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Research Article

Contrast of Homotopy and Adomian Decomposition Methods with Mittage-Leffler Function for Solving Some Nonlinear Fractional Partial Differential Equations

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Abstract

In this paper, a class of nonlinear fractional partial differential equation is considerd and solved by advanced analytical-numerical methods such as homotopy analytical and Adomian decomposition Methods and Mittag-Leffler functions. The obtained approximate solutions show that these solutions are same for the first three approximate terms u_1, u_2, u_3.

Keywords : Nonlinear fractional differential equation; Mittag-Leffler functions; Adomian Decomposition Method(ADM); Homotopy Analytical Method(HAM).

1 Introduction

Till now, various analytical methods, for example, Laplace and Fourier transforms, have been utilized to solve linear fractional differential equations [1, 2, 3, 4], but for solving nonlinear fractional differential equations, numerical methods have been used solely. Considering that Adomian decomposition method as an analytical method has successfully been applied in a variety of problems [5, 6, 7], also in the fourth section of the book [8], it is proved that in general, the homotopy analytical method logically contains Adomian decomposition method, so that the given solution by Adomian decomposition method is just a special case of the given solution by the homotopy analysis method. In this paper, we solve a class of nonlinear fractional partial differential equation, with both above-mentioned methods and compare these solutions with Mittag-Leffler functions as another method for two approximate solutions:

$$D_t^{\alpha} u(x,t) = u(x,t) + u^n(x,t). \tag{1.1}$$

2 Preliminaries

we give some necessary definitions and mathematical preliminaries of the fractional calculus and the introduction of the above-mentioned methods.

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2.1 Definition

The Riemann-Liouville fractional integral of order $\alpha > 0$ is difined as:

$$(I^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad (2.2)$$
$$(I^0 f)(t) = f(t)$$

and its fractional derivative of order$\alpha > 0$ is written as follows:

$$(D^{\alpha}(t) = (\frac{d}{dt})^n (I^{n-\alpha}f)(t) \qquad (n-1 < \alpha \leq n) \quad (2.3)$$

where n is an integer number. Regard to this paper, we consider the following modified Mittag-Lefler function:

$$h_p(x) = \sum_{k=1}^{\infty} \frac{x^{kp-1}}{(kp-1)!} = \quad (2.4)$$
$$\frac{x^{p-1}}{(p-1)!} + \frac{x^{2p-1}}{(2p-1)!} + \frac{x^{3p-1}}{(3p-1)!} + ...$$

The function (2.4) as same as Taylor expansion for e^{rx} is invariant with respect to ordinary differentiation that means $D^{(np)}h_p(x) = h_p(x)$. We will use the general form of this function to solve fractional differential equations. Hence we consider it with parameter r, that mean:

$$y(x) = h_p(x,r) = \sum_{k=1}^{\infty} \frac{r^k x^{kp-1}}{(kp-1)!}, \quad (2.5)$$

It is easy to see that

$$y^{(n)}(x) = D^{(n)}h_p(x,r) = r^n h_p(x,r), \quad (2.6)$$

By using this function, we can solve the ordinary fractional differential equations :

$$a_m y^{(\frac{m}{n})}(x) + a_{m-1}y^{(\frac{m-1}{n})}(x) + ... \quad (2.7)$$
$$+ a_1 y^{(\frac{1}{n})}(x) + a_0 y = 0.$$

by charactristic equations as same as ordinary differential equation

$$a_m r^m + a_{m-1}r^{m-1} + ... + a_1 r + a_0 = 0, \quad (2.8)$$

regarding the roots of this equation by $r_1, r_2, ..., r_m$ then general solution of equation (2.6) is

$$y(x) = c_1 h_p(x,r_1) + c_2 h_p(x,r_2) + ... \quad (2.9)$$
$$+ c_m h_p(x,r_m)$$

where $p = \frac{1}{n}$ is fractional step derivative and $\frac{m}{n}, \frac{m-1}{n}, ..., \frac{1}{n}$ show the fractional orders [9].

2.2 Adomian decomposition method

we give a brief presentation of the Adomian decomposition method (ADM).The details of this method is now well known, see for example [10, 11, 12, 13, 14, 15]. The unknown function u(x) for the solution of the equation is considered in the form of the following infinite series by ADM

$$u(x) = \sum_{i=1}^{\infty} u_i(x) \quad (2.10)$$

where the components $u_i(x)$ of the solution $u(x)$ will be determined recurrently, and the expansion of the nonlinear terms like $F(u(x))$is written as follows:

$$F(u(x)) = \sum_{n=0}^{\infty} A_n \quad (2.11)$$

wherein, A_n is the Adomian polynomials and obtained according to the following phrase:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}F(\sum_{i=0}^{\infty}\lambda^i u_i(x))|_{\lambda=0} \; n = 0,1,2,... \quad (2.12)$$

We list the formulas of the first several Adomian polynomials for the one-variable simple analytic nonlinearity $F(u)$ from A_0 to A_3, inclusively, for convenient reference as

$$\begin{cases} A_0 = F(u_0) \\ A_1 = u_1 F'(u_0) \\ A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) \\ A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \\ \frac{1}{3!}u_1^3 F^{(3)}(u_0) \end{cases} \quad (2.13)$$

The Adomian polynomials can be generated by using different algorithms such as in [10, 13, 16, 17, 15, 18, 19, 20].

For example, for the nonlinear function of u^2, we will have:

$$A_0 = u_0^2 \quad A_1 = 2u_0u_1 \quad (2.14)$$
$$A_2 = 2u_0u_2 + u_1^2$$
$$A_3 = 2u_0u_3 + 2u_1u_2$$

Now, by putting the series (2.10) and (2.11) in the nonlinear differential equations and sorting and comparing series on both sides of the equation, a recursive equation for u is obtained that subject to the initial condition $u_0(x,t)$ and recursive equation, other terms of u(x) are obtained.

2.3 Homotopy analytic method

Consider two smooth functions $f(x)$ and $g(x)$ on the real line. A linear homotopy of two such functions is itself a function

$$H(f(x), g(x), q) = (1-q)f(x) + qg(x) \quad (2.15)$$

which defined by homotopy parameter q. when $q = 0$, $H(f(x), g(x), q) = f(x)$, whereas when $q = 1$, $H(f(x), g(x), q) = g(x)$. when we evolve q from zero to one, the homotopy evolve continuously from $f(x)$ to $g(x)$. Let's consider the differential equation governed by

$$N[u(x)] = a(x) \quad (2.16)$$

where N is a nonlinear differential operator and $x \in D \subseteq R^l$.

Consider an auxiliary linear differential operator L. let us construct a homotopy of the operator $H(N, L, q)$ st $H(N, L, 0) = L$ and $H(N, L, 1) = N$ then, the homotopy itself is an operator for all $q \in [0, 1]$. Now, we expand the solution as a Taylor series, given by

$$\varphi(x, q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m \quad (2.17)$$

that the series of the solution (2.17) gives a relation between the initial guess $u_0(x)$ and the exact solution.

Furthermore, the exact solution will be given by

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) \quad (2.18)$$

To obtain the $u_m(x)$'s, one recursively solve what are known as the m-th order deformation equations, given by

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hR_m(u_0(x), ..., u_{m-1}(x), x) \quad (2.19)$$

where $\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1. \end{cases}$

and

$$R_m(u_{m-1}(x), x) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(x,q)]}{\partial q^{m-1}}|_{q=0} = \frac{1}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial q^{m-1}} N[\sum_{m=0}^{\infty} u_m(x) q^m]\right)|_{q=0} \quad (2.20)$$

3 Problem-solving with ADM

Consider the following nonlinear fractional differential equation

$$D_t^{\alpha} u(x,t) = u(x,t) + u^2(x,t) \quad (3.21)$$

First, we convert the equation (3.21) to a fractional integral equation, then we solve the integral equation with ADM. Now, By integrating both sides of the equation (3.21), the order of $\alpha - 1$ (with respect to time variable t) we have:

$$D_t^{1-\alpha} D_t^{\alpha} u(x,t) = D^{1-\alpha} u(x,t) + D_t^{1-\alpha} u^2(x,t) \quad (3.22)$$

So we have:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} u(x,\tau) d\tau + \frac{\partial}{\partial t} \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} u^2(x,\tau) d\tau \quad (3.23)$$

Then, by integrating both sides of (3.23) the in interval [0,1], we have:

$$u(x,t) = u(x,0) + \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} u(x,\tau) d\tau + \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} u^2(x,\tau) d\tau \quad (3.24)$$

Now, we solve the above integral equation with the initial condition $u(x,0) = \varphi(x)$.Due to the nonlinear term u^2, according to relation (2.14) and (2.10), (2.11) in section (2.2), we have:

$$\sum_{n=0}^{\infty} u_n(x,t) = \varphi(x) + \sum_{n=0}^{\infty} \int \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} [u_n(x,\tau) + A_n(x,\tau)] d\tau \quad (3.25)$$

In this case, we have:

$$u_{n+1}(x,t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!}[u_n(x,\tau)+A_n(x,\tau)]d\tau \quad (3.26)$$

Using (2.2) and (2.3), we have:

$$u_n(x,t) = I^{\alpha}(u_n(x,t)+A_n(x,t)) \quad (3.27)$$

So

$u_1(x,t) = I_0^{\alpha}(u_0(x,t) + A_0(x,t)) = I_0(\varphi(x)+\varphi^2(x)) = I_0^{\alpha}(\varphi+\varphi^2)$

$u_2(x,t) = I_0^{\alpha}(u_1(x,t) + A_1(x,t)) = I_0^{\alpha}(I_0^{\alpha}(\varphi+\varphi^2)+2\varphi I_0^{\alpha}(\varphi+\varphi^2))$

Due to the following relation [8]

$$D_a^q({}_aD_t^p f(t)) =_a D_t^{p+q} f(t) \quad (3.28)$$

Then $u_2(x,t)$ be obtained as follows:

$$\begin{aligned} u_2(x,t) &= I_0^{2\alpha}(\varphi+\varphi^2) \\ +2\varphi I_0^{2\alpha}(\varphi+\varphi^2) &= \\ (1+2\varphi)I_0^{2\alpha}(\varphi+\varphi^2) \end{aligned} \quad (3.29)$$

Simlarly

$$\begin{aligned} u_3(x,t) = I_0^{\alpha}(u_2(x,t)+A_2(x,t)) = \\ I_0^{\alpha}((1+2\varphi)I_0^{2\alpha}(\varphi+\varphi^2) + \\ 2\varphi(1+2\varphi)I_0^{2\alpha}(\varphi+\varphi^2) + (I_0^{\alpha}(\varphi+\varphi^2))^2) \end{aligned} \quad (3.30)$$

And as a result

$$\begin{aligned} u_3(x,t) = (1+2\varphi)^2 I_0^{3\alpha}(\varphi+\varphi^2) + \\ I_0^{\alpha}(\varphi+\varphi^2)(I_0^{\alpha}(\varphi+\varphi^2))^2 \end{aligned} \quad (3.31)$$

4 Problem-solving with HAM

Consider the following problem

$$D_t^{\alpha}u(x,t) = u(x,t) + u^2(x,t) \quad (4.32)$$

with initial condition

$$u(x,0) = \varphi(x)$$

So the nonlinear operation N is as follows:

$$N[u(x,t)] = D_t^{\alpha}u(x,t) - u(x,t) - u^2(x,t) = 0 \quad (4.33)$$

Also consider the linear operator L as follows:

$$L = \frac{D^{\alpha}}{t^{\alpha}} \quad \text{so} \quad L[\varphi(x,t,q)] = \frac{\partial^{\alpha}\varphi(x,t,q)}{\partial t^{\alpha}} \quad (4.34)$$

According to the relation (2.19), we have:

$$L[u_1(x,t)] = hR_1[u_0(x,t)] \quad (4.35)$$

let $h=-1, u_0(x,t) = u(x,0) = \varphi(x)$ then

$$L[u_1(x,t)] = -R_1[u_0(x,t)] \quad (4.36)$$

Considering to the relation (2.20):

$$R_1(u_0(x,t)) = N[\varphi(x,t,q)]|_{q=0} = \frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}}(x,t,q) - \varphi(x,t,q) - \varphi^2(x,t,q)|_{q=0} \quad (4.37)$$

Making use of (2.17) in (2.3), we have:

$$\frac{\partial^{\alpha}\varphi(x,t,q)}{\partial t^{\alpha}}|_{q=0} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}u_0(x,t) + \sum_{m=1}^{\infty}\frac{\partial^{\alpha}u_m(x,t)}{\partial t^{\alpha}}q^m \quad (4.38)$$

where $\frac{\partial^{\alpha}\varphi(x,t,q)}{\partial t^{\alpha}}|_{q=0} = 0$

It follows that

$$R_1(u_0(x,t)) = -[\varphi(x)+\varphi^2(x)] \quad (4.39)$$

From (4.36)we have:

$$L[u_1(x,t)] = [\varphi(x)+\varphi^2(x)] \quad (4.40)$$

as a result:

$$u_1(x,t) = -L^{-1}[\varphi(x)+\varphi^2(x)] \quad (4.41)$$

Because $L^{-1} = I_t^{\alpha}$ so we have:

$$u_1(x,t) = I_t^{\alpha}[\varphi(x)+\varphi^2(x)] \quad (4.42)$$

Similarly, we obtain that

$$L[u_2(x,t) - u_1(x,t)] = R_2[u_0(x,t) - u_1(x,t)] \quad (4.43)$$

Due to the linearly of the operator L, we have:

$$L[u_2(x,t)] = L[u_1(x,t)] + R_2[u_0(x,t), u_1(x,t)] \quad (4.44)$$

with making use of (2.20):

$$R_2(u_0(x,t),u_1(x,t)) = (\frac{\partial}{\partial q}[N(\varphi(x,t,q))]|_{q=0}) = (\frac{\partial^\alpha u_1}{\partial t^\alpha}) + 2\frac{\partial^\alpha u_2}{\partial t^\alpha}q + \ldots - u_1 - 2u_2 q\ldots - 2u_0u_1 - 2u_1^2 q - 4u_0u_1q + ..)|_{q=0} = \frac{\partial^\alpha u_1}{\partial t^\alpha} - u_1 - 2u_0u_1 \quad (4.45)$$

so we have:

$$\begin{aligned} u_2(x,t) &= u_1(x,t) \\ &\quad -L^{-1}(R_2(u_0(x,t),u_1(x,t)) \\ &= u_1(x,t) - u_1(x,t) + I^\alpha(u_1(x,t)) \\ &\quad +2u_0(x,t)I^\alpha(u_1(x,t)) \\ &= (1+2u_0(x,t))I^\alpha(u_1(x,t)) \\ &= (1+2u_0)I^\alpha(I^\alpha(\varphi+\varphi^2)) \end{aligned} \quad (4.46)$$

According to the relation (2.6) from [8]

$$I^\alpha(I^\alpha(\varphi+\varphi^2)) = I^{2\alpha}(\varphi+\varphi^2)) \quad (4.47)$$

So we have:

$$u_2(x,t) = (1+2\varphi)I^{2\alpha}(\varphi+\varphi^2) \quad (4.48)$$

Similary, we calculate $u_3(x,t)$ as follows:

$$\begin{aligned} L[u_3-u_2] &= -R_3(u_0,u_1,u_2) \\ \Rightarrow L[u_3] &= L[u_2] - R_3(u_0,u_1,u_2) \\ \Rightarrow u_3(x,t) &= u_2(x,t) - L^{-1}(R_3(u_0,u_1,u_2) \end{aligned} \quad (4.49)$$

Where

$$R_3(u_0,u_1,u_2) = \frac{\partial^2}{\partial q^2}(N(\varphi(x,t,q))|_{q=0} = 2\frac{\partial^\alpha u_2}{\partial t^\alpha} - 2u_2 - 2u_1^2 - 4u_0u_2 \quad (4.50)$$

So

$$u_3(x,t) = -u_2 + 2(1+2\varphi)I^\alpha(u_2) + 2I^\alpha(u_1)^2 = -(1+2\varphi)I^{2\alpha} + 2(1+2\varphi)^2 I^{3\alpha} + 2I^\alpha(I^\alpha)^2)) \quad (4.51)$$

Due to the that the given solution by the Adomian decomposition method is just a special case of the given solution by the homotopy analysis method, the solution of the $u_3(x,t)$ in (3.31) is a special case of the (4.51).

Example 4.1 *Consider the problem (4.32) with the initial condition $u(x,0) = \varphi(x)$. We see that problem-solving with both two methods Adomian decomposition and Homotopy analysis the methods leads to obtain the fractional integrals $I_0^\alpha, I_0^{2\alpha}, I_0^{3\alpha}$. So, we obtain them, here. We calculate $I_0^\alpha(\varphi+\varphi^2), I_0^{2\alpha}(\varphi+\varphi^2), I_0^{3\alpha}(\varphi+\varphi^2)$ then by putting these sentences into $u_1(x,t), u_2(x,t), \ldots,$ the approximate solution $u(x,t)$ is obtained. Let $\alpha = \frac{1}{2}, \varphi(x) = x$, we have $I^{\frac{1}{2}}(x+x^2) = I^{\frac{1}{2}}(x) + I^{\frac{1}{2}}(x^2)$ then with making use of the following relation*

$$I^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)}t^{\lambda+\alpha}, \quad \lambda > -1, \alpha > 0 \quad (4.52)$$

And considering to the properties of the gamma function, we have:

$I^{\frac{1}{2}}(x+x^2) = \frac{2(x+x^2)t^{\frac{1}{2}}}{\sqrt{\pi}}$ For $I^{2\cdot\frac{1}{2}}(x+x^2)$, we have:

$I^{2\cdot\frac{1}{2}}(x+x^2) = I^{\frac{1}{2}}(I^{\frac{1}{2}}) = I^{\frac{1}{2}}(\frac{2(x+x^2)t^{\frac{1}{2}}}{\sqrt{\pi}}) = \frac{2(x+x^2)}{\sqrt{\pi}} I^{\frac{1}{2}}(t^{\frac{1}{2}})$ by using of (4.52) we have

$$I^{\frac{1}{2}}(t^{\frac{1}{2}}) = \frac{1}{2}\sqrt{\pi}t.$$

So we have:

$I^{2\frac{1}{2}} = (x+x^2)t$ For $I^{3\cdot\frac{1}{2}}(x+x^2)$, we have: $I^{3\cdot\frac{1}{2}}(x+x^2) = I^{\frac{1}{2}}(I^{2\cdot\frac{1}{2}})(x+x^2) = I^{\frac{1}{2}}((x+x^2)t) = (x+x^2)I^{\frac{1}{2}}(t)$.

Similarly, we have:

$I^{\frac{1}{2}}(t) = \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}}$ So $I^{3\cdot\frac{1}{2}}(x+x^2) = \frac{4}{3\sqrt{\pi}}(x+x^2)t^{\frac{3}{2}}$

Now, by putting the fractional derivative obtained in (3.27), (3.31) of the sction 3, we have:
$u_1(x,t) = \frac{2(x+x^2)t^{\frac{1}{2}}}{\sqrt{\pi}}$ $u_2(x,t) = (1+2x)(x+x^2)t$ $u_3(x,t) = [\frac{4(1+2x)^2(x+x^2)}{3\sqrt{\pi}} + \frac{8(x+x^2)^3}{\pi\sqrt{\pi}}]t^{\frac{3}{2}}$ So the solution using the ADM is:

$$\begin{aligned} u(x,t) &= x + \frac{2(x+x^2)t^{\frac{1}{2}}}{\sqrt{\pi}} + \\ & (1+2x)(x+x^2)t + \\ & [\frac{4(1+2x)^2(x+x^2)}{3\sqrt{\pi}} \\ & + \frac{8(x+x^2)^3}{\pi\sqrt{\pi}}]t^{\frac{3}{2}} \end{aligned} \quad (4.53)$$

Now, by putting the fractional derivative obtained in (4.42), (4.48), (4.51) of the sction 4, we have:
$u_1(x,t) = \frac{2(x+x^2)t^{\frac{1}{2}}}{\sqrt{\pi}}$ $u_2(x,t) = (1+2x)(x+x^2)t$ $u_3(x,t) = [\frac{8}{3\sqrt{\pi}}(1+2x)^2(x+x^2) + \frac{16}{\pi\sqrt{\pi}}(x+x^2)^3]t^{\frac{3}{2}}$
So the solution using the HAM is:

$$\begin{aligned} u(x,t) = x + \frac{2(x+x^2)}{\sqrt{\pi}}t^{\frac{1}{2}} + \\ \frac{8}{\sqrt{\pi}}(1+2x)^2(x+x^2) + \\ \frac{16}{\pi\sqrt{\pi}}(x+x^2)^3 t^{\frac{3}{2}} \end{aligned} \quad (4.54)$$

In the following example, we show that the form of approximate solution for the problem (4.32) is acceptable.

Example 4.2 *Consider the following equation:*

$$D_t^{\alpha} u(x,t) = u(x,t) + u^2(x,t) \quad (4.55)$$

Regarding that, we have no any term of derivative with respect to x, therefore we can consider the equation (4.55) as an ordinary defferential equation like:

$$y^{(\alpha)} = y + y^2 \quad (4.56)$$

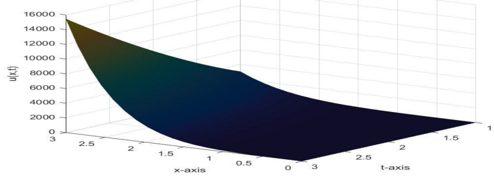

Figure 1: Graph of the solution using the Adomian Decomposition Method

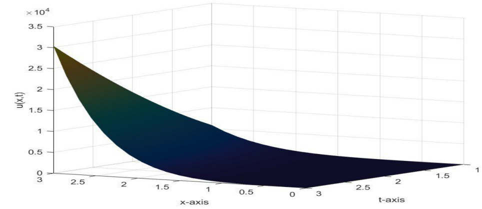

Figure 2: Graph of the solution using the Homotopy Analytic Method

where α is the fractional order derivative of y.

To find the approximate solution of (4.56), we consider the following expressions:

$$I^{\alpha}\frac{t^{k\alpha}}{(k\alpha)!} = \frac{t^{(k+1)\alpha}}{(k+1)\alpha!} + c\frac{t^{-1+\alpha}}{(-1+\alpha)!} \quad (4.57)$$

This expression is chosen by considerations about modified Mittag-Leffler function which has been introduced in [21].

We have:

$$I^{\alpha}\sum_{k=o}^{\infty}\frac{t^{k\alpha}}{(k\alpha)!} = \sum_{k=o}^{\infty}\frac{t^{(k+1)\alpha}}{(k+1)\alpha!} + c\frac{t^{-1+\alpha}}{(-1+\alpha)!} \quad (4.58)$$

Where the operation I^{α} is the fractional integral and D^{α} is the fractional derivative which has been applied in the section 2, where $D^{\alpha}(c\frac{t^{-1+\alpha}}{(-1+\alpha)!}) = 0$.

We can choose a finite term from infinite series as an approximate solution:

$y(t) = \sum_{k=o}^{N}\frac{t^{k\alpha}}{(k\alpha)!}$

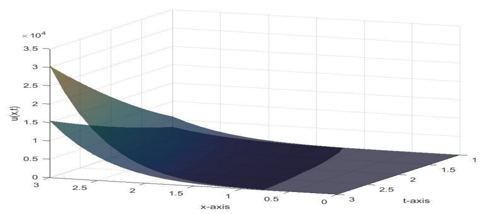

Figure 3: Graph of comparison of the obtained solutions of the above two methods

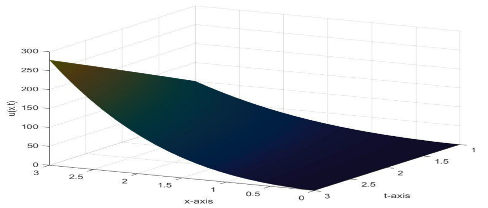

Figure 4: Graph of the solution using by three methods ADM, HAM, Mittag-Leffler function

We consider the term of y^2will be:

$$
\begin{aligned}
(\sum_{k=o}^{N}\frac{t^{k\alpha}}{(k\alpha)!})^2 &= \\
(1+\frac{t^{\alpha}}{\alpha!}+\frac{t^{2\alpha}}{(2\alpha!)}+\ldots+\frac{t^{N\alpha}}{(N\alpha!)})^2 &= \\
1+2\frac{t^{\alpha}}{\alpha!}+ & \\
(\frac{1}{\alpha!^2}+\frac{2}{(2\alpha)!})t^{2\alpha}+(\frac{2}{\alpha!\,(2\alpha)!}+ & \\
\frac{2}{(3\alpha)!}t^{3\alpha}+\ldots+\frac{t^{2N\alpha}}{(N\alpha)!^2} &= \\
1+2\frac{t^{\alpha}}{\alpha!}+(\frac{(2\alpha)!}{\alpha!}+2)\frac{t^{2\alpha}}{(2)!}+ & \\
(\frac{2(3\alpha)!}{\alpha!\,(2\alpha)!}+2)\frac{t^{3\alpha}}{(3\alpha)!} & \\
+\ldots+\frac{(2N\alpha)!}{(N\alpha)!^2}\frac{t^{2N\alpha}}{(2N\alpha)!} & \qquad (4.59)
\end{aligned}
$$

Note that the following basic formula is used: $(a+b+c)^2=a^2+2ab+(b^2+2ac)+2bc+c^2$
Similar to [21, 22], we consider the approximate solution of equation (4.52) in form of:

$$
u(x,t)=\alpha_0(x)+\alpha_1(x)\frac{t^{\alpha}}{\alpha!}+\alpha_2(x)\frac{t^{2\alpha}}{(2\alpha)!} \qquad (4.60)
$$

Regarding the initial condition $u(x,0) = \varphi(x) = \alpha_0(x)$, by getting fractional integral I_t^{α} from 4.55, we have:

$$
\begin{aligned}
u(x,t) &= \varphi(x)+I^{\alpha}(\alpha_0(x)\frac{t^{0\alpha}}{0!}+ \\
&\alpha_1(x)\frac{t^{\alpha}}{\alpha!}+\alpha_2(x)\frac{t^{2\alpha}}{(2\alpha)!})+ \\
&I^{\alpha}[\alpha_0^2+2\alpha_0(x)\alpha_1(x)\frac{t^{\alpha}}{\alpha!} \\
&+(2\alpha)!\,(\alpha_1(x)^2\frac{1}{(\alpha)\alpha!^2}+ \\
&2\alpha_0(x)\frac{\alpha_2(x)}{(2\alpha)!})\frac{t^{2\alpha}}{(2\alpha)!}+ \\
&2\alpha_1(x)\alpha_2(x)\frac{(3\alpha)!}{\alpha!\,(3\alpha)!}\frac{t^{3\alpha}}{(3\alpha)!}+ \\
&\alpha_2(x)^2\frac{(4\alpha)!}{(2\alpha)!^2}\frac{t^{4\alpha}}{(4\alpha)!}] \qquad (4.61)
\end{aligned}
$$

Therefore by considaring the oparator I^{α} according to 4.57, we have:

$$
\begin{aligned}
u(x,t) &= \alpha_0^2\frac{t^{\alpha}}{\alpha!}+2\alpha_0(x)\alpha_1(x)t^{2\alpha}(2\alpha!\,)+ \\
&(\alpha_1(x)^2\frac{(2\alpha)!}{(\alpha!\,)^2}+2\alpha_0(x)\alpha_2(x))\frac{t^{3\alpha}}{(3\alpha)!}+ \\
&2\alpha_1(x)\alpha_2(x)\frac{(3\alpha)!}{\alpha!\,(2\alpha)!}\frac{t^{4\alpha}}{(4\alpha)!}+ \\
&\alpha_2(x)^2\frac{(4\alpha)!}{(2\alpha)!^2}\frac{t^{5\alpha}}{(5\alpha)!}
\end{aligned} \qquad (4.62)
$$

Finally, the following resulted for the unknown coefficients $\alpha_j(x), j=0,1,2$ are:

$$\alpha_0(x)=\varphi(x), \alpha_1(x)=\alpha_0(x)+\alpha_0(x)^2$$

$$\alpha_1(x)=\varphi(x)+\varphi^2(x)+2\varphi(x)[\varphi(x)+\varphi^2(x)]$$

$$\alpha_2(x)=\alpha_1(x)+2\alpha_0(x)\alpha_1(x)$$

Hence the approximate solution$u(x,t)$ is:

$$
\begin{aligned}
u(x,t) &= \varphi(x)+[\varphi(x)+\varphi^2(x)]\frac{t^{\alpha}}{\alpha!}+ \\
&(\varphi+\varphi^2)(1+2\varphi)\frac{t^{2\alpha}}{(2\alpha)!}
\end{aligned} \qquad (4.63)
$$

Remark 4.1 *from the obtained solutions for u_1, u_2 in section 3, 4 we have:*

$u_1(x,t) = I_0(\varphi(x) + \varphi^2(x)) = (\varphi(x) + \varphi^2(x))I^\alpha(t^0)$

$u_2(x,t) = (1 + 2\varphi)I_0^{2\alpha}(\varphi(x) + \varphi^2(x)) = (1+2\varphi)(\varphi+\varphi^2)I^{2\alpha}(t^0)$

according to the (4.52):
$I^\alpha(t^0) = \dfrac{t^\alpha}{\alpha!}, I^{2\alpha}(t^0) = \dfrac{t^{2\alpha}}{(2\alpha)!}$

In fact, the obtained solutions with the (ADM) and (HAM) methods, corresponding to the solution of the (4.63).

Note that in the previous graphs, $x \in interval[0\ 3], t \in interval[1\ 3]$.

5 Conclusion

In this paper, we solved the nonlinear fractional partial differential equation of (1.1) in three ways. Solving the equation with these methods leads to the obtaining of fractional integrals of $I^\alpha(\varphi + \varphi^2), I^{2\alpha}(\varphi+\varphi^2), I^{3\alpha}(\varphi+\varphi^2), ...$ due to remark (4.1), the solution of the problem is in the form of 4.63 series. In fact, the problem-solving convert to the obtaining of fractional integrals that can be calculated using existing software. in the test example, we saw that the approximate termsu_0, u_1, u_2 are same by the mentions three methods.

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