}{\ell}\Big)(-1)^{j+m}t^j,$$

$$\overline{q}(t) = \underline{q}(t) =$$

$$\sum_{j=0}^{m}\binom{m}{j}\binom{m+j}{j}\Big(1+\frac{(-1)^{j}}{\ell}\Big)(-1)^{j+m}t^j.$$

2. *If m is odd, then*

$$\overline{r}(t)=\underline{r}(t)=$$
$$\sum_{j=0}^{m}\binom{m}{j}\binom{m+j}{j}\Big(1+\frac{(-1)^j}{\ell}\Big)(-1)^{j+m}t^j,$$

$$\overline{q}(t)=\underline{q}(t)=$$
$$\sum_{j=0}^{m}\binom{m}{j}\binom{m+j}{j}\Big(1+\frac{(-1)^{j+1}}{\ell}\Big)(-1)^{j+m}t^j.$$

Proof. The interval shifted Legendre polynomials $\tilde{\mathbb{L}}_{m,\ell}$ are defined on interval $[0,1)$, then using Lemma 3.1 the proof of the theorem is clear.

4 Interval Volterra-Fredholm-Hammerstein integral equation

Consider the following interval Volterra-Fredholm-Hammerstein integral

$$\begin{aligned} Y(t) &= G(t)+\lambda_1\int_{t_0}^{t}k_1(t,x)Y(x)dx \qquad (4.7)\\ &+ \lambda_2\int_{t_0}^{T}k_2(t,x)Y(x)dx\ , t\in J=[t_0,T], \end{aligned}$$

where $G: J\to I(\mathbb{R})$ is interval continuous function in J, λ_1 and λ_2 are positive constants. $k_1,k_2: J\times J\to\mathbb{R}$ such that

$$\begin{aligned} K_1^* &= \sup_{t\in J}\int_{t_0}^{t}\mid k_1(t,s)\mid dx\\ K_2^* &= \sup_{t\in J}\int_{t_0}^{T}\mid k_2(t,x)\mid dx. \end{aligned}$$

Now, we study the existence and uniqueness of solutions of problem (4.7). We define the operator T by

$$TY(t):=G(t)+\lambda_1\int_{t_0}^{t}k_1(t,x)Y(x)dx$$
$$+\ \lambda_2\int_{t_0}^{T}k_2(t,x)Y(x)dx \qquad (4.8)$$

Assume that $Y: J\to I(\mathbb{R})$ be interval continuous function on J and there exist real positive L_1, L_2 such that

$$\begin{aligned} &D\Big(\lambda_1\int_{t_0}^{t}k_1(t,x)Y(x)dx\\ &+\lambda_2\int_{t_0}^{T}k_2(t,x)Y(x)dx, \qquad (4.9)\\ &\lambda_1\int_{t_0}^{t}k_1(t,x)V(x)dx\\ &+\lambda_2\int_{t_0}^{T}k_2(t,x)V(x)dx\Big) \qquad (4.10)\\ &\le L_1\lambda_1 D\Big(\int_{t_0}^{t}k_1(t,x)Y(x)dx\\ &,\int_{t_0}^{t}k_1(t,x)V(x)dx\Big)\\ &+L_2\lambda_2 D\Big(\int_{t_0}^{T}k_2(t,x)Y(x)dx\\ &,\int_{t_0}^{T}k_2(t,x)V(x)dx\Big). \qquad (4.11) \end{aligned}$$

If a number υ such that $\mathcal{B}\le\upsilon<1$ where

$$\mathcal{B}=\Big(L_1\lambda_1K_1^*+L_2\lambda_2K_2^*\Big),$$

then the iterative procedure

$$\begin{aligned} Y_0(t) &= G(t),\\ Y_m(t) &= G(t)+\lambda_1\int_{t_0}^{t}k_1(t,x)Y_{m-1}(x)dx\\ &+ \lambda_2\int_{t_0}^{T}k_2(t,x)Y_{m-1}(x)dx,\ m\ge 1, \end{aligned} \qquad (4.12)$$

convergence to the unique solution of (4.7). In addition if $D(Y(t),0)\le M_0$ then

$$D(Y(t),Y_m(t))\le\mathcal{B}^mM_0.$$

Proof. We show that T defined by (4.8) has a fixed point. For this purpose, suppose that the integral equation has two different solution $Y(t),V(t)\in\mathbb{C}[a,b]$. Using Eq.(4.11) and the properties of distance Hausdorff (2.1), we get

Table 1: Interval Legendre polynomials in different values of t and m

m	Interval Legendre polynomials ($t<0$)	Interval Legendre Wavelets ($t>0$)
0	$[1-\frac{1}{\ell}, 1+\frac{1}{\ell}]$	$[1-\frac{1}{\ell}, 1+\frac{1}{\ell}]$
1	$\left[\left(1+\frac{1}{\ell}\right)t, \left(1-\frac{1}{\ell}\right)t\right]$	$\left[\left(1-\frac{1}{\ell}\right)t, \left(1+\frac{1}{\ell}\right)t\right]$
2	$\left[\frac{3}{2}\left(1-\frac{1}{\ell}\right)t^2-\frac{1}{2}\left(1+\frac{1}{\ell}\right), \frac{3}{2}\left(1+\frac{1}{\ell}\right)t^2-\frac{1}{2}\left(1-\frac{1}{\ell}\right)\right]$	$\left[\frac{3}{2}\left(1-\frac{1}{\ell}\right)t^2-\frac{1}{2}\left(1+\frac{1}{\ell}\right), \frac{3}{2}\left(1+\frac{1}{\ell}\right)t^2-\frac{1}{2}\left(1-\frac{1}{\ell}\right)\right]$
3	$\left[\frac{5}{2}\left(1+\frac{1}{\ell}\right)t^3-\frac{3}{2}\left(1-\frac{1}{\ell}\right)t, \frac{5}{2}\left(1-\frac{1}{\ell}\right)t^3-\frac{3}{2}\left(1+\frac{1}{\ell}\right)t\right]$	$\left[\frac{5}{2}\left(1-\frac{1}{\ell}\right)t^3-\frac{3}{2}\left(1+\frac{1}{\ell}\right)t, \frac{5}{2}\left(1+\frac{1}{\ell}\right)t^3-\frac{3}{2}\left(1-\frac{1}{\ell}\right)t\right]$

$$
\begin{aligned}
D\Big(TY(t),TV(t)\Big) \leq \\
& D\Big(G(t)+\lambda_1\int_{t_0}^{t} k_1(t,x)Y(x)dx \\
& +\lambda_2\int_{t_0}^{T} k_2(t,x)Y(x)dx)+G(t)+ \\
& \lambda_1\int_{t_0}^{t} k_1(t,x)V(x)dx \\
& \quad +\lambda_2\int_{t_0}^{T} k_2(t,x)V(x)dx\Big) \\
& \quad \leq L_1D\Big(\lambda_1\int_{t_0}^{t} k_1(t,x)Y(x)dx, \\
& \quad \lambda_1\int_{t_0}^{t} k_1(t,x)V(x)dx\Big) \\
& \quad +L_2D\Big(\lambda_2\int_{t_0}^{T} k_2(t,x)Y(x)dx, \\
& \quad \lambda_2\int_{t_0}^{T} k_2(t,x)V(x)dx\Big) \\
& \leq L_1\lambda_1\int_{t_0}^{t}|k_1(x,t)|D(V(x),Y(x))+ \\
& L_2\lambda_2\int_{t_0}^{T}|k_2(x,t)|D(Y(x),V(x)) \\
\leq \quad & L_1\lambda_1K_1^*D(Y(t),V(t)) \\
+ \quad & L_2\lambda_2K_2^*D(Y(t),V(t)) \\
\leq \quad & \Big(L_1\lambda_1K_1^*+L_2\lambda_2K_2^*\Big)D(Y(t),V(t)) \\
\leq \quad & \mathcal{B}D(Y(t),V(t)). \qquad (4.13)
\end{aligned}
$$

Since $\mathcal{B}<1$ then

$$D(TY(t),TV(t)) \quad \leq \quad D^*(Y(t),V(t)).$$

Therefore, T is a contraction mapping on $\mathbb{C}[a,b]$ and has a fixed point $TY(t)=Y(t)$. Hence the interval Volterra-Fredholm-Hammerstein integral (4.7) has a unique solution.

Now, in Eq.(4.12) by the mathematical induction method, we can see that all $\{Y_m(t)\}_{m\geq 0}$ are interval continuous mapping on J. We have

$$
\begin{aligned}
& D\Big(Y_1(t),Y_0(t)\Big) \leq D\Big(G(t) \\
& +\lambda_1\int_{t_0}^{t} k_1(t,x)Y_1(x)dx \\
& +\lambda_2\int_{t_0}^{T} k_2(t,x)Y_1(x)dx, G(t)\Big) \\
& \leq L_1D\Big(\lambda_1\int_{t_0}^{t} k_1(x,t)Y_1(x)dx, 0\Big) \\
& +L_2D\Big(\lambda_2\int_{t_0}^{T} k_2(x,t)Y_1(x)dx\Big) \\
\leq \quad & \mathcal{B}M_0. \qquad (4.14)
\end{aligned}
$$

And we obtain

$$D\Big(Y_m(t),Y_{m-1}(t)\Big) \leq \mathcal{B}D\Big(Y_{m-1}(t),Y_{m-2}(t)\Big).$$

In particular

$$D\Big(Y_2(t),Y_1(t)\Big) \leq \mathcal{B}D\Big(Y_1(t),Y_0(t)\Big) \leq \mathcal{B}^2M_0.$$

Therefore we obtain

$$D\Big(Y_m(t),Y_{m-1}(t)\Big) \leq \mathcal{B}^mM_0. \qquad (4.15)$$

It follows by mathematical induction that Eq.(4.15) holds for any $m \geq 0$. Consequently the sequence $\{Y_m(t)\}_{m\geq 0}$ is uniformly convergent. It follows that there exists a interval continuous function $Y : J \to I(\mathbb{R})$ such that $D(Y_m(t), Y(t)) \to 0$ as $m \to \infty$.

Remark 4.1 *Indeed, in previous theorem the existence and uniqueness of solution of Eq. (4.7) and the convergence of the sequence of successive approximations $Y_m(t)$ to its exact solution are proved.*

5 Interval Legendre Wavelet Method

Wavelets constitute a family of functions constructed from dilation and translation of single interval function called the mother wavelet $\psi(t)$. They are defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\Big(\frac{t-b}{a}\Big), \qquad a, b \in \mathbb{R},$$

where a is dilation parameter and b is a translation parameter.

The interval Legendre wavelets $\psi_{n,m,\ell}(t) = \psi(k, n, m, \ell, t)$ have five arguments, defined on interval $[0, 1)$ by:

$$\psi_{n,m,\ell}(t) = \tag{5.16}$$

$$\begin{cases} \left(m+\frac{1}{2}\right)^{\frac{1}{2}} 2^{\frac{k}{2}}\mathbb{L}_{m,\ell}(2^k t - 2n + 1), \\ \quad \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, \qquad \text{O.W.} \end{cases} \tag{5.17}$$

where $\ell \in \mathbb{N}$, $k \in \mathbb{Z}^+$, $n = 1, 2, 3, ..., 2^{k-1}$ and $m = 0, 1, ..., M-1$ is the order of the interval Legendre polynomials and M is a fixed positive integer.

In Table 2, we show the interval Legendre wavelets when $k = 1$ and $m = 0, 1, 2, 3$. Notice that if $k = 1$ then $n = 1$ and in Eq.(5.17) for $0 \leq t < 1$ we have

$$\begin{aligned} \psi_{1,m,\ell}(t) &= \left(m+\frac{1}{2}\right)^{\frac{1}{2}} 2^{\frac{1}{2}}\mathbb{L}_{m,\ell}(2t-1) \\ &= \left(m+\frac{1}{2}\right)^{\frac{1}{2}} 2^{\frac{1}{2}}\tilde{\mathbb{L}}_{m,\ell}(t). \end{aligned}$$

5.1 Function approximation by interval Legendre wavelets

A function $Y(t)$ defined over $[0, 1)$ can be expanded in terms of interval Legendre wavelets as

$$Y(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m,\ell}(t). \tag{5.18}$$

If the infinite series in Eq. (5.18) is truncated, then it can be written as

$$\begin{aligned} Y_{k,m}(t) &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m,\ell}(t) \\ &= C^T\ \Psi(t), \end{aligned} \tag{5.19}$$

where $\Psi(x)$ is $(2^{k-1}M \times 1)$ interval matrix, given by

$$\begin{aligned} \Psi(t) = [&\Psi_{1,0,\ell}(t), \Psi_{1,1,\ell}(t) \\ &, ..., \Psi_{1,M-1,\ell}(t), \Psi_{2,0,\ell}, ..., \Psi_{2,M-1,\ell}(t), ..., \\ &\Psi_{2^{k-1},0,\ell}, ..., \Psi_{2^{k-1},M-1,\ell}(t)]. \end{aligned} \tag{5.20}$$

Also, C is $(2^{k-1}M \times 1)$ interval matrix whose elements can be calculated from the formula

$$\begin{aligned} C_{n,m} &= < Y(t), \Psi_{n,m,\ell}(t) > \\ &= \int_0^1 \Psi_{n,m,\ell}(t)Y(t)\mathtt{d}t, \end{aligned}$$

and

$$\begin{aligned} C = [&c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, ..., c_{2,M-1}, \\ &..., c_{2^{k_1-1},0}, ..., c_{2^{k-1},M-1}]^T \end{aligned} \tag{5.21}$$

where $c_{i,j}$ are unknown interval values and the symbol T denotes transpose

5.2 The Proposed Method

Consider the following interval linear Volterra-Fredholm-Hammerstein integral equations that given by the general form

$$\begin{aligned} Y(t) &= G(t) + \lambda_1\int_0^t k_1(t,x)Y(x)dx \\ &+\lambda_2\int_0^1 k_2(t,x)Y(x)dx,\ 0 \leq t, x \leq 1, \end{aligned} \tag{5.22}$$

Table 2: Interval Legendre Wavelets for different values of m

m	Interval Legendre Wavelets($\phi_{1,m,\ell}(t)$)
0	$[1-\frac{1}{\ell},\ 1+\frac{1}{\ell}]$
1	$\sqrt{3}\Big[2\Big(1-\frac{1}{\ell}\Big)t-\Big(1+\frac{1}{\ell}\Big),\ 2\Big(1+\frac{1}{\ell}\Big)t-\Big(1-\frac{1}{\ell}\Big)\Big]$
2	$\sqrt{5}\Big[6\Big(1-\frac{1}{\ell}\Big)t^2-6\Big(1+\frac{1}{\ell}\Big)t+\Big(1-\frac{1}{\ell}\Big),\ 6\Big(1+\frac{1}{\ell}\Big)t^2-6\Big(1-\frac{1}{\ell}\Big)t+\Big(1+\frac{1}{\ell}\Big)\Big]$
3	$\Big[20\Big(1-\frac{1}{\ell}\Big)t^3-30\Big(1+\frac{1}{\ell}\Big)t^2+12\Big(1-\frac{1}{\ell}\Big)t-\Big(1+\frac{1}{\ell}\Big),\ 20\Big(1+\frac{1}{\ell}\Big)t^3-30\Big(1-\frac{1}{\ell}\Big)t^2+12\Big(1+\frac{1}{\ell}\Big)t-\Big(1-\frac{1}{\ell}\Big)\Big]$

Table 3: Numerical results for Example 6.2

	$E(t_i)$	$E(t_i)$
t	$M=4$	$M=12$
0.1	8.51357×10^{-6}	3.55271×10^{-15}
0.2	6.64364×10^{-6}	3.55271×10^{-15}
0.3	6.36872×10^{-6}	1.59872×10^{-14}
0.4	6.76613×10^{-6}	4.17444×10^{-14}
0.5	3.41857×10^{-6}	1.39444×10^{-13}
0.6	9.61076×10^{-6}	3.99686×10^{-13}
0.7	4.14359×10^{-6}	1.04006×10^{-12}
0.8	6.25683×10^{-6}	2.45674×10^{-12}
0.9	6.20908×10^{-6}	5.39213×10^{-12}
1	7.97252×10^{-6}	1.11493×10^{-11}

Table 4: Numerical results for Example 6.3

	$E(t_i)$	$E(t_i)$
t	$M=8$	$M=12$
0.1	2.10171×10^{-5}	1.60608×10^{-10}
0.2	6.76385×10^{-6}	2.51423×10^{-11}
0.3	8.76015×10^{-6}	9.75131×10^{-12}
0.4	2.21909×10^{-5}	3.65645×10^{-11}
0.5	4.32118×10^{-6}	1.31994×10^{-10}
0.6	1.85657×10^{-5}	1.52683×10^{-10}
0.7	3.47009×10^{-6}	1.33923×10^{-10}
0.8	2.61164×10^{-5}	1.63786×10^{-11}
0.9	4.85291×10^{-6}	6.86029×10^{-12}
1	5.28865×10^{-6}	2.35367×10^{-11}

where λ_1, λ_2 are positive constants and kernels $k_1(t,x), k_2(t,x)$ are given positive real functions on the interval $0 \le x, t \le 1$. In Eq. (5.22), $G(t)$ is an interval given continuous function and $Y(t)$ is an unknown interval function that should be satisfy in Eq. (5.22).

In order to use interval Legendre wavelets, we first expand $Y(t)$ by the interval Legendre wavelets as

$$Y_{k,m}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \Psi_{n,m,\ell}(t) = C^T \Phi(t), \qquad (5.23)$$

where $c_{n,m}$ are unknown. Then form Eq.(5.22) and Eq.(5.23) we can write

$$Y_{k,m}(t) = G(t) + \lambda_1 \int_0^t k_1(t,x) Y_{k,m}(x)dx + \lambda_2 \int_0^1 k_2(t,x) Y_{k,m}(x)dx. \qquad (5.24)$$

Let t_i be the set of $2^{k-1}M$ zero point of the shifted Chebyshev polynomial in $[0,1]$. Now we collocate Eq.(5.24) at t_i as

$$Y_{k,m}(t_i) = G(t_i) + \lambda_1 \int_0^{t_i} k_1(t_i,x)Y_{k,m}(x)dx$$
$$+\lambda_2 \int_0^1 k_2(t_i,x)Y_{k,m}(x)dx. \quad (5.25)$$

Gauss quadrature formulas will be used to compute the integral terms in Eq.(5.25). For this purpose, we transfer the x-intervals into $[-1,1]$ by means of the transformations

$$\tau = \frac{2}{t_i}x - 1, \quad \Rightarrow x = \frac{t_i}{2}(\tau+1), \quad t \in [0,t_i],$$
$$\eta = 2x - 1, \quad \Rightarrow x = \frac{1}{2}(\eta+1), \quad x \in [0,1],$$

So Eq.(5.25) converts to

$$Y_{k,m}(t_i) = G(t_i) +$$
$$\frac{x_i\lambda_1}{2}\int_{-1}^1 k_1(t_i,\frac{t_i}{2}(\tau+1))Y_{k,m}(\frac{t_i}{2}(\tau+1))d\tau +$$
$$\frac{\lambda_2}{2}\int_{-1}^1 k_2(t_i,\frac{1}{2}(\eta+1))Y_{k,m}(\frac{1}{2}(\eta+1))d\eta.$$

Using the Gauss quadrature formula, we estimate the integrals and gets

$$Y_{k,m}(t_i) = G(t_i) +$$
$$\frac{x_i\lambda_1}{2}\sum_{p=1}^{r}\omega_p k_1(t_i,\frac{t_i}{2}(\tau_p+1))$$
$$Y_{k,m}(\frac{t_i}{2}(\tau_p+1)) +$$
$$\frac{\lambda_2}{2}\sum_{p=1}^{r}\omega_p k_2(t_i,\frac{1}{2}(\eta_p+1))$$
$$Y_{k,m}(\frac{1}{2}(\eta_p+1)), \quad (5.26)$$

where τ_p and η_p are zeros of Legendre polynomials of degrees r, respectively, and ω_p are the corresponding weights. In this paper we consider $r = M$. Equation (5.26) gives a interval system of equation. Solving this interval linear system of equations by any method provides C.

6 Numerical experiments

In this section, we will use the above proposed method to solve different examples. Our results are compared with the exact solutions by calculating the following error function

$$E(t_i) = D\Big(Y(t_i) - \hat{Y}(t_i)\Big).$$

Where $Y(t)$ and $\hat{Y}(t)$ are the exact and approximate solutions of the integral equation, respectively. The computations associated with the examples are performed using Mathematica software.

Example 6.1 *(Numerical illustration) Consider the following interval Fredholm integral equation*

$$Y(t) = \Big[2t - \frac{1}{2}, 5t - \frac{5}{4}\Big] + \int_0^1 x^2 Y(x)dx.$$

The exact solution of this equation is $Y(t) = [2t, 5t]$.

Let us consider $k = 1$ and $r = M = 3$. Then

$$Y_{1,3}(t) = \sum_{n=1}^{1}\sum_{m=0}^{2} c_{n,m}\phi_{n,m,\ell}(t)$$
$$= c_{1,0}\phi_{1,0,\ell}(t) + c_{1,1}\phi_{1,1,\ell}(t) + c_{1,2}\phi_{1,2,\ell}(t),$$

where the values of $\phi_{1,0,\ell}(t)$, $\phi_{1,1,\ell}(t)$ and $\phi_{1,2,\ell}(t)$ are showed in Table 2 and $c_{i,j}$ are unknown interval values. We use the shifted Chebyshev points as collocation points, then $t_0 = 0.5$,$t_1 = 0.0669873$, $t_2 = 0.933013$. So, by applying the method which is discussed in detail in Subsection

5.2, we have

$$Y_{1,3}(t_i)=\begin{pmatrix}\left[1-\frac{1}{\ell},1+\frac{1}{\ell}\right]c_{1,0}+\left[\frac{-3.4641}{\ell},\frac{3.4641}{\ell}\right]c_{1,1}+\left[\frac{-12.2984-1.11803\ell}{\ell},\frac{12.2984-1.11803\ell}{\ell}\right]c_{1,2}\\ \left[1-\frac{1}{\ell},1+\frac{1}{\ell}\right]c_{1,0}+\left[\frac{-1.9641-1.5\ell}{\ell},\frac{1.9641-1.5\ell}{\ell}\right]c_{1,1}+\left[\frac{-3.195+1.39754\ell}{ell},\frac{3.195+1.39754\ell}{\ell}\right]c_{1,2}\\ \left[1-\frac{1}{\ell},1+\frac{1}{\ell}\right]c_{1,0}+\left[\frac{-4.9641+1.5\ell}{\ell},\frac{4.9641+1.5\ell}{\ell}\right]c_{1,1}+\left[\frac{-26.4329+1.39754\ell}{ell},\frac{26.4329+1.39754\ell}{\ell}\right]c_{1,2}\end{pmatrix}.$$

And

$$\begin{aligned}\Theta_{1,3}&=\int_0^1 x^2 Y_{1,3}(x)dx\\ &=\frac{1}{2}\int_{-1}^{1}\left(\frac{1}{2}(\eta+1)\right)^2 Y_{1,3}(\tfrac{1}{2}(\eta+1))\\ &=\frac{1}{2}\sum_{p=1}^{3}\omega_p\left(\frac{1}{2}(\eta_p+1)\right)^2 Y_{1,3}(\tfrac{1}{2}(\eta_p+1)),\end{aligned}$$

and we have

$$\Theta_{1,3}=\begin{pmatrix}\left[\frac{-0.333+0.333\ell}{\ell},\frac{0.333+0.333}{\ell}\right]c_{1,0}+\left[\frac{-1.443+0.288\ell}{\ell},\frac{1.443+0.288\ell}{\ell}\right]c_{1,1}+\left[\frac{-6.782+0.074\ell}{\ell},\frac{6.782+0.074\ell}{\ell}\right]c_{1,2}\\ \left[\frac{-0.333+0.333\ell}{\ell},\frac{0.333+0.333}{\ell}\right]c_{1,0}+\left[\frac{-1.443+0.288\ell}{\ell},\frac{1.443+0.288\ell}{\ell}\right]c_{1,1}+\left[\frac{-6.782+0.074\ell}{\ell},\frac{6.782+0.074\ell}{\ell}\right]c_{1,2}\\ \left[\frac{-0.333+0.333\ell}{\ell},\frac{0.333+0.333}{\ell}\right]c_{1,0}+\left[\frac{-1.443+0.288\ell}{\ell},\frac{1.443+0.288\ell}{\ell}\right]c_{1,1}+\left[\frac{-6.782+0.074\ell}{\ell},\frac{6.782+0.074\ell}{\ell}\right]c_{1,2}\end{pmatrix},$$

and

$$F(t_i)=\begin{pmatrix}\left[0.5,1.25\right]\\ \left[-0.366025,-0.915064\right]\\ \left[1.36603,3.41506\right]\end{pmatrix}.$$

Now, in interval system $Y_{k,m}(t_i)=F(t_i)+\Theta_{1,3}$, consider $\ell=50$. After solving this interval system we obtain

$$c_{1,0} = \Big[1.062057, 2.35486\Big],$$

$$c_{1,1} = \Big[0.58913, 1.41507\Big],$$

$$c_{1,2} = \Big[-7.95994, 1.29395\Big],$$

and by $Y(t) = \sum_{n=1}^{1}\sum_{m=0}^{2} c_{n,m}\phi_{n,m,\ell}(t)$, we observe that

$$Y(t) =$$

$$\Big[-1.744\times 10^{-16}, -1.489\times 10^{-16}\Big] + \Big[2t, 5t\Big] +$$

$$\Big[-1.046\times 10^{-15}t^2, 1.771\times 10^{-15}t^2\Big].$$

Numerical results will not be presented since the exact solution is obtained.

Example 6.2 *Consider the following interval Volterra-Fredholm-Hammerstein integral equation*

$$Y(t) =$$

$$\Big[3e^t - \frac{3}{2}t(4+t), 8e^2 - 4t(4+t)\Big] +$$

$$2\int_0^1 txY(x)dx + \int_0^t xe^{-x}Y(x)dx. \tag{6.27}$$

The exact solution of Eq.(6.27) is given by $Y(t) = [3e^t, 8e^t]$. Table 3 shows the approximate solutions obtained by the interval Legendre wavelets method.

Using Table 3 and Figure 1 , we can concluded that the proposed method is very efficient for numerical solutions of these problems.

Example 6.3 *Consider the following interval Volterra-Fredholm-Hammerstein integral equation*

$$Y(t) =$$

$$\Big[\Big(-9.43768 + 4.5e^{2t} - 4.5t\Big)t, \tfrac{-9}{4}t\Big(1 + e^2 - 4e^{2t} + 4t\Big)\Big] +$$

$$2\int_0^1 tY(x)dx + \int_0^t 2e^{-2x}Y(x)dx. \tag{6.28}$$

The exact solution of Eq.(6.28) is given by $Y(t) = [-4.5te^{2t}, 9te^{2t}]$. Table 4 shows the approximate solutions obtained by the interval Legendre wavelets method. By the Table 4 and Figure 2, we can see that the numerical solutions converge to the exact solution.

7 Conclusions

In this paper, a new numerical approach was introduced for interval Volterra-Fredholm-Hammerstein integral equations to approximate the numerical solution for this kind of equations. To introduce interval Legendre wavelets method, shifted Legendre polynomials were defined. Using interval Legendre wavelets method, the interval integral equation was transformed to a interval system of algebraic equations that by solving this system the approximate solution of interval Volterra-Fredholm-Hammerstein integral equations was obtained. To illustrate the technique, some examples were solved by this method.

References

[1] G. Alefeld, G. Mayer, Interval analysis: Theory and applications, *J. Comput. Appl. Math.* 121 (2000) 421-464.

[2] R. Baker Kearfott, V. Kreinovich (Eds.), Applications of Interval Computations, *Kluwer Academic Publishers,* 1996.

[3] J. A. Ferreira, F. Patrício, F. Oliveira, On the computation of the zeros of interval polynomials, *J. Comput. Appl. Math.* 136 (2001) 271-281.

[4] E. R. Hansen, On solving systems of equations using interval arithmetic, *Mathematics of Computation* 22 (1968) 374-384.

[5] E. R. Hansen, S. Sengupta. Bounding solutions of systems of equations using interval analysis, *BIT.* 21 (1981) 203-211.

[6] T. C. Hales, A proof of the Kepler conjecture, *Annals of Mathematics* 162 (2005) 1065-1185.

[7] M. Hukuhara, Intégration des applications mesurables dont la valeur est un compact convex, *Funkcial Ekvac* 10 (1967) 205-229.

[8] L. Jaulin, M. Kieffer, O. Didrit, E. Walter, Applied Interval Analysis With Examples in Parameter and State Estimation, *Robust Control and Robotics, Springer-Verlag London*, 2001.

[9] T. Johnson, W. Tucker, Rigorous parameter reconstruction for differential equations with noisy data, *Automatica* 44 (2008) 2422-2426.

[10] L. V. Kolev, Interval Methods for Circuit Analysis, *World Scientific*, 1993.

[11] V. Lakshmikantham, T. Bhaskar, J. Devi, Theory of Set Differential Equations in Metric Spaces, *Cambridge Scientific Publishers*, 2006.

[12] M. T. Malinowski, Interval differential equations with a second type Hukuhara derivative, *Appl. Math. Lett.* 24 (2011) 2118-2123.

[13] R. E. Moore, Methods and Applications of the Interval Analysis, *SIAM, Philadelphia*, 1979.

[14] R. Moore, Interval Arithmetic. Prentice-Hall, Englewood CliMs, *NJ, USA,* 1966.

[15] R. E. Moore, R. B. Kearfott, M. J. Cloud, Introduction to Interval Analysis, *SIAM*, 2009.

[16] D. Moens, D. Vandepitte, A survey of non-probabilistic uncertainty treatment in finite element analysis, *Comput. Methods Appl. Mech. Engrg.* 194 (2005) 1527-1555.

[17] A. Neumaier, Interval Methods for Systems of Equations, *Cambridge University Press,* 1990.

[18] N. S. Nedialkov, K. R. Jackson, J. D. Pryce, An effective high order interval method for validating existence and uniqueness of the solution of an IVP for an ODE, *Reliab. Comput.* 7 (2001) 449-465.

[19] A. Neumaier, Rigorous sensitivity analysis for parameter dependent systems of equations, *J. Math. Anal. Appl.* 114 (1989) 16-25.

[20] F. Patrício, J. A. Ferreira, F. Oliveira, On the interval Legendre polynomials, *Journal of Computational and Applied Mathematics* 154 (2003) 215-227.

[21] J. Rohn, Solvability of systems of interval linear equations and inequalities, *Springer, New York*, 2006.

[22] A. Salimi Samloo, E. Babolian, Numerical solution of Fractional Differential, integral and integro-differential Equations by using piecewise constant orthogonal functions , *Journal of Computational and Applied Mathematics* 214 (2007) 495-508.

[23] A. Salimi Shamloo, Parisa Hajagharezalou, Interval Interpolation by Newton's Divided Differences, *Journal of mathematics and computer science* 13 (2014) 231-237.

[24] A. Salimi Shamloo, Sanam Shahkar, Alieh Madadi, Numerical Solution of the Fredholme-Volterra Integral Equation by the Sinc Function, *American Journal of Computational Mathematics* 2 (2012) 136-142.

[25] M. Seifollahi, A.Salimi Samloo, Numerical Solution of Nonlinear Multi-Order Fractional Differential Equations by Operational Matrix Of Chebyshev Polynomails , *World Applied Programming* 3 (2013) 85-92.

[26] I. Skalna, M. V. Rama Rao, A. Pownuk, Systems of fuzzy equations in structural mechanics, *J. Comput. Appl. Math.* 218 (2008) 149-156.

[27] L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, *Nonlinear Analysis: Theory, Methods and Applications* 71 (2009) 1311-1328.

[28] . P. Sevastjanov, L. Dymova, A new method for solving interval and fuzzy equations: Linear case, *Information Sciences* 179 (2009) 925-937.

[29] A. Truong Vinh, P. Nguyen Dinh, H. Ngo Van, A note on solutions of interval-valued Volterra integral equations, *J. Integral Equations Applications* 26 (2014) 1-14. http://dx.doi.org/10.1216/JIE-2014-26-1-1/

[30] C. Wu, Z. Gong, On Henstock integrals of interval-valued functions and fuzzy-valued

functions, Fuzzy Sets and Systems 115 (2000) 377-391.

Niaz khorrami was born in Salmas in 1973. He received the B.E. degree in mathematics from the Tabriz Branch, Islamic Azad University in 1997, and M.Sc. in Applied mathematics from the Kermanshah Branch, Islamic Azad University in 2006 and Ph.D. in Applied mathematics, Analysis from the Lahijan Branch, Islamic Azad University, in 2018. He is currently a faculty member at Islamic Azad University, Salmas Branch. His research interests include numerical analysis and integral equations.

Ali Salimi Shamloo was born in Tabriz in 1972. He received the B.E. degree in mathematics from the University of Tabriz in 1995, and M.Sc. in Applied mathematics from the Tabriz University in 1998 and Ph.D. in Applied mathematics, Analysis from the Islamic Azad University, Science and Research Branch in 2008. He is currently a faculty member at Islamic Azad University, Shabestar Branch. His research interests include numerical analysis and integral equations.

Professor B. Parsa Moghaddam graduated with Ph.D. (2009), in Applied Mathematics at the University of Guilan. Since 2005 he works at the Department of Mathematics of the Islamic Azad University of Lahijan. His research focuses on complex systems, nonlinear dynamics, fractional calculus, modeling, and evolutionary computing, as well as control. In addition, he has received the Publons Peer Review Award 2017,2018 and 2019 in Mathematics.