



# A New Approach for Solving Fully Fuzzy Bilevel Linear Programming Problems

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## Abstract

This paper addresses a type of fully fuzzy bilevel linear programming (FFBLP) wherein all the coefficients and decision variables in both the objective functions and constraints are triangular fuzzy numbers. In order to obtain the fuzzy optimal solution, a new efficient method for FFBLP with unconstrained variables and parameters has been proposed. This proposed approach is based on crisp bilevel programming. Some examples have been provided to illustrate these methods.

*Keywords* : Bilevel linear programming; Triangular fuzzy numbers; Ranking function; Optimal solution; Unconstrained variables.

## 1 Introduction

The bilevel programming is a hierarchical optimization problem involving two levels wherein the upper level constraint region is implicitly specified by the lower level problem. This kind of problem is nonconvex and very hard to solve due to its structure. Over the past few decades, the bilevel programming problem has possessed a lot of consideration and has been widely applied to a variety of fields such as electricity markets [22], energy networks [8], location-allocation problem [19]. The recent surveys on this topic were given by Dempe [5],

price setting problem [11], principal-agent problems [3, 20], and so forth. Detailed study on bilevel programming problem can be found in the most recent books (Dempe [4]; Dempe et al. [6]). Considering its structure, this type of problem is nonconvex and very difficult to solve. Considering its structure, this type of problem is nonconvex and very difficult to solve. In common bilevel programming problems, it is assumed that the parameters are precisely defined, but under real conditions, the decision making coefficients and variables may not be crisp. The fuzzy sets theory is a powerful tool for dealing with inaccurate or vague information and in recent years the fuzzy bilevel programming problem wherein the coefficients are fuzzy numbers in both the objective function and constraints has been studied by some authors. Sakawa et al. [18] formulated the fuzzy bilevel linear programming problem and developed the fuzzy programming method for its solution. Zhang et al. [23] used the fuzzy Kuhn-

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Tucker, fuzzy Kth-best, and fuzzy branch and bound methods to solve fuzzy bilevel linear programming problems; however, the decision variables are not fuzzy in any of the mentioned works. Ruziyeva [16] proposed a membership function approach for solving the linear fuzzy bilevel optimization problem. Allahviranloo et al. [1] proposed a new method for solving fully fuzzy linear programming problems by the use of ranking function. Hosseinzadeh Lotfi et al. [7] transformed the fully fuzzy linear programming problem into two corresponding linear programming problems based on the concept of the symmetric triangular fuzzy number and developed a lexicography method to solve such a problem. Kumar et al. [10] proposed a new method to find the optimal fuzzy solution of the fully fuzzy linear programming problem with equal constraint. This paper aims at developing a new method for the solution of the fully fuzzy linear programming problems wherein all the coefficients and decision variables are expressed as triangular fuzzy numbers. Safaei and Saraj [17] decomposed the FFBLP problem into three deterministic linear programming problems and then obtained its fuzzy optimal solution. Ren [13] investigated the feasibility of fuzzy solutions of the generalized fuzzy bilevel linear programming problem and developed a stepwise interactive algorithm to solve the problem. Ren [14] a new approach based on deviation degree measures and a ranking function method is proposed. Ren et al. [15] converted the FFBLP problem to crisp form under different constraint feasibility degrees based on a fuzzy relation for the ranking of the fuzzy numbers. This paper has been so organized as to present the definitions and operators between triangular fuzzy numbers in Section 2, express the new approaches in the FFBLP using the ranking function in Section 3, provide some numerical examples to illustrate how methods are implemented in Section 4, and, finally, present the conclusions in Section 5.

## 2 Prerequisites

This section presents the definitions needed to obtain the results [10, 12, 9, 2].

**Definition 2.1** If  $X$  is a universal set, the fuzzy subset  $\tilde{A}$  of  $X$  can be expressed by its membership

function as follows:

$$\mu_{\tilde{A}} : X \rightarrow [0, 1]$$

where gives a real number  $\mu_{\tilde{A}}(x)$  to every element  $x \in X$  in the interval  $[0, 1]$  and the value of  $\mu_{\tilde{A}}(x)$  at  $x$  shows the membership grade of  $x$  in  $\tilde{A}$ . A fuzzy subset  $\tilde{A}$  can be characterized as a set of ordered pairs of element  $x$  and grade  $\mu_{\tilde{A}}(x)$  shown often as follows:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\}.$$

**Definition 2.2** A fuzzy number  $\tilde{A} = (a, b, c)$  will be triangular if its membership function is as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \leq x \leq b \\ \frac{(c-x)}{(c-b)}, & b \leq x \leq c \\ 0 & O.W. \end{cases}$$

**Definition 2.3** A triangular fuzzy number  $\tilde{A} = (a, b, c)$  will be non-negative only if  $a \geq 0$ ; the related non-negative set can be shown by  $F(\mathbb{R}^+)$ .

**Definition 2.4** A triangular fuzzy number  $\tilde{A} = (a, b, c)$  will be unrestricted if  $a, b, c \in \mathbb{R}$ . The set of which is shown by  $F(\mathbb{R})$ .

**Definition 2.5** if  $\tilde{A} = (a, b, c)$  and  $\tilde{B} = (d, e, f)$  are two triangular fuzzy numbers, then:

- (1)  $\tilde{A} \oplus \tilde{B} = (a, b, c) \oplus (d, e, f) = (a + d, b + e, c + f)$ ,
- (2)  $\ominus \tilde{A} = \ominus(a, b, c) = (-c, -b, -a)$ ,
- (3)  $\tilde{A} \ominus \tilde{B} = (a, b, c) \ominus (d, e, f) = (a - f, b - e, c - d)$ ,
- (4)

$$\lambda \tilde{A} = \begin{cases} (\lambda a, \lambda b, \lambda c), & \lambda \geq 0 \\ (\lambda c, \lambda b, \lambda a), & \lambda < 0. \end{cases}$$

- (5)  $\tilde{A} \otimes \tilde{B} = (\min(\gamma), be, \max(\gamma))$  where,  $\gamma = \{ad, af, cd, cf\}$
- (6)  $\tilde{B} = (d, e, f)$  be a non-negative triangular fuzzy number then:

$$\tilde{A} \otimes \tilde{B} = \begin{cases} (ad, be, cf), & a \geq 0 \\ (af, be, cf), & a < 0, c \geq 0 \\ (af, be, cd), & c < 0. \end{cases}$$

**Definition 2.6** Two triangular fuzzy numbers  $\tilde{A} = (a, b, c)$  and  $\tilde{B} = (d, e, f)$  are equal only if  $a = d, b = e$  and  $c = f$ .

**Definition 2.7** [10]. A ranking function is in the form  $F(R) \rightarrow R$ ; it maps every fuzzy number on the real-line where there is natural order. If  $\tilde{A} = (a, b, c)$  is a triangular fuzzy number, then  $R(\tilde{A}) = \frac{1}{4}(a + 2b + c)$ .

**Definition 2.8** If  $\tilde{A} = (a, b, c)$  is non-negative triangular fuzzy number and  $\tilde{B} = (e, f, g)$  is an unrestricted triangular fuzzy number, then:  
 $\tilde{A} \otimes \tilde{B} = (\min(ae, ce), bf, \max(ag, cg))$   
 where  $\min(ae, ce) = \frac{ae+ce}{2} - |\frac{ae-ce}{2}|$  and  $\max(ag, cg) = \frac{ag+cg}{2} + |\frac{ag-cg}{2}|$ . since  $(a, b, c)$  is a triangular fuzzy number (i.e.  $a \leq b \leq c$ ), so  $\min(ae, ce)$  and  $\max(ag, cg)$  can be written as:  
 $\min(ae, ce) = (\frac{a+c}{2})e - (\frac{c-a}{2})|e|$  and  $\max(ag, cg) = (\frac{a+c}{2})g + (\frac{c-a}{2})|g|$ .

**Definition 2.9** If  $\tilde{A} = (a, b, c)$  and  $\tilde{B} = (x, y, z)$  are two triangular fuzzy numbers, then :

$$\tilde{A} \otimes \tilde{B} = \begin{cases} (\min(ax, cx), by, \max(az, cz)), \\ a \geq 0 \\ (\min(cx, az), by, \max(cz, ax)), \\ a < 0, c \geq 0 \\ (\min(az, cz), by, \max(ax, cx)), \\ c < 0. \end{cases}$$

or

$$\tilde{A} \otimes \tilde{B} = \begin{cases} (\{0.5(a+c)x - 0.5(c-a)|x|\} \\ , by, \{0.5(a+c)z \\ , +0.5(c-a)|z|\}), a \geq 0 \\ (\{0.5(az+cx) - 0.5|cx-az|\} \\ , by, \{0.5(ax+cz) \\ , +0.5|cz-ax|\}), a < 0, c \geq 0 \\ (\{0.5(a+c)z - 0.5(c-a)|z|\} \\ , by, \{0.5(a+c)x \\ , +0.5(c-a)|x|\}), c < 0. \end{cases}$$

### 3 Proposed algorithms

Next, some new algorithms are proposed to find the fuzzy optimal solution for the FFBLP problem in three cases, unconstrained fuzzy coefficients corresponding to the variables and non-negative fuzzy variables, non-negative fuzzy coefficients corresponding to the variables and

unconstrained fuzzy variables and finally, unconstrained fuzzy coefficients corresponding to the variables and unconstrained fuzzy variables. Consider the following FFBLP problem and suppose the decision maker at the  $i^{th}$  level each of which controls its own decision vector,  $\tilde{x}_i$ :  
 $maximize_{\tilde{x}_1} \tilde{F}(\tilde{x}_1, \tilde{x}_2) = \tilde{c}_{11} \otimes \tilde{x}_1 \oplus \tilde{c}_{12} \otimes \tilde{x}_2$   
 s.t. (3.1)  
 $maximize_{\tilde{x}_2} \tilde{f}(\tilde{x}_1, \tilde{x}_2) = \tilde{c}_{21} \otimes \tilde{x}_1 \oplus \tilde{c}_{22} \otimes \tilde{x}_2$   
 s.t.  $\tilde{A} \otimes \tilde{x}_1 \oplus \tilde{B} \otimes \tilde{x}_2 = \tilde{b}$

where  $\tilde{x}_1, \tilde{x}_2$  are triangular fuzzy numbers,  $\tilde{A} = (\tilde{a}_{ij})_{m \times n_1}, \tilde{B} = (\tilde{b}_{ij})_{m \times n_2}, \tilde{b} = (\tilde{b}_i)_{m \times 1}$  and  $\tilde{c}_{11} = (\tilde{c}_{11j})_{1 \times n_1}, \tilde{c}_{12} = (\tilde{c}_{12j})_{1 \times n_2}, \tilde{c}_{21} = (\tilde{c}_{21j})_{1 \times n_1}, \tilde{c}_{22} = (\tilde{c}_{22j})_{1 \times n_2}, n_1 + n_2 = n, \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{b}_i \in F(R), \tilde{x}_1, \tilde{x}_2 \in F(R^+)$  for all  $1 \leq i \leq m, 1 \leq j \leq n$ .

Let the parameters  $\tilde{c}_{11}, \tilde{c}_{12}, \tilde{c}_{21}, \tilde{c}_{22}, \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{x}_1, \tilde{x}_2$  and  $\tilde{b}_i$  be the triangular fuzzy number  $(p_{1j}, q_{1j}, r_{1j}), (p_{2j}, q_{2j}, r_{2j}), (p_{3j}, q_{3j}, r_{3j}), (p_{4j}, q_{4j}, r_{4j}), (a_{ij}^1, a_{ij}^2, a_{ij}^3), (b_{ij}^1, b_{ij}^2, b_{ij}^3), (x_{1j}, y_{1j}, t_{1j}), (x_{2j}, y_{2j}, t_{2j})$  and  $(m_i, n_i, p_i)$  respectively. So the problem (3.1) can, then, be written as follows:

$$maximize_{\tilde{x}_1, \tilde{x}_2} \tilde{F}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (p_{1j}, q_{1j}, r_{1j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{j=1}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j})$$

s.t. (3.2)

$$maximize_{\tilde{x}_1, \tilde{x}_2} \tilde{f}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \oplus \sum_{j=1}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j})$$

s.t.  $\sum_{j=1}^{n_1} (a_{ij}^1, a_{ij}^2, a_{ij}^3) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{j=1}^{n_2} (b_{ij}^1, b_{ij}^2, b_{ij}^3) \otimes (x_{2j}, y_{2j}, t_{2j}) = (m_i, n_i, p_i), \forall i = 1, 2, \dots, m$ .

where  $(x_{1j}, y_{1j}, t_{1j}), (x_{2j}, y_{2j}, t_{2j})$  are triangular fuzzy numbers. Now, using Definition 2.5, we will rewrite this problem (3.2) as follows:

$$maximize_{\tilde{x}_1, \tilde{x}_2} \tilde{F}(\tilde{x}_1, \tilde{x}_2) = \sum_{\substack{j=1 \\ p_{1j} \geq 0}}^{n_1} (p_{1j}, q_{1j}, r_{1j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{\substack{j=1 \\ p_{1j} < 0 \\ r_{1j} \geq 0}}^{n_1} (p_{1j}, q_{1j}, r_{1j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{\substack{j=1 \\ p_{1j} < 0}}^{n_1} (p_{1j}, q_{1j}, r_{1j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{\substack{j=1 \\ p_{2j} \geq 0}}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j})$$

$$\begin{aligned}
 & \oplus \sum_{\substack{j=1 \\ p_{2j} < 0 \\ r_{2j} \geq 0}}^{n_2} (p_2, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \oplus \\
 & \sum_{\substack{j=1 \\ r_{2j} < 0}}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
 & \text{s.t.} \\
 & \text{maximize } \tilde{f}(\tilde{x}_1, \tilde{x}_2) = \sum_{\substack{j=1 \\ p_{3j} \geq 0}}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes \\
 & (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{\substack{j=1 \\ p_{3j} < 0 \\ r_{3j} \geq 0}}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \\
 & \oplus \sum_{\substack{j=1 \\ r_{3j} < 0}}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \\
 & \sum_{\substack{j=1 \\ p_{4j} \geq 0}}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
 & \oplus \sum_{\substack{j=1 \\ p_{4j} < 0 \\ r_{4j} \geq 0}}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \oplus \\
 & \sum_{\substack{j=1 \\ r_{4j} < 0}}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \text{ s.t.} \\
 & \sum_{\substack{j=1 \\ a_{1j}^1 \geq 0}}^{n_1} (a_{1j}^1, a_{1j}^2, a_{1j}^3) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \\
 & \sum_{\substack{j=1 \\ a_{1j}^1 < 0 \\ a_{1j}^3 \geq 0}}^{n_1} (a_{1j}^1, a_{1j}^2, a_{1j}^3) \otimes (x_{1j}, y_{1j}, t_{1j}) \\
 & \oplus \sum_{\substack{j=1 \\ a_{1j}^3 < 0}}^{n_1} (a_{1j}^1, a_{1j}^2, a_{1j}^3) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \\
 & \sum_{\substack{j=1 \\ b_{1j}^1 \geq 0}}^{n_2} (b_{1j}^1, b_{1j}^2, b_{1j}^3) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
 & \oplus \sum_{\substack{j=1 \\ b_{1j}^1 < 0 \\ b_{1j}^3 \geq 0}}^{n_2} (b_{1j}^1, b_{1j}^2, b_{1j}^3) \otimes (x_{2j}, y_{2j}, t_{2j}) \oplus \\
 & \sum_{\substack{j=1 \\ b_{1j}^1 < 0}}^{n_2} (b_{1j}^1, b_{1j}^2, b_{1j}^3) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
 & = (m_i, n_i, p_i) \forall, i = 1, 2, \dots, m,
 \end{aligned}$$

where  $(x_{1j}, y_{1j}, t_{1j}), (x_{2j}, y_{2j}, t_{2j})$  are triangular fuzzy numbers.

First, we describe an algorithm for solving a FFBLP problem with unrestricted fuzzy coefficients corresponding to the variables and non-negative fuzzy variables.

### 3.1 Algorithm 1

**Step 1.** Using Definition 2.5, the FFBLP problem (3.3) can be written as follows:

$$\begin{aligned}
 & \text{maximize } \tilde{F}(\tilde{x}_1, \tilde{x}_2) \\
 & \sum_{\substack{j=1 \\ p_{1j} \geq 0}}^{n_1} (p_{1j}x_{1j}, q_{1j}y_{1j}, r_{1j}t_{1j}) \oplus \\
 & \sum_{\substack{j=1 \\ p_{1j} < 0 \\ r_{1j} \geq 0}}^{n_1} (p_{1j}t_{1j}, q_{1j}y_{1j}, r_{1j}t_{1j})
 \end{aligned}$$

$$\begin{aligned}
 & \oplus \sum_{\substack{j=1 \\ r_{1j} < 0}}^{n_1} (p_{1j}t_{1j}, q_{1j}y_{1j}, r_{1j}x_{1j}) \oplus \\
 & \sum_{\substack{j=1 \\ p_{2j} \geq 0}}^{n_2} (p_{2j}x_{2j}, q_{2j}y_{2j}, r_{2j}t_{2j}) \\
 & \oplus \sum_{\substack{j=1 \\ p_{2j} < 0 \\ r_{2j} \geq 0}}^{n_2} (p_{2j}t_{2j}, q_{2j}y_{2j}, r_{2j}t_{2j}) \oplus \\
 & \sum_{\substack{j=1 \\ r_{2j} < 0}}^{n_2} (p_{2j}t_{2j}, q_{2j}y_{2j}, r_{2j}x_{2j}) \\
 & \text{s.t.} \\
 & \text{maximize } \tilde{f}(\tilde{x}_1, \tilde{x}_2) \\
 & \sum_{\substack{j=1 \\ p_{3j} \geq 0}}^{n_1} (p_{3j}x_{1j}, q_{3j}y_{1j}, r_{3j}t_{1j}) \oplus \\
 & \sum_{\substack{j=1 \\ p_{3j} < 0 \\ r_{3j} \geq 0}}^{n_1} (p_{3j}t_{1j}, q_{3j}y_{1j}, r_{3j}t_{1j}) \\
 & \oplus \sum_{\substack{j=1 \\ r_{3j} < 0}}^{n_1} (p_{3j}t_{1j}, q_{3j}y_{1j}, r_{3j}x_{1j}) \oplus \\
 & \sum_{\substack{j=1 \\ p_{4j} \geq 0}}^{n_2} (p_{4j}x_{2j}, q_{4j}y_{2j}, r_{4j}t_{2j}) \\
 & \oplus \sum_{\substack{j=1 \\ p_{4j} < 0 \\ r_{4j} \geq 0}}^{n_2} (p_{4j}t_{2j}, q_{4j}y_{2j}, r_{4j}t_{2j}) \oplus \\
 & \sum_{\substack{j=1 \\ r_{4j} < 0}}^{n_2} (p_{4j}t_{2j}, q_{4j}y_{2j}, r_{4j}x_{2j}) \\
 & \text{s.t.} \quad \sum_{\substack{j=1 \\ a_{ij}^1 \geq 0}}^{n_1} (a_{ij}^1x_{1j}, a_{ij}^2y_{1j}, a_{ij}^3t_{1j}) \oplus \\
 & \sum_{\substack{j=1 \\ a_{ij}^1 < 0 \\ a_{ij}^3 \geq 0}}^{n_1} (a_{ij}^1t_{1j}, a_{ij}^2y_{1j}, a_{ij}^3t_{1j}) \\
 & \oplus \sum_{\substack{j=1 \\ a_{ij}^3 < 0}}^{n_1} (a_{ij}^1t_{1j}, a_{ij}^2y_{1j}, a_{ij}^3x_{1j}) \oplus \\
 & \sum_{\substack{j=1 \\ b_{ij}^1 \geq 0}}^{n_2} (b_{ij}^1x_{2j}, b_{ij}^2y_{2j}, b_{ij}^3t_{2j}) \\
 & \oplus \sum_{\substack{j=1 \\ b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0}}^{n_2} (b_{ij}^1t_{2j}, b_{ij}^2y_{2j}, b_{ij}^3t_{2j}) \oplus \\
 & \sum_{\substack{j=1 \\ b_{ij}^1 < 0}}^{n_2} (b_{ij}^1t_{2j}, b_{ij}^2y_{2j}, b_{ij}^3x_{2j}) \\
 & = (m_i, n_i, p_i), \forall i = 1, 2, \dots, m.
 \end{aligned}$$

where  $(x_{1j}, y_{1j}, t_{1j}), (x_{2j}, y_{2j}, t_{2j})$  are non-negative triangular fuzzy numbers.

**Step 2.** Using Definition 2.7, the fuzzy objective function at each level of the FFBLP problem found in Step 1, can be transformed into the crisp objective function at each level and the FFBLP problem can be written as follows:

$$\begin{aligned}
 & \text{maximize } F \\
 & x_{1j}, y_{1j}, t_{1j}
 \end{aligned}$$

$$\begin{aligned}
 & R[\sum_{\substack{j=1 \\ p_{1j} \geq 0}}^{n_1} (p_{1j}x_{1j}, q_{1j}y_{1j}, r_{1j}t_{1j})] \quad \oplus \quad \text{maximize } F = \frac{1}{4}[\sum_{\substack{j=1 \\ p_{1j} \geq 0}}^{n_1} (p_{1j}x_{1j}, q_{1j}y_{1j}, r_{1j}t_{1j})] \quad \oplus \\
 & R[\sum_{\substack{j=1 \\ p_{1j} < 0 \\ r_{1j} \geq 0}}^{n_1} (p_{1j}t_{1j}, q_{1j}y_{1j}, r_{1j}t_{1j})] \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ p_{1j} < 0 \\ r_{1j} \geq 0}}^{n_1} (p_{1j}t_{1j}, q_{1j}y_{1j}, r_{1j}t_{1j})] \\
 & \oplus \quad R[\sum_{\substack{j=1 \\ r_{1j} < 0}}^{n_1} (p_{1j}t_{1j}, q_{1j}y_{1j}, r_{1j}x_{1j})] \quad \oplus \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ r_{1j} < 0}}^{n_1} (p_{1j}t_{1j}, q_{1j}y_{1j}, r_{1j}x_{1j})] \quad \oplus \\
 & R[\sum_{\substack{j=1 \\ p_{2j} \geq 0}}^{n_2} (p_{2j}x_{2j}, q_{2j}y_{2j}, r_{2j}t_{2j})] \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ p_{2j} \geq 0}}^{n_2} (p_{2j}x_{2j}, q_{2j}y_{2j}, r_{2j}t_{2j})] \\
 & \oplus R[\sum_{\substack{j=1 \\ p_{2j} < 0 \\ r_{2j} \geq 0}}^{n_2} (p_{2j}t_{2j}, q_{2j}y_{2j}, r_{2j}t_{2j})] \quad \oplus \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ p_{2j} < 0 \\ r_{2j} \geq 0}}^{n_2} (p_{2j}t_{2j}, q_{2j}y_{2j}, r_{2j}t_{2j})] \quad \oplus \\
 & R[\sum_{\substack{j=1 \\ r_{2j} < 0}}^{n_2} (p_{2j}t_{2j}, q_{2j}y_{2j}, r_{2j}x_{2j})] \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ r_{2j} < 0}}^{n_2} (p_{2j}t_{2j}, q_{2j}y_{2j}, r_{2j}x_{2j})] \\
 & \text{s.t.} \quad \text{maximize } f = R[\sum_{\substack{j=1 \\ p_{3j} \geq 0}}^{n_1} (p_{3j}x_{1j}, q_{3j}y_{1j}, r_{3j}t_{1j})] \quad \oplus \quad \text{maximize } f = \frac{1}{4}[\sum_{\substack{j=1 \\ p_{3j} \geq 0}}^{n_1} (p_{3j}x_{1j}, q_{3j}y_{1j}, r_{3j}t_{1j})] \quad \oplus \\
 & R[\sum_{\substack{j=1 \\ p_{3j} < 0 \\ r_{3j} \geq 0}}^{n_1} (p_{3j}t_{1j}, q_{3j}y_{1j}, r_{3j}t_{1j})] \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ p_{3j} < 0 \\ r_{3j} \geq 0}}^{n_1} (p_{3j}t_{1j}, q_{3j}y_{1j}, r_{3j}t_{1j})] \\
 & \oplus R[\sum_{\substack{j=1 \\ r_{3j} < 0}}^{n_1} (p_{3j}t_{1j}, q_{3j}y_{1j}, r_{3j}x_{1j})] \quad \oplus \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ r_{3j} < 0}}^{n_1} (p_{3j}t_{1j}, q_{3j}y_{1j}, r_{3j}x_{1j})] \quad \oplus \\
 & R[\sum_{\substack{j=1 \\ p_{4j} \geq 0}}^{n_2} (p_{4j}x_{2j}, q_{4j}y_{2j}, r_{4j}t_{2j})] \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ p_{4j} \geq 0}}^{n_2} (p_{4j}x_{2j}, q_{4j}y_{2j}, r_{4j}t_{2j})] \\
 & \oplus R[\sum_{\substack{j=1 \\ p_{4j} < 0 \\ r_{4j} \geq 0}}^{n_2} (p_{4j}t_{2j}, q_{4j}y_{2j}, r_{4j}t_{2j})] \quad \oplus \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ p_{4j} < 0 \\ r_{4j} \geq 0}}^{n_2} (p_{4j}t_{2j}, q_{4j}y_{2j}, r_{4j}t_{2j})] \quad \oplus \\
 & R[\sum_{\substack{j=1 \\ r_{4j} < 0}}^{n_2} (p_{4j}t_{2j}, q_{4j}y_{2j}, r_{4j}x_{2j})] \quad \oplus \quad \frac{1}{4}[\sum_{\substack{j=1 \\ r_{4j} < 0}}^{n_2} (p_{4j}t_{2j}, q_{4j}y_{2j}, r_{4j}x_{2j})] \\
 & \text{s.t.} \quad \sum_{\substack{j=1 \\ a_{ij}^1 \geq 0}}^{n_1} (a_{ij}^1x_{1j}, a_{ij}^2y_{1j}, a_{ij}^3t_{1j}) \quad \oplus \quad \text{s.t.} \quad \sum_{\substack{j=1 \\ a_{ij}^1 \geq 0}}^{n_1} (a_{ij}^1x_{1j}) + \sum_{\substack{j=1 \\ a_{ij}^1 < 0 \\ a_{ij}^3 \geq 0}}^{n_1} (a_{ij}^1t_{1j}) + \\
 & \sum_{\substack{j=1 \\ a_{ij}^1 < 0 \\ a_{ij}^3 \geq 0}}^{n_1} (a_{ij}^1t_{1j}, a_{ij}^2y_{1j}, a_{ij}^3t_{1j}) \quad \oplus \quad \sum_{\substack{j=1 \\ a_{ij}^3 < 0}}^{n_1} (a_{ij}^1t_{1j}) + \sum_{\substack{j=1 \\ b_{ij}^1 \geq 0}}^{n_2} (b_{ij}^1x_{2j}) \\
 & \oplus \sum_{\substack{j=1 \\ a_{ij}^3 < 0}}^{n_1} (a_{ij}^1t_{1j}, a_{ij}^2y_{1j}, a_{ij}^3x_{1j}) \quad \oplus \quad + \sum_{\substack{j=1 \\ b_{ij}^1 < 0}}^{n_2} (b_{ij}^1t_{2j}) + \sum_{\substack{j=1 \\ b_{ij}^1 < 0}}^{n_2} (b_{ij}^1t_{2j}) = m_i, \forall i = \\
 & \sum_{\substack{j=1 \\ b_{ij}^1 \geq 0}}^{n_2} (b_{ij}^1x_{2j}, b_{ij}^2y_{2j}, b_{ij}^3t_{2j}) \quad \oplus \quad \sum_{\substack{j=1 \\ b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0}}^{n_2} (b_{ij}^1t_{2j}) + \sum_{\substack{j=1 \\ a_{ij}^1 \geq 0}}^{n_1} (a_{ij}^2y_{1j}) + \sum_{\substack{j=1 \\ a_{ij}^1 < 0 \\ a_{ij}^3 \geq 0}}^{n_1} (a_{ij}^2y_{1j}) + \sum_{\substack{j=1 \\ a_{ij}^3 < 0}}^{n_1} (a_{ij}^2y_{1j}) + \\
 & \oplus \sum_{\substack{j=1 \\ b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0}}^{n_2} (b_{ij}^1t_{2j}, b_{ij}^2y_{2j}, b_{ij}^3t_{2j}) \quad \oplus \quad \sum_{\substack{j=1 \\ b_{ij}^1 \geq 0}}^{n_2} (b_{ij}^2y_{2j}) \\
 & \sum_{\substack{j=1 \\ b_{ij}^1 < 0}}^{n_2} (b_{ij}^1t_{2j}, b_{ij}^2y_{2j}, b_{ij}^3x_{2j}) \quad \oplus \quad + \sum_{\substack{j=1 \\ b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0}}^{n_2} (b_{ij}^2y_{2j}) + \sum_{\substack{j=1 \\ b_{ij}^1 < 0}}^{n_2} (b_{ij}^2y_{2j}) = n_i, \forall i = \\
 & = (m_i, n_i, p_i) \forall i = 1, 2, \dots, m, \quad \text{1, 2, } \dots, m. \\
 & \quad \text{where } (x_{1j}, y_{1j}, t_{1j}), (x_{2j}, y_{2j}, t_{2j}) \text{ are non-negative triangular fuzzy numbers.} \\
 & \quad \text{Step 3. Using Definitions 2.6, 2.7, the FFBLP problem, obtained in Step 2, is transformed into the following crisp bilevel linear programming problem:} \\
 & \sum_{\substack{j=1 \\ a_{ij}^1 \geq 0}}^{n_1} (a_{ij}^3t_{1j}) + \sum_{\substack{j=1 \\ a_{ij}^1 < 0 \\ a_{ij}^3 \geq 0}}^{n_1} (a_{ij}^3t_{1j}) + \sum_{\substack{j=1 \\ a_{ij}^3 < 0}}^{n_1} (a_{ij}^3x_{1j}) + \\
 & \sum_{\substack{j=1 \\ b_{ij}^1 \geq 0}}^{n_2} (b_{ij}^3t_{2j})
 \end{aligned}$$

$$+ \sum_{\substack{j=1 \\ b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0}}^{n_2} (b_{ij}^3 t_{2j}) + \sum_{\substack{j=1 \\ b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0}}^{n_2} (b_{ij}^3 x_{2j}) = p_i, \forall i =$$

$$1, 2, \dots, m. y_{1j} - x_{1j} \geq 0, t_{1j} - y_{1j} \geq 0, y_{2j} - x_{2j} \geq 0, t_{2j} - y_{2j} \geq 0, x_{1j} \geq 0, x_{2j} \geq 0, \forall j = 1, \dots, n.$$

**Step 4.** Find the optimal solution  $x_{1j}, y_{1j}, t_{1j}, x_{2j}, y_{2j}, t_{2j}$ , by solving the bilevel linear programming problem obtained in Step 3.

**Step 5.** Find the fuzzy optimal solution by putting the values of  $x_{1j}, y_{1j}, t_{1j}, x_{2j}, y_{2j}, t_{2j}$  in  $\tilde{x}_1 = (x_{1j}, y_{1j}, t_{1j}), \tilde{x}_2 = (x_{2j}, y_{2j}, t_{2j})$ .

**Step 6.** Find the fuzzy optimal value by putting  $\tilde{x}_1, \tilde{x}_2$  in:  $\tilde{F}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (p_{1j}, q_{1j}, r_{1j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{j=1}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j})$   
 $\tilde{f}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{j=1}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j})$ .

Now, we will present an algorithm for the FFBLP problem with non-negative fuzzy coefficients corresponding to the variables and unrestricted fuzzy variables .

### 3.2 Algorithm 2

**Step 1.** Using Definition 2.8, the FFBLP problem (3) can be written as follows:

$$\begin{aligned} & \underset{(x_{1j}, y_{1j}, t_{1j})}{\text{maximize}} \tilde{F}(\tilde{x}_1, \tilde{x}_2) \\ & \sum_{j=1}^{n_1} (\min(p_{1j}x_{1j}, r_{1j}t_{1j}), q_{1j}y_{1j}, \\ & \text{max}(p_{1j}t_{1j}, r_{1j}t_{1j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(p_{2j}x_{2j}, r_{2j}t_{2j}), q_{2j}y_{2j}, \\ & \text{max}(p_{2j}t_{2j}, r_{2j}t_{2j})) \\ & \text{s.t.} \\ & \underset{(x_{2j}, y_{2j}, t_{2j})}{\text{maximize}} \tilde{f}(\tilde{x}_1, \tilde{x}_2) \\ & \sum_{j=1}^{n_1} (\min(p_{3j}x_{1j}, r_{3j}t_{1j}), q_{3j}y_{1j}, \\ & \text{max}(p_{3j}t_{1j}, r_{3j}t_{1j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(p_{4j}x_{2j}, r_{4j}t_{2j}), q_{4j}y_{2j}, \\ & \text{max}(p_{4j}t_{2j}, r_{4j}t_{2j})) \\ & \text{s.t.} \sum_{j=1}^{n_1} (\min(a_{ij}^1 x_{1j}, a_{ij}^3 x_{1j}), a_{ij}^2 y_{1j}, \\ & \text{max}(a_{ij}^1 t_{1j}, a_{ij}^3 t_{1j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(b_{ij}^1 x_{2j}, b_{ij}^3 x_{2j}), b_{ij}^2 y_{1j}, \\ & \text{max}(b_{ij}^1 t_{2j}, b_{ij}^3 t_{2j})) \\ & = (m_i, n_i, p_i) \forall i = 1, 2, \dots, m. \\ & (x_{1j}, y_{1j}, t_{1j}), (x_{2j}, y_{2j}, t_{2j}) \in F(R), j = 1, \dots, n. \end{aligned}$$

**Step 2.** Using Definition 2.8, the FFBLP

problem obtained in Step 1 can be written as follows:

$$\begin{aligned} & \underset{(x_{1j}, y_{1j}, t_{1j})}{\text{maximize}} \tilde{F}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (\frac{1}{2}(p_{1j} + r_{1j})x_{1j} - \\ & \frac{1}{2}(r_{1j} - p_{1j})|x_{1j}|, q_{1j}y_{1j}, \frac{1}{2}(p_{1j} + r_{1j})t_{1j} \\ & + \frac{1}{2}(r_{1j} - p_{1j})|t_{1j}|) \oplus \sum_{j=1}^{n_2} (\frac{1}{2}(p_{2j} + r_{2j})x_{2j} - \\ & \frac{1}{2}(r_{2j} - p_{2j})|x_{2j}|, q_{2j}y_{2j}, \\ & \frac{1}{2}(p_{2j} + r_{2j})t_{2j} + \frac{1}{2}(r_{2j} - p_{2j})|t_{2j}|) \end{aligned}$$

s.t.

$$\begin{aligned} & \underset{(x_{2j}, y_{2j}, t_{2j})}{\text{maximize}} \tilde{f}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (\frac{1}{2}(p_{3j} + r_{3j})x_{1j} - \\ & \frac{1}{2}(r_{3j} - p_{3j})|x_{1j}|, q_{3j}y_{1j}, \frac{1}{2}(p_{3j} + r_{3j})t_{1j} \\ & + \frac{1}{2}(r_{3j} - p_{3j})|t_{1j}|) \oplus \sum_{j=1}^{n_2} (\frac{1}{2}(p_{4j} + r_{4j})x_{2j} - \\ & \frac{1}{2}(r_{4j} - p_{4j})|x_{2j}|, q_{4j}y_{2j}, \\ & \frac{1}{2}(p_{4j} + r_{4j})t_{2j} + \frac{1}{2}(r_{4j} - p_{4j})|t_{2j}|) \\ & \text{s.t.} \sum_{j=1}^{n_1} (\frac{1}{2}(a_{ij}^1 + a_{ij}^3)x_{1j} - \frac{1}{2}(a_{ij}^3 - \\ & a_{ij}^1)|x_{1j}|, a_{ij}^2 y_{1j}, \frac{1}{2}(a_{ij}^1 + a_{ij}^3)t_{1j} + \frac{1}{2}(a_{ij}^3 - a_{ij}^1)|t_{1j}|) \\ & \oplus \sum_{j=1}^{n_2} (\frac{1}{2}(b_{ij}^1 + b_{ij}^3)x_{2j} - \frac{1}{2}(b_{ij}^3 - \\ & b_{ij}^1)|x_{2j}|, b_{ij}^2 y_{2j}, \frac{1}{2}(b_{ij}^1 + b_{ij}^3)t_{2j} + \frac{1}{2}(b_{ij}^3 - b_{ij}^1)|t_{2j}|) \\ & = (m_i, n_i, p_i) \forall i = 1, 2, \dots, m. \\ & (x_{1j}, y_{1j}, t_{1j}), (x_{2j}, y_{2j}, t_{2j}) \in F(R), j = 1, \dots, n. \end{aligned}$$

**Step 3.** Using Definitions 2.6, 2.7, the FFBLP problem, obtained in Step 2, is transformed into the following crisp bilevel nonlinear programming problem:

$$\begin{aligned} & \underset{x_{1j}, y_{1j}, t_{1j}}{\text{maximize}} F = \frac{1}{4} [\sum_{j=1}^{n_1} (\frac{1}{2}(p_{1j} + r_{1j})x_{1j} - \\ & \frac{1}{2}(r_{1j} - p_{1j})|x_{1j}| + 2(q_{1j}y_{1j}) + \frac{1}{2}(p_{1j} + r_{1j})t_{1j} \\ & + \frac{1}{2}(r_{1j} - p_{1j})|t_{1j}|)] + \frac{1}{4} [\sum_{j=1}^{n_2} (\frac{1}{2}(p_{2j} + r_{2j})x_{2j} - \\ & \frac{1}{2}(r_{2j} - p_{2j})|x_{2j}| \\ & + 2(q_{2j}y_{2j}) + \frac{1}{2}(p_{2j} + r_{2j})t_{2j} + \frac{1}{2}(r_{2j} - p_{2j})|t_{2j}|)] \\ & \text{s.t.} \\ & \underset{x_{2j}, y_{2j}, t_{2j}}{\text{maximize}} f = \frac{1}{4} [\sum_{j=1}^{n_1} (\frac{1}{2}(p_{3j} + r_{3j})x_{1j} - \frac{1}{2}(r_{3j} - \\ & p_{3j})|x_{1j}| + 2(q_{3j}y_{1j}) + \frac{1}{2}(p_{3j} + r_{3j})t_{1j} \\ & + \frac{1}{2}(r_{3j} - p_{3j})|t_{1j}|)] + \frac{1}{4} [\sum_{j=1}^{n_2} (\frac{1}{2}(p_{4j} + r_{4j})x_{2j} - \\ & \frac{1}{2}(r_{4j} - p_{4j})|x_{2j}| \\ & + 2(q_{4j}y_{2j}) + 2(\frac{1}{2}(p_{4j} + r_{4j})t_{2j}) + \frac{1}{2}(r_{4j} - p_{4j})|t_{2j}|)] \\ & \text{s.t.} \sum_{j=1}^{n_1} (\frac{1}{2}(a_{ij}^1 + a_{ij}^3)x_{1j} - \frac{1}{2}(a_{ij}^3 - \\ & a_{ij}^1)|x_{1j}| + \sum_{j=1}^{n_2} (\frac{1}{2}(b_{ij}^1 + b_{ij}^3)x_{2j} \\ & - \frac{1}{2}(b_{ij}^3 - b_{ij}^1)|x_{2j}|) = m_i, \forall i = 1, \dots, m. \\ & \sum_{j=1}^{n_1} a_{ij}^2 y_{1j} + \sum_{j=1}^{n_2} b_{ij}^2 y_{2j} = n_i \forall i = 1, \dots, m. \\ & \sum_{j=1}^{n_1} (\frac{1}{2}(a_{ij}^1 + a_{ij}^3)x_{1j} + \frac{1}{2}(a_{ij}^3 - \\ & a_{ij}^1)|t_{1j}| + \sum_{j=1}^{n_2} (\frac{1}{2}(b_{ij}^1 + b_{ij}^3)x_{2j} \\ & + \frac{1}{2}(b_{ij}^3 - b_{ij}^1)|t_{2j}|) = p_i \forall i = 1, \dots, m. \end{aligned}$$

$$y_{1j} - x_{1j} \geq 0, y_{2j} - x_{2j} \geq 0, t_{1j} - y_{1j} \geq 0, t_{2j} - y_{2j} \geq 0$$

$$y_{1j}, x_{1j}, t_{1j}, y_{2j}, x_{2j}, t_{2j} \in R, \forall : 1 \leq i \leq m, 1 \leq j \leq n, \forall j = 1, \dots, n.$$

**Step 4.** Find the optimal solution  $x_{1j}, y_{1j}, t_{1j}, x_{2j}, y_{2j}, t_{2j}$ , by solving the bilevel nonlinear programming problem obtained in Step 3.

**Step 5.** Find the fuzzy optimal solution by putting the values of  $x_{1j}, y_{1j}, t_{1j}, x_{2j}, y_{2j}, t_{2j}$  in  $\tilde{x}_1 = (x_{1j}, y_{1j}, t_{1j}), \tilde{x}_2 = (x_{2j}, y_{2j}, t_{2j})$ .

**Step 6.** Find the fuzzy optimal value by putting  $\tilde{x}_1, \tilde{x}_2$  in:  $\tilde{F}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (p_{1j}, q_{1j}, r_{1j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{j=1}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j})$   
 $\tilde{f}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{j=1}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j})$ .

And finally, we will present an algorithm which is used for all FFBLP problems with unconstrained fuzzy coefficients corresponding to the variables and unconstrained fuzzy variables.

### 3.3 Algorithm 3

**Step 1.** Using Definition 2.9, the FFBLP problem (3.1) can be written as follows:

$$\begin{aligned} & \underset{(x_{1j}, y_{1j}, t_{1j})}{\text{maximize}} \tilde{F}(\tilde{x}_1, \tilde{x}_2) \\ & \sum_{j=1}^{n_1} (\min(p_{1j}x_{1j}, r_{1j}x_{1j})), q_{1j}y_{1j} \\ & \quad , \max(p_{1j}t_{1j}, r_{1j}t_{1j}) \\ & \oplus \sum_{j=1}^{n_1} (\min(r_{1j}x_{1j}, p_{1j}t_{1j}), q_{1j}y_{1j}, \\ & \quad \begin{matrix} p_{1j} < 0 \\ r_{1j} \geq 0 \end{matrix} \\ & \max(r_{1j}t_{1j}, p_{1j}x_{1j})) \\ & \oplus \sum_{j=1}^{n_1} (\min p_{1j}t_{1j}, r_{1j}t_{1j}), q_{1j}y_{1j}, \\ & \quad \begin{matrix} p_{1j} < 0 \\ r_{1j} < 0 \end{matrix} \\ & \max(p_{1j}x_{1j}, r_{1j}x_{1j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(p_{2j}x_{2j}, r_{2j}x_{2j}), q_{2j}y_{2j}, \\ & \quad \begin{matrix} p_{2j} \geq 0 \\ r_{2j} \geq 0 \end{matrix} \\ & \max(p_{2j}t_{2j}, r_{2j}t_{2j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(r_{2j}x_{2j}, p_{2j}t_{2j}), q_{2j}y_{2j}, \\ & \quad \begin{matrix} p_{2j} < 0 \\ r_{2j} \geq 0 \end{matrix} \\ & \max(r_{2j}t_{2j}, p_{2j}x_{2j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(p_{2j}t_{2j}, r_{2j}t_{2j}), q_{2j}y_{2j}, \\ & \quad \begin{matrix} p_{2j} < 0 \\ r_{2j} < 0 \end{matrix} \\ & \max(p_{2j}x_{2j}, r_{2j}x_{2j})) \\ & \text{s.t.} \end{aligned}$$

$$\begin{aligned} & \underset{(x_{2j}, y_{2j}, t_{2j})}{\text{maximize}} \tilde{f}(\tilde{x}_1, \tilde{x}_2) \\ & \sum_{j=1}^{n_1} (\min(p_{3j}x_{1j}, r_{3j}x_{1j})), q_{3j}y_{1j}, \\ & \quad \begin{matrix} p_{3j} \geq 0 \\ r_{3j} \geq 0 \end{matrix} \\ & \max(p_{3j}t_{1j}, r_{3j}t_{1j})) \\ & \oplus \sum_{j=1}^{n_1} (\min(r_{3j}x_{1j}, p_{3j}t_{1j}), q_{3j}y_{1j}, \\ & \quad \begin{matrix} p_{3j} < 0 \\ r_{3j} \geq 0 \end{matrix} \\ & \max(r_{3j}t_{1j}, p_{3j}x_{1j})) \\ & \oplus \sum_{j=1}^{n_1} (\min(p_{3j}t_{1j}, r_{3j}t_{1j}), q_{3j}y_{1j}, \\ & \quad \begin{matrix} p_{3j} < 0 \\ r_{3j} < 0 \end{matrix} \\ & \max(p_{3j}x_{1j}, r_{3j}x_{1j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(p_{4j}x_{2j}, r_{4j}x_{2j}), q_{4j}y_{2j}, \\ & \quad \begin{matrix} p_{4j} \geq 0 \\ r_{4j} \geq 0 \end{matrix} \\ & \max(p_{4j}t_{2j}, r_{4j}t_{2j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(r_{4j}x_{2j}, p_{4j}t_{2j}), q_{4j}y_{2j}, \\ & \quad \begin{matrix} p_{4j} < 0 \\ r_{4j} \geq 0 \end{matrix} \\ & \max(r_{4j}t_{2j}, p_{4j}x_{2j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(p_{4j}t_{2j}, r_{4j}t_{2j}), q_{4j}y_{2j}, \\ & \quad \begin{matrix} p_{4j} < 0 \\ r_{4j} < 0 \end{matrix} \\ & \max(p_{4j}x_{2j}, r_{4j}x_{2j})) \\ & \sum_{j=1}^{n_1} (\min(a_{ij}^1x_{1j}, a_{ij}^3x_{1j}), a_{ij}^2y_{1j}, \\ & \quad \begin{matrix} a_{ij}^1 \geq 0 \\ a_{ij}^3 \geq 0 \end{matrix} \\ & \max(a_{ij}^1t_{1j}, a_{ij}^3t_{1j})) \\ & \oplus \sum_{j=1}^{n_1} (\min(a_{ij}^3x_{1j}, a_{ij}^1t_{1j}), a_{ij}^2y_{1j}, \\ & \quad \begin{matrix} a_{ij}^1 < 0 \\ a_{ij}^3 \geq 0 \end{matrix} \\ & \max(a_{ij}^3t_{1j}, a_{ij}^1x_{1j})) \\ & \oplus \sum_{j=1}^{n_1} (\min(a_{ij}^1t_{1j}, a_{ij}^3t_{1j}), a_{ij}^2y_{1j}, \\ & \quad \begin{matrix} a_{ij}^1 < 0 \\ a_{ij}^3 < 0 \end{matrix} \\ & \max(a_{ij}^1x_{1j}, a_{ij}^3x_{1j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(b_{ij}^1x_{2j}, b_{ij}^3x_{2j}), b_{ij}^2y_{2j}, \\ & \quad \begin{matrix} b_{ij}^1 \geq 0 \\ b_{ij}^3 \geq 0 \end{matrix} \\ & \max(b_{ij}^1t_{2j}, b_{ij}^3t_{2j})) \\ & \oplus \sum_{j=1}^{n_2} ((\min(b_{ij}^3x_{2j}, b_{ij}^1t_{2j}), b_{ij}^2y_{2j}, \\ & \quad \begin{matrix} b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0 \end{matrix} \\ & \max(b_{ij}^3t_{2j}, b_{ij}^1x_{2j})) \\ & \oplus \sum_{j=1}^{n_2} (\min(b_{ij}^1t_{2j}, b_{ij}^3t_{2j}), b_{ij}^2y_{2j}, \\ & \quad \begin{matrix} b_{ij}^1 < 0 \\ b_{ij}^3 < 0 \end{matrix} \\ & \max(b_{ij}^1x_{2j}, b_{ij}^3x_{2j})) \\ & = (m_i, n_i, p_i), \forall i = \\ & 1, 2, \dots, m, (x_{1j}, y_{1j}, t_{1j}), (x_{2j}, y_{2j}, t_{2j}) \in \\ & F(R), j = 1, \dots, n. \\ & \text{Step 2. Using Definition 2.9, the problem in} \\ & \text{Step 1, can be written as follows:} \\ & \underset{(x_{1j}, y_{1j}, t_{1j})}{\text{maximize}} \tilde{F}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (0.5(p_{1j} + r_{1j})x_{1j} - \\ & \quad \begin{matrix} p_{1j} \geq 0 \\ r_{1j} \geq 0 \end{matrix} \\ & 0.5(r_{1j} - p_{1j})|x_{1j}|, q_{1j}y_{1j}, 0.5(p_{1j} + r_{1j})t_{1j} \end{aligned}$$

**Table 1:** Results of solving three FFBLP problems by the proposed algorithms.

Example	fuzzy optimal solution	lower level fuzzy optimal value	upper level fuzzy optimal value
4.1	$\tilde{x}_1 = (1, 2, 3)$	$\tilde{x}_2 = (4, 5, 6)$	$\tilde{f}(\tilde{x}_1, \tilde{x}_2) = (13, 32, 39)$
4.2	$\tilde{x}_1 = (-3, -2, \frac{7}{5})$	$\tilde{x}_2 = (1, 2, \frac{121}{25})$	$\tilde{f}(\tilde{x}_1, \tilde{x}_2) = (-14, 2, \frac{156}{5})$
4.3	$\tilde{x}_1 = (1, \frac{3}{2}, 2)$	$\tilde{x}_2 = (-3, -2, \frac{1}{2})$	$\tilde{f}(\tilde{x}_1, \tilde{x}_2) = (-13, -\frac{5}{2}, \frac{7}{2})$

$$\begin{aligned}
 &+0.5(r_{1j}-p_{1j})|t_{1j}| \oplus \sum_{\substack{j=1 \\ p_{1j}<0 \\ r_{1j}\geq 0}}^{n_1} (0.5(r_{1j}x_{1j}+p_{1j}t_{1j}) - \\
 &0.5|r_{1j}x_{1j}-p_{1j}t_{1j}|, q_{1j}y_{1j}, \\
 &0.5(r_{1j}t_{1j}+p_{1j}x_{1j})+0.5|r_{1j}t_{1j}-p_{1j}x_{1j}|) \oplus \\
 &\sum_{\substack{j=1 \\ r_{1j}<0}}^{n_1} (0.5(p_{1j}+r_{1j})t_{1j} \\
 &-0.5(r_{1j}-p_{1j})|t_{1j}|, q_{1j}y_{1j}, 0.5(p_{1j}+r_{1j})x_{1j}+ \\
 &0.5(r_{1j}-p_{1j})|x_{1j}|) \\
 &\oplus \sum_{\substack{j=1 \\ p_{2j}\geq 0}}^{n_2} (0.5(p_{2j}+r_{2j})x_{2j}-0.5(r_{2j}- \\
 &p_{2j})|x_{2j}|, q_{2j}y_{2j}, 0.5(p_{2j}+r_{2j})t_{2j} \\
 &+0.5(r_{2j}-p_{2j})|t_{2j}|) \oplus \sum_{\substack{j=1 \\ p_{2j}<0 \\ r_{2j}\geq 0}}^{n_2} (0.5(r_{2j}x_{2j}+ \\
 &p_{2j}t_{2j})-0.5|r_{2j}x_{2j}-p_{2j}t_{2j}|, \\
 &q_{2j}y_{2j}, 0.5(r_{2j}t_{2j}+p_{2j}x_{2j})+0.5|r_{2j}t_{2j}-p_{2j}x_{2j}|) \oplus \\
 &\sum_{\substack{j=1 \\ r_{2j}<0}}^{n_2} (0.5(p_{2j}+r_{2j})t_{2j} \\
 &-0.5(r_{2j}-p_{2j})|t_{2j}|, q_{2j}y_{2j}, 0.5(p_{2j}+r_{2j})x_{2j})+ \\
 &0.5(r_{2j}-p_{2j})|x_{2j}|) \\
 &s.t. \\
 &maximize \tilde{f}(\tilde{x}_1, \tilde{x}_2) = \sum_{\substack{j=1 \\ p_{3j}\geq 0}}^{n_1} (0.5(p_{3j}+r_{3j})x_{1j}- \\
 &(x_{2j}, y_{2j}, t_{2j})) \\
 &0.5(r_{3j}-p_{3j})|x_{1j}|, q_{3j}y_{1j}, 0.5(p_{3j}+r_{3j})t_{1j} \\
 &+0.5(r_{3j}-p_{3j})|t_{1j}| \oplus \sum_{\substack{j=1 \\ p_{3j}<0 \\ r_{3j}\geq 0}}^{n_1} (0.5(r_{3j}x_{1j}+p_{3j}t_{1j})- \\
 &0.5|r_{3j}x_{1j}-p_{3j}t_{1j}|, q_{3j}y_{1j}, \\
 &0.5(r_{3j}t_{1j}+p_{3j}x_{1j})+0.5|r_{3j}t_{1j}-p_{3j}x_{1j}|) \oplus \\
 &\sum_{\substack{j=1 \\ r_{3j}<0}}^{n_1} (0.5(p_{3j}+r_{3j})t_{1j} \\
 &-0.5(r_{3j}-p_{3j})|t_{1j}|, q_{3j}y_{1j}, 0.5(p_{3j}+r_{3j})x_{1j}+ \\
 &0.5(r_{3j}-p_{3j})|x_{1j}|) \\
 &\oplus \sum_{\substack{j=1 \\ p_{4j}\geq 0}}^{n_2} (0.5(p_{4j}+r_{4j})x_{2j}-0.5(r_{4j}- \\
 &p_{4j})|x_{2j}|, q_{4j}y_{2j}, 0.5(p_{4j}+r_{4j})t_{2j} \\
 &+0.5(r_{4j}-p_{4j})|t_{2j}|) \oplus \sum_{\substack{j=1 \\ p_{4j}<0 \\ r_{4j}\geq 0}}^{n_2} (0.5(r_{4j}x_{2j}+ \\
 &p_{4j}t_{2j})-0.5|r_{4j}x_{2j}-p_{4j}t_{2j}|, \\
 &q_{4j}y_{2j}, 0.5(r_{4j}t_{2j}+p_{4j}x_{2j})+0.5|r_{4j}t_{2j}-p_{4j}x_{2j}|) \oplus \\
 &\sum_{\substack{j=1 \\ r_{4j}<0}}^{n_2} (0.5(p_{4j}+r_{4j})t_{2j} \\
 &-0.5(r_{4j}-p_{4j})|t_{2j}|, q_{4j}y_{2j}, 0.5(p_{4j}+r_{4j})x_{2j})+ \\
 &0.5(r_{4j}-p_{4j})|x_{2j}|)
 \end{aligned}$$

$$\begin{aligned}
 &s.t. \quad \sum_{\substack{j=1 \\ a_{ij}^1 \geq 0}}^{n_1} (0.5(a_{ij}^1 + a_{ij}^3)x_{1j} - 0.5(a_{ij}^3 - \\
 &a_{ij}^1)|x_{1j}|, a_{ij}^2y_{1j}, 0.5(a_{ij}^1 + a_{ij}^3)t_{1j} + 0.5(a_{ij}^3 - \\
 &a_{ij}^1)|t_{1j}|) \\
 &\oplus \sum_{\substack{j=1 \\ a_{ij}^1 < 0 \\ a_{ij}^3 \geq 0}}^{n_1} (0.5(a_{ij}^3x_{1j} + a_{ij}^1t_{1j}) - 0.5|a_{ij}^3x_{1j} - \\
 &a_{ij}^1t_{1j}|, a_{ij}^2y_{1j}, 0.5(a_{ij}^1x_{1j} + a_{ij}^3t_{1j}) + 0.5|a_{ij}^3t_{1j} - \\
 &a_{ij}^1x_{1j}|) \\
 &\oplus \sum_{\substack{j=1 \\ a_{ij}^3 < 0}}^{n_1} (0.5(a_{ij}^1 + a_{ij}^3)t_{1j} - 0.5(a_{ij}^3 - \\
 &a_{ij}^1)|t_{1j}|, a_{ij}^2y_{1j}, 0.5(a_{ij}^1 + a_{ij}^3)x_{1j} + 0.5(a_{ij}^3 - \\
 &a_{ij}^1)|x_{1j}|) \\
 &\oplus \sum_{\substack{j=1 \\ b_{ij}^1 \geq 0}}^{n_2} ((0.5(b_{ij}^1 + b_{ij}^3)x_{2j} - 0.5(b_{ij}^3 - \\
 &b_{ij}^1)|x_{2j}|, b_{ij}^2y_{2j}, 0.5(b_{ij}^1 + b_{ij}^3)t_{2j} + 0.5(b_{ij}^3 - b_{ij}^1)|t_{2j}|) \\
 &\oplus \sum_{\substack{j=1 \\ b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0}}^{n_2} (0.5(b_{ij}^3x_{2j} + b_{ij}^1t_{2j}) - 0.5|b_{ij}^3x_{2j} - \\
 &b_{ij}^1t_{2j}|, b_{ij}^2y_{2j}, 0.5(b_{ij}^1x_{2j} + b_{ij}^3t_{2j}) + 0.5|b_{ij}^3t_{2j} - \\
 &b_{ij}^1x_{2j}|) \\
 &\oplus \sum_{\substack{j=1 \\ b_{ij}^3 < 0}}^{n_2} (0.5(b_{ij}^1 + b_{ij}^3)t_{2j} - 0.5(b_{ij}^3 - \\
 &b_{ij}^1)|t_{2j}|, b_{ij}^2y_{2j}, 0.5(b_{ij}^1 + b_{ij}^3)x_{2j} + 0.5(b_{ij}^3 - b_{ij}^1)|x_{2j}|) \\
 &= (m_i, n_i, p_i), \forall i \\
 &= 1, 2, \dots, m, (x_{1j}, y_{1j}, t_{1j}), (x_{2j}, y_{2j}, t_{2j}) \in \\
 &F(R), j = 1, \dots, n.
 \end{aligned}$$

**Step 3.** Using Definitions 2.6,2.7, the FFBLP problem, obtained in Step 2, is transformed into the following crisp bilevel linear programming problem:

$$\begin{aligned}
 &maximize F = \frac{1}{4}[\sum_{\substack{j=1 \\ p_{1j}\geq 0}}^{n_1} (0.5(p_{1j}+r_{1j})x_{1j}- \\
 &x_{1j}, y_{1j}, t_{1j}) \\
 &0.5(r_{1j}-p_{1j})|x_{1j}|+2q_{1j}y_{1j}+0.5(p_{1j}+r_{1j})t_{1j} \\
 &+0.5(r_{1j}-p_{1j})|t_{1j}|] + \frac{1}{4}[\sum_{\substack{j=1 \\ p_{1j}<0 \\ r_{1j}\geq 0}}^{n_1} (0.5(r_{1j}x_{1j}+ \\
 &p_{1j}t_{1j})-0.5|r_{1j}x_{1j}-p_{1j}t_{1j}|+2q_{1j}y_{1j} \\
 &+0.5(r_{1j}t_{1j}+p_{1j}x_{1j})+0.5|r_{1j}t_{1j}-p_{1j}x_{1j}|)] +
 \end{aligned}$$



$$\begin{aligned}
 & \frac{1}{4} [\sum_{\substack{j=1 \\ r_{1j} < 0}}^{n_1} (0.5(p_{1j} + r_{1j})t_{1j} \\
 & - 0.5(r_{1j} - p_{1j})|t_{1j}| + 2q_{1j}y_{1j} + 0.5(p_{1j} + r_{1j})x_{1j} + \\
 & 0.5(r_{1j} - p_{1j})|x_{1j}|)] \\
 & + \frac{1}{4} [\sum_{\substack{j=1 \\ p_{2j} \geq 0}}^{n_2} (0.5(p_{2j} + r_{2j})x_{2j} - 0.5(r_{2j} - \\
 & p_{2j})|x_{2j}| + 2q_{2j}y_{2j} + 0.5(p_{2j} + r_{2j})t_{2j} \\
 & + 0.5(r_{2j} - p_{2j})|t_{2j}|)] + \frac{1}{4} [\sum_{\substack{j=1 \\ p_{2j} < 0 \\ r_{2j} \geq 0}}^{n_2} (0.5(r_{2j}x_{2j} + \\
 & p_{2j}t_{2j}) - 0.5|r_{2j}x_{2j} - p_{2j}t_{2j}| \\
 & + 2q_{2j}y_{2j} + 0.5(r_{2j}t_{2j} + p_{2j}x_{2j}) + 0.5|r_{2j}t_{2j} - \\
 & p_{2j}x_{2j}|)] + \frac{1}{4} [\sum_{\substack{j=1 \\ r_{2j} < 0}}^{n_2} (0.5(p_{2j} + r_{2j})t_{2j} \\
 & - 0.5(r_{2j} - p_{2j})|t_{2j}| + 2q_{2j}y_{2j} + 0.5(p_{2j} + r_{2j})x_{2j}) + \\
 & 0.5(r_{2j} - p_{2j})|x_{2j}|] \\
 & \text{s.t.} \\
 & \text{maximize } f = \frac{1}{4} [\sum_{\substack{j=1 \\ p_{3j} \geq 0}}^{n_1} (0.5(p_{3j} + r_{3j})x_{1j} - \\
 & 0.5(r_{3j} - p_{3j})|x_{1j}| + 2q_{3j}y_{1j} + 0.5(p_{3j} + r_{3j})t_{1j} \\
 & + 0.5(r_{3j} - p_{3j})|t_{1j}|)] + \frac{1}{4} [\sum_{\substack{j=1 \\ p_{3j} < 0 \\ r_{3j} \geq 0}}^{n_1} (0.5(r_{3j}x_{1j} + \\
 & p_{3j}t_{1j}) - 0.5|r_{3j}x_{1j} - p_{3j}t_{1j}| + 2q_{3j}y_{1j}2 \\
 & + 0.5(r_{3j}t_{1j} + p_{3j}x_{1j}) + 0.5|r_{3j}t_{1j} - p_{3j}x_{1j}|)] + \\
 & \frac{1}{4} [\sum_{\substack{j=1 \\ r_{3j} < 0}}^{n_1} (0.5(p_{3j} + r_{3j})t_{1j} \\
 & - 0.5(r_{3j} - p_{3j})|t_{1j}| + 2q_{3j}y_{1j} + 0.5(p_{3j} + r_{3j})x_{1j} + \\
 & 0.5(r_{3j} - p_{3j})|x_{1j}|)] \\
 & + \frac{1}{4} [\sum_{\substack{j=1 \\ p_{4j} \geq 0}}^{n_2} (0.5(p_{4j} + r_{4j})x_{2j} - 0.5(r_{4j} - \\
 & p_{4j})|x_{2j}| + 2q_{4j}y_{2j} + 0.5(p_{4j} + r_{4j})t_{2j} \\
 & + 0.5(r_{4j} - p_{4j})|t_{2j}|)] + \frac{1}{4} [\sum_{\substack{j=1 \\ p_{4j} < 0 \\ r_{4j} \geq 0}}^{n_2} (0.5(r_{4j}x_{2j} + \\
 & p_{4j}t_{2j}) - 0.5|r_{4j}x_{2j} - p_{4j}t_{2j}| \\
 & + 2q_{4j}y_{2j} + 0.5(r_{4j}t_{2j} + p_{4j}x_{2j}) + 0.5|r_{4j}t_{2j} - \\
 & p_{4j}x_{2j}|)] + \frac{1}{4} [\sum_{\substack{j=1 \\ r_{4j} < 0}}^{n_2} (0.5(p_{4j} + r_{4j})t_{2j} \\
 & - 0.5(r_{4j} - p_{4j})|t_{2j}| + 2q_{4j}y_{2j} + 0.5(p_{4j} + r_{4j})x_{2j}) + \\
 & 0.5(r_{4j} - p_{4j})|x_{2j}|] \\
 & \text{s.t.} \quad \sum_{\substack{j=1 \\ a_{ij}^1 \geq 0}}^{n_1} (0.5(a_{ij}^1 + a_{ij}^3)x_{1j} - 0.5(a_{ij}^3 - \\
 & a_{ij}^1)|x_{1j}|) + \sum_{\substack{j=1 \\ a_{ij}^1 < 0 \\ a_{ij}^3 \geq 0}}^{n_1} (0.5(a_{ij}^3x_{1j} + a_{ij}^1t_{1j}) - \\
 & 0.5|a_{ij}^3x_{1j} - a_{ij}^1t_{1j}|) \\
 & + \sum_{\substack{j=1 \\ a_{ij}^3 < 0}}^{n_1} (0.5(a_{ij}^1 + a_{ij}^3)t_{1j} - 0.5(a_{ij}^3 - a_{ij}^1)|t_{1j}|) + \\
 & \sum_{\substack{j=1 \\ b_{ij}^1 \geq 0}}^{n_2} (0.5(b_{ij}^1 + b_{ij}^3)x_{2j} - 0.5(b_{ij}^3 - b_{ij}^1)|x_{2j}|) \\
 & + \sum_{\substack{j=1 \\ b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0}}^{n_2} (0.5(b_{ij}^3x_{2j} + b_{ij}^1t_{2j}) - 0.5|b_{ij}^3x_{2j} - b_{ij}^1t_{2j}|) +
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{\substack{j=1 \\ b_{ij}^1 < 0}}^{n_2} (0.5(b_{ij}^1 + b_{ij}^3)t_{2j} - 0.5(b_{ij}^3 - b_{ij}^1)|t_{2j}|) = m_i, \\
 & \forall i = 1, \dots, m. \\
 & \sum_{j=1}^{n_1} (a_{ij}^3y_{1j}) + \sum_{j=1}^{n_2} (b_{ij}^3y_{2j}) = n_i, \forall i = 1, \dots, m. \\
 & \sum_{\substack{j=1 \\ a_{ij}^1 \geq 0}}^{n_1} (0.5(a_{ij}^1 + a_{ij}^3)t_{1j} + 0.5(a_{ij}^3 - a_{ij}^1)|t_{1j}|) + \\
 & \sum_{\substack{j=1 \\ a_{ij}^1 < 0 \\ a_{ij}^3 \geq 0}}^{n_1} (0.5(a_{ij}^1x_{1j} + a_{ij}^3t_{1j}) + 0.5|a_{ij}^3t_{1j} - a_{ij}^1x_{1j}|) \\
 & + \sum_{\substack{j=1 \\ a_{ij}^3 < 0}}^{n_1} (0.5(a_{ij}^1 + a_{ij}^3)x_{1j} + 0.5(a_{ij}^3 - a_{ij}^1)|x_{1j}|) + \\
 & \sum_{\substack{j=1 \\ b_{ij}^1 \geq 0}}^{n_2} (0.5(b_{ij}^1 + b_{ij}^3)t_{2j} + 0.5(b_{ij}^3 - b_{ij}^1)|t_{2j}|) \\
 & + \sum_{\substack{j=1 \\ b_{ij}^1 < 0 \\ b_{ij}^3 \geq 0}}^{n_2} (0.5(b_{ij}^1x_{2j} + b_{ij}^3t_{2j}) + 0.5|b_{ij}^3t_{2j} - b_{ij}^1x_{2j}|) + \\
 & \sum_{\substack{j=1 \\ b_{ij}^1 < 0}}^{n_2} (0.5(b_{ij}^1 + b_{ij}^3)x_{2j} + 0.5(b_{ij}^3 - b_{ij}^1)|x_{2j}|) = p_i, \\
 & \forall i = 1, \dots, m., y_{1j} - x_{1j} \geq 0, y_{2j} - x_{2j} \geq \\
 & 0, t_{1j} - y_{1j} \geq 0, t_{2j} - y_{2j} \geq 0 \\
 & , y_{1j}, x_{1j}, t_{1j}, y_{2j}, x_{2j}, t_{2j} \in R, \forall : 1 \leq i \leq m, 1 \leq \\
 & j \leq n
 \end{aligned}$$

**Step 4.** Find the optimal solution  $x_{1j}, y_{1j}, t_{1j}, x_{2j}, y_{2j}, t_{2j}$ , by solving the bilevel nonlinear programming problem obtained in Step 3.

**Step 5.** Find the fuzzy optimal solution by putting the values of  $x_{1j}, y_{1j}, t_{1j}, x_{2j}, y_{2j}, t_{2j}$  in  $\tilde{x}_1 = (x_{1j}, y_{1j}, t_{1j}), \tilde{x}_2 = (x_{2j}, y_{2j}, t_{2j})$ .

**Step 6.** Find the fuzzy optimal value by putting  $\tilde{x}_1, \tilde{x}_2$  in:  $\tilde{F}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (p_{1j}, q_{1j}, r_{1j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{j=1}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j})$   
 $\tilde{f}(\tilde{x}_1, \tilde{x}_2) = \sum_{j=1}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{1j}, y_{1j}, t_{1j}) \oplus \sum_{j=1}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j})$ .

## 4 Numerical examples

Here, we give some numerical examples and solve them by proposed algorithms in previous section.

Examples 4.1, 4.2 and 4.3 have been solved by the algorithms 1, 2 and 3 respectively. Table 1, shows the results of FFBLP problems solved by the proposed algorithms.

**Example 4.1** Solve the following FF-BLP problem using the first algorithm: maximize  $\tilde{F}(\tilde{x}_1, \tilde{x}_2) = (1, 6, 9) \otimes \tilde{x}_1 \oplus (2, 3, 8) \otimes \tilde{x}_2$

s.t. maximize  $\tilde{f}(\tilde{x}_1, \tilde{x}_2) = (1, 6, 3) \otimes \tilde{x}_1 \oplus (3, 4, 5) \otimes \tilde{x}_2$   
 s.t.  $(2, 3, 4) \otimes \tilde{x}_1 \oplus (1, 2, 3) \otimes \tilde{x}_2 = (6, 16, 30)$   
 $(-1, 1, 2) \otimes \tilde{x}_1 \oplus (1, 3, 4) \otimes \tilde{x}_2 = (1, 17, 30)$ ,  
 where  $\tilde{x}_1, \tilde{x}_2$  are non-negative triangular fuzzy numbers.

**Example 4.2** Solve the following FF-BLP problem using the second algorithm:  
 maximize  $\tilde{F}(\tilde{x}_1, \tilde{x}_2) = (1, 2, 3) \otimes \tilde{x}_1 \oplus (2, 4, 6) \otimes \tilde{x}_2$   
 s.t. maximize  $\tilde{f}(\tilde{x}_1, \tilde{x}_2) = (1, 2, 5) \otimes \tilde{x}_1 \oplus (1, 3, 4) \otimes \tilde{x}_2$   
 s.t.  $(0, 1, 2) \otimes \tilde{x}_1 \oplus (1, 3, 5) \otimes \tilde{x}_2 = (-5, 4, 27)$   
 $(2, 4, 7) \otimes \tilde{x}_1 \oplus (2, 3, 5) \otimes \tilde{x}_2 = (-19, -2, 34)$ ,  
 where  $\tilde{x}_1, \tilde{x}_2$  are unrestricted triangular fuzzy numbers.

**Example 4.3** Solve the following FFBLP problem using the third algorithm:  
 maximize  $\tilde{F}(\tilde{x}_1, \tilde{x}_2) = (-1, 2, 3) \otimes \tilde{x}_1 \oplus (2, 3, 4) \otimes \tilde{x}_2$   
 s.t. maximize  $\tilde{f}(\tilde{x}_1, \tilde{x}_2) = (-2, 1, 2) \otimes \tilde{x}_1 \oplus (1, 2, 3) \otimes \tilde{x}_2$   
 s.t.  $(-3, -2, 0) \otimes \tilde{x}_1 \oplus (6, 7, 8) \otimes \tilde{x}_2 = (-30, -17, -3)$   
 $(2, 4, 6) \otimes \tilde{x}_1 \oplus (-2, -1, 2) \otimes \tilde{x}_2 = (-4, 8, 18)$ , where  $\tilde{x}_1, \tilde{x}_2$  are unrestricted triangular fuzzy numbers.

## 5 Conclusion

This paper presents new algorithms to find the fuzzy optimal solution of the FFBLP problem with equality constraints and non-negative fuzzy variables or unconstrained fuzzy variables. In these algorithms, the fuzzy bilevel linear programming problem is transformed to a deterministic bilevel programming problem using the ranking function method, and the solution of the deterministic problem is achieved using the common Kth-best method. The proposed methods are quite useful in solving the real-world problems where the information is inexact. To obtain the fuzzy optimal solution, new efficient algorithms have been proposed for FFBLP problems and some numerical examples have been solved to illustrate the proposed methods.

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