Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 14, No. 2, 2022 Article ID IJIM-1467, 11 pages
DOI: http://dx.doi.org/10.30495/ijim.2022.19296
Research Article

# Existence of Weak Solutions to a Kind of System of Fractional Semi-Linear Fredholm-Volterra Boundary Value Problem

E. Shivanian *†

Received Date: 2020-10-20 Revised Date: 2021-03-09 Accepted Date: 2021-06-25

**Abstract**

This article is devoted to study the weak solutions of a class of nonlinear system of fractional boundary value problems including both Volterra and Fredholm linear integral terms. This system of fractional semi-linear Fredholm-Volterra integro-differential equations does have a gradient of a nonlinear source term as well. We apply the critical point theory and the variational structure to prove the existence of at least three distinct weak solutions to the system. Furthermore, it is presented an example to verify the legitimacy and applicability of the theory.

*Keywords* : System of fractional semi-linear Fredholm-Volterra integro-differential equations; Dirichlet condition; Weak solution; Critical point theory; Variational method.

## 1 Introduction

The fractional derivative and fractional differential equations have been implemented to describe some phenomena in physics and engineering. On the other hand, to describe the process having delay, it is not best tool to employ an ordinary or partial differential equation. A convenient approach to resolve these kind of problems is to apply integro-differential equations. Numerous mathematical models of physical phenomena end up with integro-differential equations [1, 2, 3, 5, 6, 31, 32]. Not many studies are available in the literature to deal with fractional integro-differential equations. Balachandran et al. [8, 9] investigated the existence results for some kinds of fractional integro-differential equations using fixed point theory. The existence of positive solutions to the nonlinear fractional differential equations with nonlocal fractional integro-differential boundary conditions on an unbounded domain has been investigated by applying the Leray-Schauder nonlinear alternative theorem [36]. Rahimkhani et al. [30] suggested non-integer order Bernoulli functions to deal with fractional Fredholem-Volterra integral differential problems. In [10, 21, 25, 28] Lie group technique and some other techniques based on Galerkin approach, optimum q-HAM and spectral-collocation method have been applied for solving fractional integro-differential equations. As mentioned in precedent paragraph, a widely

*Corresponding author. shivanian@sci.ikiu.ac.ir, Tel:+98(912)6825371.
†Department of Applied Mathematics, Imam Khomeini International University, Qazvin, Iran.

accepted method to study the existence of solutions to the nonlinear non-integer BVPs is the fixed point theory which has been employed prosperously on many models, e.g. [4, 11, 37]. However, another superb method is the calculus of variation which has been very successful in the investigation of the existence of solutions to the differential equation with type of integer order. The necessary condition for applying this method is that the given boundary value problem should possess a variational structure on some convenient Sobolev spaces. The readers are referred to [7, 13, 14, 15, 16, 17, 18, 20, 23, 24, 26, 27, 29, 33, 34, 35, 38] and the references therein to receive more information in this research line.

Consider the following nonlinear fractional semi-linear Fredholm-Volterra integro-differential equations

$$\begin{aligned}
&{}_yD_Y^{\vartheta_i}\left(a_i(y)\,{}_0D_y^{\vartheta_i}w_i(y)\right) = \\
&\lambda F_{w_i}\left(y, w_1(y), ..., w_n(y)\right) + \\
&\int_0^Y k_{1,i}\left(y,\varpi\right) w_i(\varpi)\mathrm{d}\varpi + \\
&\int_0^y k_{2,i}\left(y,\varpi\right) w_i(\varpi)\mathrm{d}\varpi,\ y\in(0,Y), \\
&i = 1, ..., n; \\
&w_i(y) = \int_0^Y k_{1,i}\left(y,\varpi\right) w_i(\varpi)\mathrm{d}\varpi + \\
&\int_0^y k_{2,i}\left(y,\varpi\right) w_i(\varpi)\mathrm{d}\varpi, \\
&y\in(0,Y),\ i = 1, ..., n; \\
&w_i(0) = w_i(Y) = 0,\ i = 1, ..., n.
\end{aligned}$$

where $\lambda$ is a positive real parameter, $0 < \vartheta_i \leq 1$ , $a_i(y) \in L^\infty[0,Y]$, $\bar{a}_i = essinf_{[0,Y]}a_i(y) > 0$ and $F : [0,Y]\times\mathbb{R}^n \to \mathbb{R}$ is a function to the extent that $F(y, w_1, ..., w_n)$ is continuous with regards to $t$ and continuously differentiable with regards to $w_i$ i.e. $F(\cdot, w_1, ..., w_n) \in C[0,Y]$ and $F(y,\cdot,...,\cdot) \in C^1(\mathbb{R}^n)$, and further, $k_{1,i}(\cdot,\cdot), k_{2,i}(\cdot,\cdot) \in C([0,Y],[0,Y])$ hence the kernels $k_{1,i}$ and $k_{2,i}$ are bounded by for example $L_i$ and $M_i$, respectively. Moreover, $F_s$ indicates the partial-derivative of $F$ to the $s$ and, ${}_yD_Y^\gamma$ and ${}_0D_y^\gamma$ are the right and left Riemann-Liouville none-integer derivatives of order $\gamma$ as given by [22]

$$\begin{aligned}
&{}_0D_y^\gamma w(y) = \\
&\frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{w(z)}{(y-z)^\gamma}\mathrm{d}z, \qquad (1.1)\\
&{}_yD_Y^\gamma w(y) = \\
&-\frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}y}\int_y^Y \frac{w(z)}{(z-y)^\gamma}\mathrm{d}z. \qquad (1.2)
\end{aligned}$$

Heidarkhani et. al. [19] have investigated the above nonlinear system of fractional differential equations in the case of having nonlinear function of $w_i(y)$ with some specific properties instead of integral terms on the right hand side. By using a local minimization principle they proved the existence of at least one weak solution. Motivated by the paper [19], in the current work, we utilize well-known theorem established by Bonanno and Marano [12] to study the existence of at the lowest three different weak solutions to the nonlinear model of non-integer partial Fredholm-Volterra integal-differential equations (1.1).

## 2 Some essential remarks

Consider the following variant of three critical points theorem due to Bonanno and Marano [12] to discuss the existence and multiplicity of weak solutions to the system of fractional partial Fredholm-Volterra integro-differential equations (1.1), this theorem would help to prove the existence of at least three distinct weak solutions.

**Theorem 2.1.** *(see [12], Theorem 3.6). Let $X$ be a reflexive real Banach space and $\Delta : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable in which its Gâteaux derivative has a continuous inverse on $X^*$, moreover, suppose that $\nabla : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional in which its Gâteaux derivative is compact, such that $\Delta(0) = \nabla(0) = 0$. Suppose also there exist $r \in \mathbb{R}$ and $w_1 \in X$ with $0 < r < \Delta(w_1)$, satisfying*

1. $\sup_{w\in\Delta^{-1}(]-\infty,r])}\nabla(u) < r\frac{\nabla(w_1)}{\Delta(w_1)}$

2. $\forall\lambda\in\Lambda_r := \left]\frac{\Delta(w_1)}{\nabla(w_1)}, \frac{r}{\sup_{w\in\Delta^{-1}(]-\infty,r])}\nabla(u)}\right[$, *the functional* $\Delta-\lambda\nabla$ *is coercive.*

*Then, for each* $\lambda\in\Lambda_r$ *the functional* $\Delta-\lambda\nabla$ *admits at least three distinct critical points in* $X$*.*

Take $C_0^\infty([0,Y],\mathbb{R}^n)$ be all functions $v\in C^\infty([0,Y],\mathbb{R}^n)$ so that $v(0)=v(Y)=0$ with the normal norm $\|v\|_\infty = \max_{y\in[0,Y]}|v(y)|$. Let $L^p([0,Y],\mathbb{R}^n)$ supplied with its normal norm $\|v\|_{L^p} = \left(\int_0^Y |v(z)|^p \mathrm{d}z\right)^{\frac{1}{p}}$. We use the following useful Lemma to define the weak solution for the system (1.1).

**Definition 2.1.** *Take* $0<\vartheta_i\le 1, i=1,...,n,$ *be non-integer derivative space* $E_0^{\vartheta_i}$ *defining by the closure of* $C_0^\infty([0,Y],\mathbb{R})$ *i.e.* $E_0^{\vartheta_i}=\overline{C_0^\infty([0,Y],\mathbb{R})}$ *with regards to the following weighted norm*

$$\|w\|_{\vartheta_i} = \left(\int_0^Y a_i(y)\left|{}_0D_y^{\vartheta_i}w(y)\right|^2 dy + \int_0^Y |w(y)|^2\, dy\right)^{\frac{1}{2}},$$
$$\forall w\in E_0^{\vartheta_i}, i=1,...,n. \qquad (2.3)$$

**Lemma 2.1.** *(zee [20]). If* $\frac{1}{2}<\vartheta_i\le 1$ *then* $\forall w\in E_0^{\vartheta_i},\ i=1,...,n,$ *they satisfy*

1.
$$\|w\|_{L^2} \le \frac{Y^{\vartheta_i}}{\Gamma(\vartheta_i+1)}\|{}_0D_y^{\vartheta_i}w\|_{L^2}, \qquad (2.4)$$

2.
$$\|w\|_\infty \le \frac{Y^{\vartheta_i-\frac{1}{2}}}{\Gamma(\vartheta_i)\sqrt{2\vartheta_i-1}}\|{}_0D_y^{\vartheta_i}w\|_{L^2}. \qquad (2.5)$$

**Remark 2.1.** *It is straightforward, from the above Lemma 2.1, to extract the inequalities for* $i=1,...,n$ *as*

1.
$$\|w\|_{L^2} \le \frac{Y^{\vartheta_i}}{\Gamma(\vartheta_i+1)\sqrt{\bar{a}_i}} \times \left(\int_0^Y a_i(y)\left|{}_0D_y^{\vartheta_i}w(y)\right|^2 dy\right)^{\frac{1}{2}} \qquad (2.6)$$

2.
$$\|w\|_\infty \le \frac{Y^{\vartheta_i-\frac{1}{2}}}{\Gamma(\vartheta_i)\sqrt{\bar{a}_i(2\vartheta_i-1)}} \times \left(\int_0^Y a_i(y)\left|{}_0D_y^{\vartheta_i}w(y)\right|^2 dy\right)^{\frac{1}{2}} \qquad (2.7)$$

The equation (2.3) and inequality (2.6) yield that the norm defined by equation (2.3) is equivalent to the following norm

$$\|w\|_{\vartheta_i} = \left(\int_0^Y a_i(y)\left|{}_0D_y^{\vartheta_i}w(y)\right|^2 \mathrm{d}y\right)^{\frac{1}{2}},$$
$$\forall w\in E_0^{\vartheta_i}, i=1,...,n. \qquad (2.8)$$

We work with the norm (2.8) in what follows. Now, define $X=\prod_{i=1}^{i=n}E_0^{\vartheta_i}$ augmented to the norm

$$\|W\|_X = \sum_{i=1}^n \|w_i\|_{\vartheta_i},\quad w_i\in E_0^{\vartheta_i},$$
$$W=(w_1,w_2,...,w_n)\in X. \qquad (2.9)$$

**Definition 2.2.** *It is called* $W=(w_1,w_2,...,w_n)\in X$ *as the weak solution of the system (1.1) if we have*

$$\sum_{i=1}^n\int_0^Y a_i(y)\,{}_0D_y^{\vartheta_i}w_i(y)\,{}_0D_y^{\vartheta_i}v_i(y)\,dy - \sum_{i=1}^n\int_0^Y\int_0^Y k_{1,i}(y,\varpi)\,w_i(\varpi)v_i(y)\,d\varpi\,dy - \sum_{i=1}^n\int_0^Y\int_0^y k_{2,i}(y,\varpi)\,w_i(\varpi)v_i(y)\,d\varpi\,dy - \lambda\int_0^Y\sum_{i=1}^n F_{w_i}(y,w_1(y),...,w_n(y))\,v_i(y)\,dy = 0, \qquad (2.10)$$

*for every* $V=(v_1,v_2,...,v_n)\in X$*.*

Let us, define $H_i(w_i(y)) = \frac{1}{2}\int_0^Y k_{1,i}(y,\varpi)\,w_i(\varpi)w_i(y)\mathrm{d}\varpi + \frac{1}{2}\int_0^y k_{2,i}(y,\varpi)\,w_i(\varpi)w_i(y)\mathrm{d}\varpi$ for all $x\in\mathbb{R}$ and for $1\le i\le n$ where we have used it for our aim in the remain discussions. The key point of this definition is that we use

$$w_i(y) = \int_0^Y k_{1,i}(y,\varpi)\,w_i(\varpi)\mathrm{d}\varpi + \int_0^y k_{2,i}(y,\varpi)\,w_i(\varpi)\mathrm{d}\varpi, \qquad (2.11)$$

to take Gâteaux derivative of $H_i(w_i(y))$ as follows

$$H_i'(w_i(y))(v_i(y)) = \int_0^Y k_{1,i}(y,\varpi)\, w_i(\varpi) v_i(y) d\varpi + \int_0^y k_{2,i}(y,\varpi)\, w_i(\varpi) v_i(y) d\varpi. \quad (2.12)$$

# 3 Main result

Before we bring our main result, let us to set the following notations which are key tools in our remain discussion.

$$\theta_i = \max\{L_i, M_i\}, \quad (3.13)$$

$$M = \max_{1\le i\le n} \quad (3.14)$$

$$\left\{\frac{Y^{2\vartheta_i-1}}{(\Gamma(\vartheta_i))^2 \bar{a}_i(2\vartheta_i-1) - 2\theta_i Y^{2\vartheta_i+1}}\right\}, \quad (3.15)$$

$$\Upsilon(c) = \{\Xi = (\varsigma_1, \varsigma_2, ..., \varsigma_n) \in \mathbb{R}^n : \frac{1}{2}\sum_{i=1}^n \varsigma_i^2 \le c\}, \quad c>0, \quad (3.16)$$

$$\sigma = \min_{1\le i\le n}\{\sigma_i\}, \quad (3.17)$$

$$\sigma_i = 1 - \frac{2\theta_i Y^{2\vartheta_i+1}}{(\Gamma(\vartheta_i))^2 \bar{a}_i(2\vartheta_i-1)}, \quad (3.18)$$

$$L = \max_{1\le i\le n}\left\{\frac{Y^{2\vartheta_i}}{\sigma(\Gamma(\vartheta_i+1))^2 \bar{a}_i}\right\}, \quad (3.19)$$

**Theorem 3.1.** *Consider $F:[0,Y]\times\mathbb{R}^n \to \mathbb{R}$ as a function in order that $F(\cdot, w_1, ..., w_n) \in C[0,Y]$, $F(y,\cdot,...,\cdot) \in C^1(\mathbb{R}^n)$ and $F(y,0,...,0)=0$ for all $y\in[0,Y]$. By the way, let $r>0$ be a constant and $\Omega(y) = (\omega_1(y), ..., \omega_n(y))$ a vector-valued function such that we have*

**(H0)** $\frac{1}{2} < \vartheta_i \le 1$;

**(H1)** $\theta_i < \frac{(\Gamma(\vartheta_i))^2 \bar{a}_i(2\vartheta_i-1)}{2Y^{2\vartheta_i+1}}$;

**(H2)** $\sum_{i=1}^n \|\omega_i\|_{\vartheta_i}^2 \ge 2r + 2\sum_{i=1}^n \int_0^Y H_i(\omega_i(y))\,dy$;

**(H3)** $\frac{\int_0^Y \sup_{(\varsigma_1,...,\varsigma_n)\in\Upsilon(Mr)} F(y,\varsigma_1,...,\varsigma_n)\,dy}{r} < \frac{2\int_0^Y F(y,\omega_1(y),...,\omega_n(y))\,dy}{\sum_{i=1}^n \|\omega_i\|_{\vartheta_i}^2 - 2\sum_{i=1}^n \int_0^Y H_i(\omega_i(y))\,dy}$;

**(H4)** $\liminf_{\forall i:|\varsigma_i|\to+\infty} \frac{F(y,\varsigma_1,...,\varsigma_n)}{\sum_{i=1}^n |\varsigma_i|^2} < \int_0^Y \sup_{(\varsigma_1,...,\varsigma_n)\in\Upsilon(Mr)} F(y,\varsigma_1,...,\varsigma_n)\,dy_{\overline{2Lr}}$.

*Then, for those $\lambda$'s being a member of the interval*

$$\Lambda = \left[\frac{\sum_{i=1}^n \left(\|\omega_i\|_{\vartheta_i}^2 - 2\int_0^Y H_i(\omega_i(y))\,dy\right)}{2\int_0^Y F(y,\omega_1(y),...,\omega_n(y))\,dy}, \frac{r}{\int_0^Y \sup_{(\varsigma_1,...,\varsigma_n)\in\Upsilon(Mr)} F(y,\varsigma_1,...,\varsigma_n)\,dy}\right]. \quad (3.20)$$

*the system (1.1) has at least three different weak solutions in $X$.*

**Proof.** To establish the above Theorem, we would apply Theorem 2.1. Consider $X = \prod_{i=1}^{i=n} E_0^{\vartheta_i}$ equipped with the norm $\|W\|_X$ defined by (2.9). It can be easily verified that $X$ is a reflexive and separable Banach space. Now, for any given $W = (w_1(y), ..., w_n(y)) \in X$, we construct the functionals $\Delta, \nabla : X \to \mathbb{R}$ as follows:

$$\Delta(W) = \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^n \int_0^Y H_i(w_i(y))dy, \quad (3.21)$$

$$\nabla(W) = \int_0^Y F(y, w_1(y), ..., w_n(y))dy. \quad (3.22)$$

The functionals $\Delta$ and $\nabla$ are obviously well-defined, Gâteaux differentiable and further, their Gâteaux derivatives are achieved by

$$\Delta'(W)(V) = \sum_{i=1}^n \int_0^Y a_i(y)\, {}_0D_y^{\vartheta_i} w_i(y)\, {}_0D_y^{\vartheta_i} v_i(y) dy - \sum_{i=1}^n \int_0^Y \int_0^Y k_{1,i}(y,\varpi)\, w_i(\varpi) v_i(y) d\varpi dy - \sum_{i=1}^n \int_0^Y \int_0^y k_{2,i}(y,\varpi)\, w_i(\varpi) v_i(y) d\varpi dy,$$
$$\forall V = (v_1, v_2, ..., v_n) \in X, \quad (3.23)$$

$$\nabla'(W)(V) = \int_0^Y \sum_{i=1}^n F_{w_i}(y, w_1(y), ..., w_n(y))\, v_i(y) dy,$$
$$\forall V = (v_1, v_2, ..., v_n) \in X, \quad (3.24)$$

as a matter of fact, $\Delta'(W), \nabla'(W) \in X^*$ in where $X^*$ is dual space of $X$. It can be easily verified that the functional $\Delta$ is sequentially weakly lower semicontinuous and its Gâteaux derivative does have a continuous inverse on $X^*$. Further, since $|k_{1,i}(y,\varpi)| \leq L_i$ and $|k_{2,i}(y,\varpi)| \leq M_i$ for $0 \leq t, \varpi \leq Y$, one can obtain

$$\begin{aligned} H_i(w_i(y)) &= \frac{1}{2}\int_0^Y k_{1,i}(y,\varpi)\, w_i(\varpi)w_i(y)\mathrm{d}\varpi \\ &+ \frac{1}{2}\int_0^y k_{2,i}(y,\varpi)\, w_i(\varpi)w_i(y)\mathrm{d}\varpi \\ &\leq \frac{1}{2}w_i(y)YL_i\|w_i\|_\infty \\ &+ \frac{1}{2}w_i(y)tM_i\|w_i\|_\infty \\ &\leq \frac{1}{2}YL_i\|w_i\|_\infty^2 + \frac{1}{2}YM_i\|w_i\|_\infty^2 \\ &\leq \theta_i Y\|w_i\|_\infty^2. \end{aligned} \quad (3.25)$$

Inspections on (2.7), (2.8) and (3.21) leads us to

$$\begin{aligned} \Delta(W) &= \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^n \int_0^Y H_i(w_i(y))\mathrm{d}y \\ &\geq \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^n \int_0^Y \theta_i Y\|w_i\|_\infty^2 \mathrm{d}y \\ &= \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^n \theta_i Y^2\|w_i\|_\infty^2 \\ &\geq \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \\ &\sum_{i=1}^n \frac{\theta_i Y^{2\vartheta_i+1}}{(\Gamma(\vartheta_i))^2\,\bar{a}_i(2\vartheta_i-1)}\|w_i\|_{\vartheta_i}^2 \\ &= \frac{1}{2}\sum_{i=1}^n \left(1 - \frac{2\theta_i Y^{2\vartheta_i+1}}{(\Gamma(\vartheta_i))^2\,\bar{a}_i(2\vartheta_i-1)}\right)\|w_i\|_{\vartheta_i}^2 \\ &= \frac{1}{2}\sum_{i=1}^n \sigma_i\|w_i\|_{\vartheta_i}^2 \\ &\geq \frac{\sigma}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 = \frac{\sigma}{2}\|W\|_X, \end{aligned} \quad (3.26)$$

then $\sigma$ is positive because each $\sigma_i$ is positive by assumption (H1), thus

$$\lim_{\|W\|_X \to +\infty} \Delta(W) = +\infty$$

hence it is coercive. Now, our next step is to prove the functional $\nabla$ is definitely sequentially weakly upper semicontinuous and its derivative $\nabla' : X \to X^*$ is a compact operator. If $\lim_{m\to+\infty} W_m \rightharpoonup W$ in $X$ where $W_m(y) = (w_{m,1}(y), ..., w_{m,n}(y))$, then absolutely $W_m$ converges uniformly to $W$ on the interval $[0, Y]$. Therefore

$$\begin{aligned} &\limsup_{m\to+\infty} \nabla(W_m) \leq \\ &\int_0^Y \limsup_{m\to+\infty} F(y, w_{m,1}(y), ..., w_{m,n}(y))\,\mathrm{d}y \\ &= \int_0^Y F(y, w_1(y), ..., w_n(y))\mathrm{d}y = \\ &\nabla(W), \end{aligned} \quad (3.27)$$

hence we are led to this fact that $\nabla$ is sequentially weakly upper semi-continuous. Moreover, it holds

$$\lim_{m\to+\infty} F(y, w_{m,1}(y), ..., w_{m,n}(y))$$

$$= F(y, w_1(y), ..., w_n(y)) \text{ for all } y \in [0, Y]$$

because $F(y, \cdot, ..., \cdot) \in C^1(\mathbb{R}^n)$. Now, the Lebesgue control convergence theorem yields $\nabla'(W_m) \to \nabla'(W)$ strongly, so it results in that $\nabla'$ is strongly continuous on $X$. Therefore, $\nabla' : X \to X^*$ would be a compact operator.

If we allow $W_0(y) = (0, ..., 0)$ and $W_1(y) = \Omega(y)$ then assumption (H2) implies immediately that

$$\begin{aligned} 0 < r &\leq \frac{1}{2}\sum_{i=1}^n \|\omega_i\|_{\vartheta_i}^2 - \\ &\sum_{i=1}^n \int_0^Y H_i(\omega_i(y))\mathrm{d}y = \Delta(W_1), \end{aligned} \quad (3.28)$$

also it is easy to see that $\Delta(W_0(y)) = \nabla(W_0(y)) = 0$ by definitions (3.21) and (3.22), which are required assumptions in Theorem 2.1. Eqs. (2.7), (2.8), (3.15) and (3.16), would result in:

$$\begin{aligned} &\Delta^{-1}(]-\infty, r]) = \{W \in X : \Delta(W) \leq r\} \\ &= \left\{ W \in X : \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 \right. \\ &\left. - \sum_{i=1}^n \int_0^Y H_i(w_i(y))\mathrm{d}y \leq r \right\} \end{aligned}$$

$$\subseteq \left\{ W \in X : \frac{1}{2}\sum_{i=1}^{n}\|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^{n}\int_0^Y \theta_i Y \|w_i\|_\infty^2 \mathrm{d}y \le r \right\}$$
$$\subseteq \left\{ W \in X : \sum_{i=1}^{n} \frac{(\Gamma(\vartheta_i))^2\, \bar{a}_i(2\vartheta_i-1)}{2Y^{2\vartheta_i-1}}\|w_i\|_\infty^2 - \sum_{i=1}^{n}\theta_i Y^2\|w_i\|_\infty^2 \mathrm{d}y \le r \right\}$$
$$\subseteq \left\{ W \in X : \sum_{i=1}^{n} \frac{(\Gamma(\vartheta_i))^2\, \bar{a}_i(2\vartheta_i-1) - 2\theta_i Y^{2\vartheta_i+1}}{2Y^{2\vartheta_i-1}}\|w_i\|_\infty^2 \le r \right\}$$
$$\subseteq \left\{ W \in X : \frac{1}{2M}\sum_{i=1}^{n}\|w_i\|_\infty^2 \le r \right\}$$
$$\subseteq \left\{ W \in X : \frac{1}{2}\sum_{i=1}^{n}|w_i(y)|^2 \le Mr, \text{ for all } y \in [0,Y] \right\}$$
$$\subseteq \Upsilon(Mr),$$

which leads to the following without postponing

$$\begin{aligned}\sup_{W\in\Delta^{-1}(]-\infty,r])} \nabla(W) &= \sup_{W\in\Delta^{-1}(]-\infty,r])} \int_0^Y F\left(y, w_1(y), ..., w_n(y)\right) \mathrm{d}y \\ &\le \sup_{\Xi\in\Upsilon(Mr)} \int_0^Y F(y, \varsigma_1, ..., \varsigma_n)\mathrm{d}y \\ &= \int_0^Y \sup_{\Xi\in\Upsilon(Mr)} F(y, \varsigma_1, ..., \varsigma_n)\mathrm{d}y, \end{aligned} \tag{3.29}$$

and then, from (H3), we would have

$$\begin{aligned}\frac{\sup_{W\in\Delta^{-1}(]-\infty,r])} \nabla(W)}{r} &= \frac{\sup_{W\in\Delta^{-1}(]-\infty,r])} \int_0^Y F\left(y, w_1(y), ..., w_n(y)\right) \mathrm{d}y}{r} \\ &\le \frac{\int_0^Y \sup_{\Xi\in\Upsilon(Mr)} F(y, \varsigma_1, ..., \varsigma_n)\mathrm{d}y}{r} \\ &< \frac{2\int_0^Y F(y, \omega_1(y), ..., \omega_n(y))\mathrm{d}y}{\sum_{i=1}^n \|\omega_i\|_{\vartheta_i}^2 - 2\sum_{i=1}^n \int_0^Y H_i(\omega_i(y))\mathrm{d}y} \\ &= \frac{\nabla(\Omega(y))}{\Delta(\Omega(y))} = \frac{\nabla(W_1)}{\Delta(W_1)}, \end{aligned} \tag{3.30}$$

thus, $\sup_{W\in\Delta^{-1}(]-\infty,r])} \nabla(W) < r\frac{\nabla(W_1)}{\Delta(W_1)}$ and hence the hypothesis (1) of Theorem 2.1 is valid.

Furthermore, the hypothesis (H4) direct us to set two constants $\mu, \varepsilon \in \mathbb{R}$ with the properties

$$\frac{\mu}{\sigma} < \frac{\int_0^Y \sup_{\Xi\in\Upsilon(Mr)} F(y, \varsigma_1, ..., \varsigma_n)\mathrm{d}y}{r}, \tag{3.31}$$

and

$$\forall \Xi \in \mathbb{R}^n : \ F(y, \varsigma_1, ..., \varsigma_n) \le \frac{\mu}{2L\sigma}\sum_{i=1}^n |\varsigma_i|^2 + \varepsilon, \tag{3.32}$$

for all $y \in [0,Y]$. Then clearly for a fixed $W = (w_1(y), ..., w_n(y)) \in X$, we have

$$F(y, w_1(y), ..., w_n(y)) \le \frac{\mu}{2L\sigma}\sum_{i=1}^n |w_i(y)|^2 + \varepsilon, \tag{3.33}$$

for all $y \in [0,Y]$. Now, we show the coercivity of the functional $\Delta(W) - \lambda\nabla(W)$, suppose $\lambda \in \Lambda$, then bringing into accounts (2.6), (2.7), (3.19), (3.20), (3.31) and (3.33), we would have

$$\begin{aligned}\Delta(W) - \lambda\nabla(W) &= \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^n \int_0^Y H_i(w_i(y))\mathrm{d}y - \lambda\int_0^Y F\left(y, w_1(y), ..., w_n(y)\right)\mathrm{d}y \\ &\ge \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^n \int_0^Y \theta_i Y \|w_i\|_\infty^2 \mathrm{d}y - \lambda\int_0^Y F\left(y, w_1(y), ..., w_n(y)\right)\mathrm{d}y \\ &\ge \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^n \theta_i Y^2 \|w_i\|_\infty^2 \mathrm{d}y - \frac{\lambda\mu}{2L\sigma}\int_0^Y \left(\sum_{i=1}^n |w_i(y)|^2\right)\mathrm{d}y - \lambda Y\varepsilon \\ &= \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^n \theta_i Y^2 \|w_i\|_\infty^2 \mathrm{d}y - \frac{\lambda\mu}{2L\sigma}\sum_{i=1}^n \|w_i\|_{L^2}^2 - \lambda Y\varepsilon \\ &\ge \frac{1}{2}\sum_{i=1}^n \|w_i\|_{\vartheta_i}^2 - \sum_{i=1}^n \frac{\theta_i Y^{2\vartheta_i+1}}{(\Gamma(\vartheta_i))^2\, \bar{a}_i(2\vartheta_i-1)}\|w_i\|_{\vartheta_i}^2\end{aligned}$$

$$-\frac{\lambda\mu}{2L\sigma}\sum_{i=1}^{n}\frac{Y^{2\vartheta_i}}{(\Gamma(\vartheta_i+1))^2\bar{a}_i}$$

$\|w_i\|^2_{\vartheta_i} - \lambda Y\varepsilon$
$= 1\frac{}{2\sum_{i=1}^{n}\sigma_i\|w_i\|^2_{\vartheta_i} -}$
$\lambda\mu\frac{}{2L\sigma\sum_{i=1}^{n}\frac{Y^{2\vartheta_i}}{(\Gamma(\vartheta_i+1))^2\bar{a}_i}\|w_i\|^2_{\vartheta_i} - \lambda Y\varepsilon}$
$\geq \frac{1}{2}\sum_{i=1}^{n}\sigma\|w_i\|^2_{\vartheta_i} - \frac{\lambda\mu}{2}\sum_{i=1}^{n}\|w_i\|^2_{\vartheta_i} - \lambda Y\varepsilon$
$\geq \frac{1}{2}\left(\sigma - \frac{\mu r}{\int_0^Y \sup_{\Xi\in\Upsilon(Mr)} F(y,\varsigma_1,\ldots,\varsigma_n)\mathrm{d}y}\right)$
$\times\sum_{i=1}^{n}\|w_i\|^2_{\vartheta_i} - \lambda Y\varepsilon$, since term $\left(\sigma - \frac{\mu r}{\int_0^Y \sup_{\Xi\in\Upsilon(Mr)} F(y,\varsigma_1,\ldots,\varsigma_n)\mathrm{d}y}\right)$ is clearly positive from (3.31), then

$$\lim_{\|W\|_X\to+\infty}(\Delta(W)-\lambda\nabla(W)) = +\infty. \tag{3.30}$$

Hence, we conclude that $\Delta - \lambda\nabla$ is coercive so the hypothesis (2) of Theorem 2.1 is also established. Then, let us apply Theorem 2.1 and consider that the weak solutions of the model (1.1) are precisely the solution of the equality $\Delta'(W) - \lambda\nabla'(W) = 0$. We result in this fact that the system (1.1) accepts to have at least three distinct weak solutions in $X$ for $\lambda \in \Lambda$ and then the proof is perfect.

## 4 Illustrative example

Let us to discuss on the following nonlinear system of fractional semi-linear partial Fredholm-Volterra integro-differential equations

$$
\begin{aligned}
&{}_yD_1^{0.75}\left((1+y^2)_0D_y^{0.75}u(y)\right) = \\
&\lambda F_u\left(y,u(y),v(y),w(y)\right) + \int_0^1 \frac{1}{6}y\varpi u(\varpi)\mathrm{d}\varpi \\
&+ \int_0^y \frac{1}{6}(\sinh y)\varpi u(\varpi)\mathrm{d}\varpi, 0<y<1; \\
&u(y) = \int_0^1 \frac{1}{6}y\varpi u(\varpi)\mathrm{d}\varpi + \\
&\int_0^y \frac{1}{6}(\sinh y)\varpi u(\varpi)\mathrm{d}\varpi, 0<y<1; \\
&{}_yD_1^{0.8}\left((0.5+y)_0D_y^{0.8}v(y)\right) = \\
&\lambda F_v\left(y,u(y),v(y),w(y)\right) + \int_0^1 \frac{1}{16}y\varpi v(\varpi)\mathrm{d}\varpi \\
&+ \int_0^y \frac{1}{16}y(\sinh y)\varpi v(\varpi)\mathrm{d}\varpi, 0<y<1; \\
&v(y) = \int_0^1 \frac{1}{16}y\varpi v(\varpi)\mathrm{d}\varpi + \\
&\int_0^y \frac{1}{16}y(\sinh y)\varpi v(\varpi)\mathrm{d}\varpi, 0<y<1; \\
&{}_yD_1^{0.9}\left((1+y)_0D_y^{0.9}w(y)\right) = \\
&\lambda F_w\left(y,u(y),v(y),w(y)\right) + \int_0^1 \frac{1}{8}y\varpi w(\varpi)\mathrm{d}\varpi \\
&+ \int_0^y \frac{1}{8}y^2(\sinh y)\varpi w(\varpi)\mathrm{d}\varpi, 0<y<1; \\
&w(y) = \int_0^1 \frac{1}{8}y\varpi w(\varpi)\mathrm{d}\varpi + \\
&\int_0^y \frac{1}{8}y^2(\sinh y)\varpi w(\varpi)\mathrm{d}\varpi, 0<y<1; \\
&w(0)=w(1)=0,\ v(0)=v(1)=0, \\
&w(0)=w(1)=0,
\end{aligned}
$$

where

$$
F\left(y,u,v,w\right) = (1+y^2)\times
\begin{cases}
\left(u^2+v^2+w^2\right)^2, & u^2+v^2+w^2 \leq 1; \\
2\sqrt{u^2+v^2+w^2} & \\
-\left(u^2+v^2+w^2\right), & u^2+v^2+w^2 > 1,
\end{cases}
\tag{4.31}
$$

to exclude some important results through Theorem 3.1. The above system enforce us to set $Y = 1$, $\vartheta_1 = 0.75, \vartheta_2 = 0.8, \vartheta_3 = 0.9$, $a_1(y) = 1+y^2$, $a_2(y) = 0.5+y$ and $a_3(y) = 1+y$. There is no doubt that $F$ is continuous with regards to $t$ and continuously differentiable with regards to $u, v$ and $w$. Also, $F(y,0,0,0) = 0$ and some easy calculations result in that

$$
\begin{aligned}
&\bar{a}_1 = 1, \bar{a}_2 = 0.5, \bar{a}_3 = 1, \sigma \cong 0.556043, \\
&M \cong 3.55077, L \cong 20.5695.
\end{aligned}
\tag{4.32}
$$

Besides, we do not hesitate to define

$$H_1(u(y)) = \frac{1}{12}\int_0^1 t\varpi u(\varpi)u(y)\mathrm{d}\varpi +$$

$$\frac{1}{12}\int_0^y (\sinh y)\varpi u(\varpi)u(y)\mathrm{d}\varpi,$$

$$H_2(v(y)) = \frac{1}{32}\int_0^1 t\varpi v(\varpi)v(y)\mathrm{d}\varpi +$$

$$\frac{1}{32}\int_0^y t(\sinh y)\varpi v(\varpi)v(y)\mathrm{d}\varpi,$$

$$H_3(w(y)) = \frac{1}{16}\int_0^1 t\varpi w(\varpi)w(y)\mathrm{d}\varpi +$$

$$\frac{1}{16}\int_0^y t^2(\sinh y)\varpi w(\varpi)w(y)\mathrm{d}\varpi.$$

Moreover, $\theta_1 = L_1 = M_1 = \frac{1}{6}$, $\theta_2 = L_2 = M_2 = \frac{1}{16}$ and $\theta_3 = L_3 = M_3 = \frac{1}{8}$, thus the assumption (H1) is satisfied because of

$$\frac{2\theta_1 Y^{2\vartheta_1+1}}{(\Gamma(\vartheta_1))^2\,\bar{a}_1(2\vartheta_1-1)} = 0.443957 < 1,$$

$$\sigma_1 = 0.556043,$$

$$\frac{2\theta_2 Y^{2\vartheta_2+1}}{(\Gamma(\vartheta_2))^2\,\bar{a}_2(2\vartheta_2-1)} = 0.307405 < 1,$$

$$\sigma_2 = 0.692595,$$

$$\frac{2\theta_3 Y^{2\vartheta_3+1}}{(\Gamma(\vartheta_3))^2\,\bar{a}_3(2\vartheta_3-1)} = 0.273651 < 1,$$

$$\sigma_3 = 0.726349.$$

Consider $\omega_1(y) = \Gamma(1.25)y(1-y)$, $\omega_2(y) = \Gamma(1.2)y(1-y)$, $\omega_3(y) = \Gamma(1.1)y(1-y)$ and $r = 0.0001$ to better apply Theorem 3.1. We see that $\omega_i(0) = \omega_i(1) = 0, i = 1,2,3$ and more

$${}_0D_y^{0.75}\omega_1(y) = \frac{1}{5}(5-8y)\sqrt[4]{y}, \quad (4.33)$$

$${}_0D_y^{0.8}\omega_2(y) = \frac{1}{3}(3-5y)\sqrt[5]{y}, \quad (4.34)$$

$${}_0D_y^{0.9}\omega_3(y) = \frac{1}{11}(11-20y)\sqrt[10]{y}, \quad (4.35)$$

then, we are let to

$$\begin{aligned} \|\omega_1(y)\|_{0.75}^2 &\cong 0.158153,\\ \|\omega_2(y)\|_{0.8}^2 &\cong 0.138783,\\ \|\omega_3(y)\|_{0.9}^2 &\cong 0.318772, \end{aligned} \quad (4.36)$$

by some direct calculations, therefore, $\|\omega_1(y)\|_{0.75}^2 + \|\omega_2(y)\|_{0.8}^2 + \|\omega_3(y)\|_{0.9}^2 \cong 0.145889$. On the other hand, by some not difficult integrations, we obtain:

$$H_1(\omega_1(y)) = \frac{1}{144}(y-1)y^2\Gamma\left(\frac{5}{4}\right)^2$$

$$\times\left(y^2(3y-4)\sinh(y)-1\right),$$

$$H_2(\omega_2(y)) = \frac{1}{384}(y-1)y^2\Gamma\left(\frac{6}{5}\right)^2$$

$$\times\left(y^3(3y-4)\sinh(y)-1\right),$$

$$H_3(\omega_3(y)) = \frac{1}{192}(y-1)y^2\Gamma\left(\frac{11}{10}\right)^2$$

$$\times\left(y^4(3y-4)\sinh(y)-1\right),$$

then

$$2\int_0^1 \left(H_1(\omega_1(y)) + H_2(\omega_2(y)) + H_3(\omega_3(y))\right)\mathrm{d}y$$

$$= 0.00299678,$$

thus satisfying the condition (H2) of Theorem 3.1 is clarified. Now, having a look at definition (4.31) and bringing into account $\omega_1^2(y) + \omega_2^2(y) + \omega_3^2(y) < 0.17$ for all $y \in [0, Y]$, we conclude the following inequality

$$\frac{\int_0^1 \sup_{(\varsigma_1,\varsigma_2,\varsigma_3)\in\Upsilon(Mr)} F(y,\varsigma_1,\varsigma_2,\varsigma_3)\mathrm{d}y}{r} = \frac{16}{3}M^2 r \cong 0.0753051 < 0.186711 \cong \frac{2\int_0^1 F(y,\omega_1(y),\omega_2(y),\omega_3(y))\mathrm{d}y}{\left(\begin{array}{c}\|\omega_1(y)\|_{0.75}^2+\|\omega_2(y)\|_{0.8}^2+\|\omega_3(y)\|_{0.9}^2\\ -2\int_0^1\left(H_1(\omega_1)+H_2(\omega_2)+H_3(\omega_3)\right)\mathrm{d}y\end{array}\right)}, \quad (4.37)$$

so, the condition (H3) of Theorem 3.1 holds too. Eventually, obviously

$$\liminf_{|\varsigma_1|\to+\infty,|\varsigma_2|\to+\infty,|\varsigma_3|\to+\infty} \frac{F(y,\varsigma_1,\varsigma_2,\varsigma_3)}{|\varsigma_1|^2+|\varsigma_2|^2+|\varsigma_3|^2} = -1 < 0.0183051 \cong \frac{\int_0^1 \sup_{(\varsigma_1,\varsigma_2,\varsigma_3)\in\Upsilon(Mr)} F(y,\varsigma_1,\varsigma_2,\varsigma_3)\mathrm{d}y}{2Lr}, \quad (4.38)$$

Thus, the condition (H4) of Theorem 3.1 is satisfied. Therefore, based on Theorem 3.1 the nonlinear system (4.31) does have at least three distinct weak solutions in the space $E_0^{0.75}\times E_0^{0.8}\times E_0^{0.9}$ for each $\lambda \in ]5.35586, 13.2793[$.

## References


[1] S. Abbasbandy, E. Shivanian, Application of the variational iteration method for system of nonlinear volterra's integro-differential equations, *Mathematical and computational applications* 14 (2009) 147-158.

[2] S. Abbasbandy, E. Shivanian, Application of variational iteration method for nth-order integro-differential equations, *Zeitschrift für Naturforschung A* 64 (2009) 439-444.

[3] S. Abbasbandy, E. Shivanian, Series solution of the system of integro-differential equations, *Zeitschrift für Naturforschung A* 64 (2009) 811-818.

[4] R. P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Applicandae Mathematicae* 109 (2010) 973-1033.

[5] B. Ahmad, J. J. Nieto, Riemann-liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, *Boundary Value Problems* 1 (2011) 36-46.

[6] M. Aslefallah, E. Shivanian. Nonlinear fractional integro-differential reaction-diffusion equation via radial basis functions, *Eur Phys J Plus* 130 (2015) 1-9.

[7] C. Bai, Infinitely many solutions for a perturbed nonlinear fractional boundary-value problem, *Electronic Journal of Differential Equations* 136 (2013) 1-12.

[8] K. Balachandran, J. Dauer, P. Balasubramaniam, Controllability of semilinear integrodi erential systems in banach spaces, *Journal of Mathematical Systems, Estimation and Control* 6 (1996) 1-10.

[9] K. Balachandran, J. J. Trujillo, The nonlocal cauchy problem for nonlinear fractional integrodifferential equations in banach spaces, *Nonlinear Analysis: Theory, Methods & Applications* 72 (2010) 4587-4593.

[10] D. Baleanu, R. Darzi, B. Agheli, New study of weakly singular kernel fractional fourth-order partial integro-differential equations based on the optimum q-homotopic analysis method, *Journal of Computational and Applied Mathematics* 320 (2017) 193-201.

[11] M. Benchohra, S. Hamani, S. Ntouyas, Boundary value problems for differential equations with fractional order and non-local conditions, *Nonlinear Analysis: Theory, Methods & Applications* 71 (2009) 2391-2396.

[12] G. Bonanno, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Applicable Analysis* 89 (2010) 1-10.

[13] J. Chen, X. Tang, Existence and multiplicity of solutions for some fractional boundary value problem via critical point theory, *In Abstract and Applied Analysis, volume 2012,* Hindawi Publishing Corporation, 2012.

[14] J. Chu, S. Heidarkhani, A. Salari, G. Caristi, Weak solutions and energy estimates for singular p-laplacian-type equations, *Journal of Dynamical and Control Systems* 11 (2017) 1-13.

[15] J. N. Corvellec, V. Motreanu, C. Saccon, Doubly resonant semilinear elliptic problems via nonsmooth critical point theory, *Journal of Differential Equations* 248 (2010) 2064-2091.

[16] A. Firouzjai, G. Afrouzi, S. Talebi, Existence results for kirchhoff type systems with singular nonlinearity, *Opuscula Mathematica* 38 (2018) 187-199.

[17] S. Heidarkhani, G. A. Afrouzi, S. Moradi, G. Caristi, Existence of multiple solutions for a perturbed discrete anisotropic equation, *Journal of Difference Equations and Applications* 22 (2017) 1-17.

[18] S. Heidarkhani, Y. Zhao, G. Caristi, G. A. Afrouzi, S. Moradi, Infinitely many solutions

for perturbed impulsive fractional differential systems, *Applicable Analysis* 96 (2017) 1401-1424.

[19] S. Heidarkhani, Y. Zhou, G. Caristi, G. A. Afrouzi, S. Moradi. Existence results for fractional differential systems through a local minimization principle, *Computers & Mathematics with Applications* 2016.

[20] F. Jiao, Y. Zhou, Existence results for fractional boundary value problem via critical point theory, *International Journal of Bifurcation and Chaos* 22 (2012) 125-138.

[21] M. Kamrani, Convergence of galerkin method for the solution of stochastic fractional integro differential equations, *Optik-International Journal for Light and Electron Optics* 127 (2016) 10049-10057.

[22] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, *Elsevier Science*, Amsterdam, (2006).

[23] F. Li, Z. Liang, Q. Zhang. Existence of solutions to a class of nonlinear second order two-point boundary value problems, *Journal of mathematical analysis and applications* 312 (2005) 357-373.

[24] Y. N. Li, H. R. Sun, Q. G. Zhang, Existence of solutions to fractional boundary-value problems with a parameter, *Electronic Journal of Differential Equations* 141 (2013) 1-12.

[25] X. Ma, C. Huang, Spectral collocation method for linear fractional integro-differential equations, *Applied Mathematical Modelling* 38 (2014) 1434-1448.

[26] J. Mawhin, Critical Point Theory and Hamiltonian Systems, *Springer-Verlag*, Berlin/New York, 1989.

[27] J. J. Nieto, D. O'Regan, Variational approach to impulsive differential equations, *Nonlinear Analysis: Real World Applications* 10 (2009) 680-690.

[28] S. Pashayi, M. Hashemi, S. Shahmorad, Analytical lie group approach for solving fractional integro-differential equations, *Communications in Nonlinear Science and Numerical Simulation* 51 (2017) 66-77.

[29] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *American Mathematical Soc.* 65 (1986) 129-142.

[30] P. Rahimkhani, Y. Ordokhani, E. Babolian, Fractional-order bernoulli functions and their applications in solving fractional fredholemvolterra integro-differential equations, *Applied Numerical Mathematics* 122 (2017) 66-81.

[31] E. Rawashdeh, Numerical solution of fractional integro-differential equations by collocation method, *Applied Mathematics and Computation* 176 (2006) 1-6.

[32] E. Shivanian, Analysis of meshless local radial point interpolation (mlrpi) on a nonlinear partial integro-differential equation arising in population dynamics, *Engineering Analysis with Boundary Elements* 37 (2013) 1693-1702.

[33] H. R. Sun, Q. G. Zhang, Existence of solutions for a fractional boundary value problem via the mountain pass method and an iterative technique, *Computers & Mathematics with Applications* 64 (2012) 3436-3443.

[34] C. L. Tang, X. P. Wu, Some critical point theorems and their applications to periodic solution for second order hamiltonian systems, *Journal of Differential Equations* 248 (2010) 660-692.

[35] W. Xie, J. Xiao, Z. Luo, Existence of solutions for fractional boundary value problem with nonlinear derivative dependence, *In Abstract and Applied Analysis*, volume 2014, Hindawi Publishing Corporation, (2014).

[36] L. Zhang, B. Ahmad, G. Wang, R. P. Agarwal, M. Al-Yami, W. Shammakh, Nonlo-

cal integrodifferential boundary value problem for nonlinear fractional differential equations on an unbounded domain, *In Abstract and Applied Analysis*, volume 2013, Hindawi Publishing Corporation, (2013).

[37] S. Zhang, Positive solutions to singular boundary value problem for nonlinear fractional differential equation, *Computers & Mathematics with Applications* 59 (2010) 1300-1309.

[38] Y. Zhao, H. Chen, B. Qin, Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods, *Applied Mathematics and Computation* 257 (2015) 417-427.

Dr. Elyas Shivanian was born in Zanjan province, Iran in August 26, 1982. He started his master course in applied mathematics in 2005 at Amirkabir University of Technology and has finished MSc. thesis in the field of fuzzy linear programming in 2007. After his Ph.D. in the field of prediction of multiplicity of solutions to the boundary value problems, at Imam Khomeini International University in 2012, he became Assistant Professor at the same university. His research interests are analytical and numerical solutions of ODEs, PDEs and IEs. He has published several papers on these subjects. He also has published some papers in other fields, for more information see please http://scholar.google.com/citations?user=MFncks8AAAAJ&hl=en