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Research Article

Solution of Nonlinear Fredholm-Volterra Integral Equations via Block-Pulse Functions

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Abstract

In this work, a new simple direct method to solve nonlinear Fredholm-Volterra integral equations is presented. By using Block-pulse (BP) functions, their operational matrices and Taylor expansion a nonlinear Fredholm- Volterra integral equation converts to a nonlinear system. Some numerical examples illustrate accuracy and reliability of our solutions.

Keywords : Nonlinear Fredholm-Volterra integral equation; Block-pulse functions; Operational matrices.

1 Introduction

In many branches of mathematics and sciences we are Encountered Integral equation. Integral equation have been applied in mathematics, physics, engineering, continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, communication theory, mathematical economics, population genetics and radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics etc. Therefore, find an acceptable solution to these equations is necessary. Many researchers focused on them and presented different methods to solve them including analytical methods, numerical methods or mixed methods. See [1, 2, 3, 4, 5, 6, 7]. In this paper, we consider general form of a nonlinear Fredholm-Volterra integral equation as follows:

$$\begin{cases} \lambda u(x) = f(x) + \lambda_1 \int_a^b k_1(x,t) G(u(t))dt \\ \qquad +\lambda_2 \int_a^x k_2(x,t) H(u(t))dt \\ u^{j}(a) = v_i, \qquad j = 0, \cdots, n-1. \end{cases} \quad (1.1)$$

where $\lambda \in \{0,1\}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Also, $G(t), H(t)$ are smooth functions and $u(t)$ is unknown function. Inaddition, all of the functions belong to $\ell^2\,[a,b])$ or $\ell^2([a,b] \times [a,b]$. Recently, it has become common to employ a sequence of polynomials or functions in order to solve integral equations. In these methods, integral equations are converted to a system whose solution leads us to the solution of a given integral equation. Fredholm integral equation of the first kind is an ill-posed problem and the common and effective methods to solve them numerically are based on wavelets [8, 9]. Volterra integral equation of

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the first kind is easier to solve. Babolian and his coworkers presented a direct method [10] and an iterative method [11] to solve these equations in linear case based on Block-pulse functions which our method is a general case of them. Almost all of the methods which use to solve nonlinear Fredholm-Volterra integral equations are in particular case or are dependent to choose of points [12, 13, 14, 15]. However our method is in general case and is free of the points. In this article, a new direct method based on BP functions and Taylor expansion are presented. This method converts a nonlinear integral equation to a system.

2 Preliminaries

BP functions are famous functions that many authors used them to solve different equations. See [10, 11, 12].

Definition 2.1 *Suppose m be a positive integer number, an m-set of BPFs defined over $[0,T)$ as [10]:*

$$\phi_i(t) = \begin{cases} 1, & \frac{iT}{m} \leqslant t < \frac{(i+1)T}{m}, \\ 0, & otherwise. \end{cases} \quad (2.2)$$

where $i = 0, 1, \cdots, m-1$ and $\phi_i(t)$ is the ith BPF and consider $h = \dfrac{T}{m}$. In this paper, for convince we consider our interval $[0,1)$.For an arbitrary interval $[a,b)$, it is sufficient to use the change of variable $t = \dfrac{b-a}{2}x + \dfrac{b+a}{2}$.
We denote $\Phi_m(t)$ as an m-vector as follows:

$$\Phi_m(t) = \begin{bmatrix} \phi_0(t) & \phi_1(t) & \cdots & \phi_{m-1}(t) \end{bmatrix}^T \quad (2.3)$$

where $t \in [0,1)$.
It is easy to see the BPFs have many properties that most important of them are disjointness, orthogonality, and completeness [10, 11]. For disjointness property, 2.2 gives:

$$\phi_i(t).\phi_j(t) = \begin{cases} 0, & i \neq j, \\ \phi_i(t), & i = j. \end{cases} \quad (2.4)$$

The second property is orthonormality, it is obvious that

$$\int_0^1 \phi_i(t)\phi_j(t)dt = h\delta_{ij}, \quad (2.5)$$

where δ_{ij} is the Kroneker delta.

for every $f \in \ell^2[0,1]$ when $m \to \infty$, Parsevals identity holds:

$$\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|\phi_i(t)\|^2,$$

where

$$f_i = \frac{1}{h}\int_0^1 f(t)\phi_i(t)dt. \quad (2.6)$$

3 BPFs expansion and operational matrices

Theorem 3.1 *[16] Supposes $H = \ell^2[a,b]$ that is a Hilbert space with the inner product that is defined by $\langle f, g\rangle = \int_a^b f(t)g(t)dt$ and $Y = Span\{y_1, y_2, \cdots, y_m\}$. Let f be an arbitrary element in$H$. Since Y is a finite dimensional and closed subspace, it is a complete subset of H . So, f has the unique best approximation out of Y .*

Consider $f \in \ell^2[0,1]$, with respect to BPFs on $[0,1)$ and we can write

$$f(t) = \sum_{i=0}^{\infty} f_i\phi_i(t) \simeq \sum_{i=0}^{m} f_i\phi_i(t) \quad (3.7)$$

$$= F^T\Phi_m(t) = \Phi_m^T F, \quad (3.8)$$

where $F = \begin{bmatrix} f_0 & f_1 & \cdots & f_{m-1} \end{bmatrix}^T$ and f_i is defined as 2.6.
Theorem 3.1 guarantees uniqueness of coefficients. Now, assume $k(x,t) \in \ell^2([0,1) \times [0,1))$ be a two dimensional function. With respect to BPFs we can write

$$k(x,t) \simeq \Phi_m^T K \Phi_m, \quad (3.9)$$

where

$$K_{ij} = m^2 \int_0^1 \int_0^1 k(x,t)\phi_i(x)\phi_j(t)dxdt, \quad (3.10)$$

$i, j = 0, 1, \cdots, m-1$.

Lemma 3.1 *Let m be an integer and $\Phi_m(t)$ defined as 2.3. Then:*

$$\int_0^1 \Phi_m(x)\Phi_m^T(x)dx = hI_m \tag{3.11}$$

Proof. With respect 2.5 it is obvious.

Lemma 3.2 *Let V be an m-vector then*

$$\Phi_m(t)\Phi_m^T(t)V = \tilde{V}\Phi_m(t) \tag{3.12}$$

where $\tilde{V}$ is an $m \times m$ diagonal matrix such that $\tilde{V}_{ii} = V_i$, for $i = 0, 1, ..., m-1$.

Proof. see[10, 11].

Lemma 3.3 *For every $m \times m$ matrix B we can write:*

$$\Phi_m^T(t)B\Phi_m(t) = \hat{B}\Phi_m(t), \tag{3.13}$$

where $\hat{B}$ is an m-vector such that $\hat{B}_i = B_{ii}$, for $i = 0, 1, ..., m-1$.

Proof. see [10, 11].

Lemma 3.4 *Let $0 \le x \le 1$ and $\Phi_m(t)$ defined as 2.3 Then:*

$$\int_0^x \Phi_m(t)dt \simeq P\Phi_m(x) \tag{3.14}$$

Where P , operational matrix of integration, is an $m \times m$upper triangular matrix and can be presented as $P=\frac{h}{2}\begin{bmatrix} 1 & 2 & \cdots & 2 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$.

Proof. See [10, 11]

4 Main Idea

Lemma 4.1 *Suppose $f(x), g(x) \in \ell^2[0,1)$, then $f(x)g(x) = H^T\Phi_m(x)$. Where*

$$H = \begin{bmatrix} f_0g_0 & f_1g_1 & \cdots & f_{m-1}g_{m-1} \end{bmatrix}^T,$$
$$F = \begin{bmatrix} f_0 & f_1 & \cdots & f_{m-1} \end{bmatrix}^T$$
$$and G = \begin{bmatrix} g_0 & g_1 & \cdots & g_{m-1} \end{bmatrix}^T$$

Proof. With respect to 3.7 we can write: $f(x)g(x) = F^T\Phi_m(x)\Phi_m^T(x)G$. Now 3.14 implies

$$F^T\Phi_m(x)\Phi_m^T(x)G = F^T Diag(G)\Phi_m(x)$$
$$= \begin{bmatrix} f_0g_0 & f_1g_1 & \cdots & f_{m-1}g_{m-1} \end{bmatrix}\Phi_m(x).$$

Lemma 4.2 *If $f(x) \in \ell^2[0,1)$ then $f^n(x) = H^T\Phi_m(x)$, where $H = [f_0^n \dots f_{m-1}^n]^T$.*

Proof. with respect to lemma 4.1 it is obvious.

Now, if $\psi(x) \in C^\infty[0,1]$ then$\psi(x)$ has Taylor expansion as follows:

$$\psi(x) = \textstyle\sum_{i=0}^{\infty} a_i x^i.$$

Suppose $u(x) \in \ell^2[0,1)$e an arbitrary function 3.7 gives:

$$u(x) \simeq U^T\Phi_m(x) = \Phi_m^T(x)U,$$

where $U = \begin{bmatrix} u_0 & u_1 & \cdots & u_{m-1} \end{bmatrix}$. Now

$$\psi(u(x)) = \sum_{i=0}^{\infty} a_i u^i(x) \approx \sum_{i=0}^{N} a_i u^i(x). \tag{4.15}$$

Where N is the number of Taylor terms. with respect to 3.7, $\psi(u(x))$ has a unique expansion as follows: $\psi(u(x)) = \Psi^T\phi_m(x)$,where $\Psi = \begin{bmatrix} \Psi_0 & \Psi_1 & \cdots & \Psi_{m-1} \end{bmatrix}$. Lemma 4.2 and Eq. 4.15 indicate:

$$\Psi_j = \textstyle\sum_{i=0}^{N} a_i u_j^i, j = 0, 1, ..., m-1.$$

The results mentioned in previous sections are used to obtain a direct efficient method to solve nonlinear Fredholm-Volterra integral equation

5 Direct method to solve nonlinear Fredholm-Volterra integral equation

In 1.1 suppose $u(x) \in \ell^2[0,1)$ is unknown, $k_1(x,t), k_2(x,t) \in \ell^2([0,1) \times [0,1))$ are known and $G(x), H(x) \in C^\infty[0,1)$, moreover $\lambda \in \{0,1\}$and λ_1, λ_2 are two real parameters. At first, consider Fredholm term in 1.1 as follows:

$$\int_0^1 k_1(x,t)G(u(t))dt \tag{5.16}$$

Table 1: Results of Example 6.1.

x	$m=100$	e_{100}	$m=120$	e_{120}	exact solution
0.00	0.00499731331	4.99×10^{3}	0.0041645073	4.16×10^{3}	0.00
0.10	0.10494632640	4.94×10^{3}	0.1041153163	4.11×10^{3}	0.10
0.20	0.2049097628	4.90×10^{3}	0.2040801536	4.08×10^{3}	0.20
0.30	0.30489870480	4.89×10^{3}	0.3040698597	4.06×10^{3}	0.30
0.40	0.4049191294	4.91×10^{3}	0.4040901963	4.09×10^{3}	0.40
0.50	0.5049694998	4.96×10^{3}	0.5041394396	4.13×10^{3}	0.50
0.60	0.6050384501	5.03×10^{3}	0.6042060663	4.20×10^{3}	0.60
0.70	0.7051025934	5.10×10^{3}	0.7042665638	4.26×10^{3}	0.70
0.80	0.8051244822	5.12×10^{3}	0.8042833912	4.28×10^{3}	0.80
0.90	0.9050507458	5.05×10^{3}	0.9042031178	4.20×10^{3}	0.90

Table 2: Results of Example 6.2.

x	$m=32$	e_{32}	$m=64$	e_{64}	exact solution
0.00	0.989584	1.04×10^{2}	0.994792	5.20×10^{3}	1.000000
0.10	0.891689	1.31×10^{2}	0.905768	9.31×10^{4}	0.904837
0.20	0.820394	1.66×10^{3}	0.824711	5.98×10^{3}	0.818731
0.30	0.739236	1.58×10^{3}	0.735426	5.39×10^{3}	0.740818
0.40	0.680130	9.81×10^{3}	0.669613	7.07×10^{4}	0.670320
0.50	0.600213	6.31×10^{3}	0.603372	3.15×10^{3}	0.606531
0.60	0.540837	7.95×10^{3}	0.549376	5.64×10^{4}	0.548812
0.70	0.497594	1.00×10^{3}	0.500213	3.62×10^{3}	0.496585
0.80	0.448370	9.59×10^{4}	0.446058	3.27×10^{3}	0.449329
0.90	0.412520	5.95×10^{3}	0.406141	4.29×10^{4}	0.406570

Table 3: Results of Example 6.3.

x	$m=50, N=40$	$e_{50,40}$	$m=100, N=20$	$e_{100,20}$	exact solution
0.00	0.006667	1.04×10^{2}	0.003333	5.20×10^{3}	0.00
0.10	0.1134	1.31×10^{2}	0.1033	9.31×10^{4}	0.10
0.20	0.2066	1.66×10^{3}	0.2033	5.98×10^{3}	0.20
0.30	0.3135	1.58×10^{3}	0.3033	5.39×10^{3}	0.30
0.40	0.4064	9.81×10^{3}	0.4032	7.07×10^{4}	0.40
0.50	0.5138	6.31×10^{3}	0.5031	3.15×10^{3}	0.50
0.60	0.6059	7.95×10^{3}	0.603	5.64×10^{4}	0.60
0.70	0.7144	1.00×10^{3}	0.7028	3.62×10^{3}	0.70
0.80	0.8052	9.59×10^{4}	0.8027	3.27×10^{3}	0.80
0.90	0.9155	5.95×10^{3}	0.9024	4.29×10^{4}	0.90

Approximating functions $k_1(x,t)$ and $G(u(t))$ with respect to BPFs gives:

$$k_1(x,t) = \Phi_m^T(x) K_1 \Phi_m(t) \; G(u(t)) = \Phi_m^T(t) G \tag{5.17}$$

where the matrixK_1 is BPFs coefficients of $k_1(x,t)$ and the vector G is BPFs coefficients of $G(u(t))$ such that

$$G_j = \sum_{i=0}^{N} a_i u_j^i, j = 0, 1, ..., m-1.$$

Substituting 5.17 into 5.16 and using 3.11 gives:

$$\int_0^1 \Phi_m^T(x) K_1 \Phi_m(t) \Phi_m^T(t) G dt = h \Phi_m^T(x) K_1 G. \tag{5.18}$$

Table 4: Results of Example 6.4.

x	$m=50$	e_{50}	$m=100$	e_{100}	exact solution
0.00	0.009999	1.04×10^2	0.005	5.00×10^3	0.00
0.10	0.11	1.31×10^2	0.105	5.00×10^3	0.10
0.20	0.21	1.66×10^3	0.205	5.00×10^3	0.20
0.30	0.31	1.58×10^3	0.305	5.00×10^3	0.30
0.40	0.4099	9.81×10^3	0.405	5.00×10^3	0.40
0.50	0.5099	6.31×10^3	0.505	5.00×10^3	0.50
0.60	0.6099	7.95×10^3	0.605	5.00×10^3	0.60
0.70	0.7099	1.00×10^3	0.705	5.00×10^3	0.70
0.80	0.8099	9.59×10^4	0.805	5.00×10^3	0.80
0.90	0.9099	5.95×10^3	0.905	5.00×10^3	0.90

Table 5: Results of Example 6.5.

x	$m=50, N=10$	$e_{50,10}$	$m=100, N=10$	$e_{100,10}$	exact solution
0.00	0.99	1.00×10^2	0.995	5.00×10^3	1.0
0.10	0.89	1.00×10^2	0.895	5.00×10^3	0.9
0.20	0.7899	1.01×10^2	0.795	5.00×10^3	0.8
0.30	0.6899	1.01×10^2	0.695	5.00×10^3	0.7
0.40	0.5899	1.01×10^2	0.595	5.00×10^3	0.6
0.50	0.4899	1.01×10^2	0.495	5.00×10^3	0.5
0.60	0.3899	1.01×10^2	0.395	5.00×10^3	0.4
0.70	0.2899	1.01×10^2	0.295	5.00×10^3	0.3
0.80	0.1899	1.01×10^2	0.195	5.00×10^3	0.2
0.90	0.08989	1.01×10^2	0.09497	5.03×10^3	0.1

Table 6: Results of Example 6.6.

x	$m=50$	e_{50}	$m=100$	e_{100}	exact solution
0.0	0.009643	9.64×10^2	0.005043	5.04×10^{-3}	0.000000
0.1	0.1229	1.23×10^2	0.1167	6.18×10^{-3}	0.1105170918
0.2	0.2593	1.5×10^{-2}	0.2517	7.42×10^{-3}	0.2442805516
0.3	0.4229	1.79×10^2	0.4138	8.84×10^{-3}	0.4049576424
0.4	0.6181	2.13×10^{-2}	0.6073	6.97×10^{-3}	0.5967298792
0.5	0.8497	2.53×10^2	0.8368	1.25×10^{-2}	0.8243606355
0.6	1.123	2.97×10^{-2}	1.108	1.47×10^{-2}	1.0932712800
0.7	1.445	3.53×10^2	1.427	1.73×10^{-2}	1.4096268950
0.8	1.822	4.15×10^{-2}	1.801	2.05×10^{-2}	1.7804327420
0.9	2.262	4.83×10^2	2.237	$2,33\times10^{-2}$	2.213642800

Now, in 1.1 the Volterra term is:

$$\int_0^x k_2(x,t)H(u(t))dt \tag{5.19}$$

Approximating functions $k_2(x,t)$ and $H(u(t))$ with respect to BPFs gives:

$$k_2(x,t) = \Phi_m^T(x)K_2\Phi_m(t)$$

$$\text{H(u(t))}=\Phi_m^T(t)H(5.20)$$

whereK_2 is the matrix of $k_2(x,t)$ and the vector H is BPFs coefficients of $H(u(t))$ such that $H_j = \sum_{i=0}^N b_i u_j^i, j = 0,1,...,m-1$. Now, Substituting 5.19 into 5.18 gives:

$$\int_0^x \Phi_m^T(x)K_2\Phi_m(t)\Phi_m^T(t)Hdt \tag{5.21}$$

Table 7: Results of Example 6.7.

x	$m=50, N=10$	$e_{50,10}$	$m=100, N=10$	$e_{100,10}$	exact solution
0.0	0.0076	7.60×10^{3}	0.003894	3.89×10^{-3}	0.0
0.1	0.1119	1.19×10^{2}	0.1041	4.1×10^{-3}	0.1
0.2	0.2084	8.4×10^{-3}	0.2043	4.3×10^{-3}	0.2
0.3	0.3113	1.13×10^{2}	0.3044	4.4×10^{-3}	0.3
0.4	0.4089	8.4×10^{-3}	0.4045	4.5×10^{-3}	0.4
0.5	0.5108	1.08×10^{2}	0.5046	4.6×10^{-3}	0.5
0.6	0.6093	9.3×10^{-3}	0.6047	4.7×10^{-3}	0.6
0.7	0.7106	1.06×10^{2}	0.7047	4.7×10^{-3}	0.7
0.8	0.8095	9.5×10^{-3}	0.8048	4.8×10^{-3}	0.8
0.9	0.9103	1.03×10^{2}	0.9048	4.8×10^{-3}	0.9

Using Eq. 3.12 and 3.14 follows:

$$\int_0^x \Phi_m^T(x)K_2\Phi_m(t)\Phi_m^T(t)H dt \tag{5.22}$$

$$= \Phi_m^T(x)K_2\tilde{H}\int_0^x \Phi_m(t)dt \tag{5.23}$$

$$= \Phi_m^T(x)K_2\tilde{H}_u P\Phi_m \tag{5.24}$$

Let $H_u = K_2\tilde{H}P$ be an $m\times m$ matrix. Volterra term of 1.1 gives the following matrix form:

$$\int_0^x k_2(x,t)H(u(t))dt = \Phi_m^T(x)\hat{H}_u. \tag{5.25}$$

Where where $\hat{H}_u$ is an m-vector with components equal to the diagonal entries of matrix H_u. The other terms in (1.1) approximate by BPFs as follows:

$$u(x) = \Phi_m^T(x)U, f(x) = \Phi_m^T(x)F, \tag{5.26}$$

Where U and F are m-vectors which computed by 2.6. replacing 5.17, 5.22 and in 1.1 gives the final matrix form of nonlinear Fredholm-Volterra integral equation as follows:

$$\lambda U - \lambda_1 h K_1 G \lambda_2 \hat{H}_u = F.$$

6 Numerical examples

We are going to apply our method to some numerical examples. We selected examples from different references, so our results can be compared with the results from other methods. Also, the results of each example are shown in a table. We denote $e_{m,N}$ or e_m to show the absolute error with respect to the values of m and N at a given point.

Example 6.1 *[9] Consider*

$$\int_0^1 sin(xt)u(t)dt = \frac{sinx - xcosx}{x^2}$$

with the exact solution $u(x)=x$.
This equation is a linear Fredholm integral equation of the first kind. Table 1 shows the exact solution, our approximations and absolute error in some special points.

Example 6.2 *[10, 11] Consider following linear Volterra integral equation of the first kind with the exact solutionu$(x)=e^{-x}$:*

$$\int_0^x e^{x+t}dt = xe^x, 0\le x\le 1.$$

Table 2 shows some results about Example 6.2.

Example 6.3 *$u(x)=x$ is the exact solution of the following nonlinear Volterra integral equation of the first kind:*

$$\int_0^x 2cos(x-t)sin(u(t))dt = xsinx, 0\le x\le 1.$$

The results of applying the method at some points are shown in the Table 3.

Example 6.4 *$u(x)=x$ is the exact solution of the following nonlinear Fredholm integral equation of the second kind:*

$$\frac{5}{6}x + \int_0^1 xt^2u^3(t)dt = u(x)$$

Table 4 shows the results obtained in some points.

Example 6.5 *Consider following nonlinear Fredholm-Volterra integral equation with the exact solution* $u(x) = 1 - x$*:*

$$u(x) = \frac{1}{12}(19 - 28x - 6sinxsin1 \\ -6xcos1sinx + 6sin1cosx) \\ + \int_0^x sin(x-t)cos(u(t))dt \\ + \int_0^1 (1+u^2(t))(x-t)dt$$

results are presented in Table 5.

Example 6.6 *The following integral equation:*

$$u(x) = e^x - x - 1 + \int_0^x u(t)dt + \int_0^1 xu(t)dt$$

is a linear Fredholm-Volterra integral equation with the exact solution $u(x) = xe^x$*.*

Example 6.7 *Suppose* $u(x) = x$ *be the exact solution of the following nonlinear Fredholm-Volterra integral equation of the first kind.*

$$\frac{1}{5}(sinx - 2cosx + 2e^{2x}) \\ +\frac{1}{2}e^x(1 - e^{-1}(cos1 + sin1)) \\ = \int_0^x cos(x-t)e^{2u(t)}dt \\ + \int_0^1 e^{x-t}sinu(t)dt.$$

Table 7 shows results of Example 6.7.

7 conclusion

In this article introduced an efficient and general method to solve integral equation. This method can solve any type of integral equations. Applying vector forms of BPFs, operational matrices and Taylor expansion provide a direct method to solve integral equations. Examples show ability of our method to solve all of the type of integral equations, linear and nonlinear, Fredholm, Volterra and Fredholm-Volterra, also, first kind and second kind. In addition, this approach can be extended to solve nonlinear Fredholm-Volterra integro-differential equations. Computation of absolute errors confirms our method is convergence and stable.

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