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Research Article

Uncertain BCC Data Envelopment Analysis Model with Belief Degree: A case study in Iranian Banks

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Abstract

Data Envelopment Analysis (DEA) is recognized as a robust analytical tool extensively utilized in measuring the relative efficiency of a group of Decision Making Units (DMUs) with multiple inputs and outputs. The DEA models require inputs and outputs equipped with precise information. However in real-world situations inputs and outputs may be unstable and complicated thus unable to be accurately measured. This problem resulted in the investigation of uncertain DEA models. The BCC model is studied in this paper in an uncertain environment where uncertain inputs and outputs were belief degree-based uncertainty useful for the cases for which no historical information of an uncertain event is available. As the solution method the uncertain BCC model was converted to a crisp form using two approaches of expected value method (EV) and expected value and chance-constrained method (EVCC) separately. Finally, an applied example regarding the Iranian Banking system is presented to document the proposed models.

Keywords : Data envelopment analysis; Uncertainty theory; Belief degree; BCC model; Uncertainty; Iranian Banks.

1 Introduction

As a strong analytical tool data envelopment analysis (DEA) is a method for evaluating the relation efficiency of decision-making units (DMUs) originally developed by Charens et al. [4] within a printed-paper named CCR. They extended the nonparametric method introduced by Farrell [13] to gauge DMUs with multiple inputs and outputs. Afterward Banker et al. (1984) introduced the BCC model. In addition to CCR and BCC there are several models that discuss DEA from several perspectives: RAM by Cooper et al. [7] slack-adjust by Sueyoshi [28] additive model by Ali and Seiford et al. [1] SBM model by Tone [30] and FDH model by Deprins et al. [11] all of which are DEA basic models.

In classical DEA models DMUs are evaluated by considering input and output values in order to

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measure rational efficiency as compared to different DMUs; eventually the measure to which rational efficiency belong is obtained (0,1). The first DEA models expected that inputs and outputs be measured by precise values. However in numerous situations such as when producing a system or in preparation process the banking system insurance industry and financial service system inputs and outputs are unstable and complicated and therefore cannot be accurately measured. Consequently many analysts attempted to model DEA with completely different questioning hypotheses.

The possible hypothesis is the earliest principles which may be used to build stochastic DEA models. Sengupta [29] summed up the stochastic DEA model utilizing expected value. Moreover Banker [3] consolidated applied mathematics elements underneath DEA in order to develop a statistical method. Several papers [6, 8, 14, 25] utilized chance-constrained programming to DEA so as to introduce stochastic varieties to information. Fuzzy outlook is another theory in which the hypothesis has been incorporated to cope with the uncertainty in DEA. United with the DEA innovator Cooper et al. [8, 9] introduced a technique to deal with inaccurate information such as moderate data adjectival data and ratio moderate data in DEA. Furthermore Kao and Liu [17] designed a technique to discover the membership function of fuzzy performance marks when each input and output are fuzzy numbers. Entani et al. [12] proposed an interval potency DEA model by pessimistic and idealistic values. Several researchers have introduced the possibility measure into DEA [15, 18].

As confirmed by several studies human uncertainty does not come with the same fuzziness. Because of the shortages of Fuzzy Theory Liu [19] introduced Uncertain Theory and refined it in 2010 as an understandable mathematical structure for confronting uncertainty in data which serves as a strong alternative to the probability theory when one has to restrict the information in the face of insufficient trusted data. The belief degree function is associated with an underlying concept of this theory built according to the experts' opinion. Optimization problems including uncertain data can be even more interesting and realistic in uncertain environments with uncertain values for parameters and even variables.

To tackle such problems with uncertain parameters any approach based on randomness fuzzy theory stochastic programming probability theory and so on can be applied in the face of historical information of the parameters. In such cases the uncertain manner of the problem is estimated from the historical data as the probability function random number fuzzy number etc. On the other hand for cases in which no historical information for an uncertain event exists uncertainty theory based on belief degree has been applied to solve the problem. This uncertainty theory can be explained by a simple example.

Consider a bridge and strength. At first it is assumed that no destructive experiment is allowed for the bridge. Thus there is no sample regarding the strength of the bridge. In this case no statistical methods exists for estimating its probability distribution. Therefore there is choice but to invite bridge engineers to evaluate the belief degrees about the bridge's strength.

Some basic concepts of the belief degree-based uncertainty theory will be explained in Section 2 and a complete study of this topic can be found in Liu [19]. The belief degree depends heavily on personal knowledge (even including preferences) concerning the event. When changes the personal knowledge the belief degree changes as well. Different people may produce different belief degrees. The question is which belief degree is correct a question which may be answered as follows: All belief degrees are wrong but some are useful. A belief degree becomes "correct" only when it is close enough to the frequency of the indeterminate quantity which however does not usually occur. Numerous surveys demonstrated that human beings usually estimate a much wider range of values than the object actually takes. This human conservatism makes the belief degrees deviate far from the frequency. Thus all belief degrees are wrong compared with the frequency. Nevertheless it is undeniable that these belief degrees are indeed helpful for decision-making.

Wen et al.[31] applied the uncertain theory for the first time to rewrite the DEA model in uncertainty condition and then published a paper on the sensitivity and stability of the additive model in terms of uncertainty. Wen et al. [31] also introduced a new Additive model with uncertain inputs and outputs. Additionally Wen et al. [31] developed the DEA model with uncertainty index ranking for criteria. Liu et al. [23] also published a paper to evaluate DMU with uncertain inputs and outputs. These studies are nevertheless insufficient for describing programming with uncertain data [5, 10, 16, 24, 26, 32, 25] and new models are often introduced to create a new method.

This article proposed a BCC of efficiency in DEA. In contrast to the previous model measures based on the proportional reduction (enlargement) of input (output) vectors. The present paper assumed inputs and outputs to be uncertain variables in BCC model and then to deal with uncertainty downside recommend two completely different models to crisp this new model. Afterward the uncertain BCC model is applied to an Iranian Bank to see how the target customer accepts a replacement-banking product before promotion and announcement phase.

The paper proceeds as follows: In Section 2 some preliminary knowledge of uncertainty theory and basic notions of some DEA models are reviewed. In Section 3 several new uncertain DEA models are introduced and their new structures are verified. In Section 4 the crisp equivalents of the model are presented. Finally a practical example of the Iranian Banking system with the uncertain BCC model is introduced.

2 Preliminaries

Here discuss basic concept and present uncertain variables. Let Γ be a nonempty set and L an σ-algebra over Γ. Each element $\Lambda \in \mathrm{L}$ is called an event. A set function $\mathcal{M}\{\Lambda\} \in [0,1]$ is known as an uncertain measure if it satisfies the following three axioms [19]:

1. $\mathcal{M}\{\Lambda\} = 1$ for the universal set Γ.
2. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^{\mathrm{c}}\} = 1$ for any event Λ.
3. For every countable subadditive of events, Λ_{i} we have $\mathcal{M}\{\bigcup_{i=1}^{\infty}\} \leqslant \sum_{i=1}^{\infty} \Lambda_{\mathrm{i}}$.

Definition 2.1. *[19] The set function $\mathcal{M}$ called an uncertain measure if it contents the duality normality and subadditivity axioms.*

The uncertain measure has the following attributes:

1. $\mathcal{M}\{\emptyset\} = 0$.
2. $0 \leqslant \mathcal{M}\{\Lambda\} \leqslant 1$ for any event Λ.
3. $\mathcal{M}\{\Lambda_1\} \leqslant \mathcal{M}\{\Lambda_2\}$ for any events $\Lambda_1 \subset \Lambda_2$.

The triplet $(\Gamma, L, \mathcal{M})$ called an uncertainty space. In order to define product uncertain measure Liu [22] proposed the fourth axiom as follow:
Let $(\Gamma_k, L_k, \mathcal{M}_k)$ be uncertainty space for $k = 1, 2, \dots$ then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying $\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_{\mathrm{k}}\} = \bigwedge_{\mathrm{k}=1}^{\infty} \mathcal{M}_{\mathrm{k}}\{\Lambda_{\mathrm{k}}\}$.

Definition 2.2. *[19]. An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, L, \mathcal{M})$ to the set of real numbers i.e. for any Borel set B of real numbers the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event.*

An uncertainty distribution function is used to characterize an uncertain variable and is defined as follows [19].

$$\varphi(x) = \mathcal{M}\{\xi \leq x\} \qquad \forall x, x \in \mathbb{R}$$

Example 2.1. *The uncertain variable with linear uncertainty distribution is defined as follow:*

$$\varphi(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x \geq b \end{cases}$$

for convenience it is denoted in the paper by $\xi \sim \mathcal{L}(a,b)$ where $a < b$.

Definition 2.3. *An uncertainty distribution φ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \varphi(x) < 1$, and*

$$\lim_{x \to -\infty} \varphi(x) = 0, \lim_{x \to +\infty} \varphi(x) = 1.$$

Example 2.2. *The inverse uncertainty distribution of linear uncertain variable $\mathcal{L}(a,b)$ is $\varphi^{-1}(\alpha) = (1-\alpha)a + \alpha b$.*

Theorem 2.1. *[21] Let $\xi_1, \xi_2, \dots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\varphi_1, \varphi_2, \dots, \varphi_n$ respectively. If f is a strictly increasing with* *respect to* $\xi_1, \xi_2, ..., \xi_m$ *and strictly decreasing with respect to* $\xi_{m+1}, \xi_{m+2}, ..., \xi_n$, *then* $\xi = f(\xi_1, \xi_2, ..., \xi_n)$ *is an uncertain variable with an inverse uncertainty distribution* $\varphi^{-1}(\alpha) = f(\varphi_1^{-1}(\alpha), ..., \varphi_m^{-1}(\alpha), \varphi_{m+1}^{-1}(1-\alpha), ..., \varphi_n^{-1}(1-\alpha))$.

Theorem 2.2. *[21]Let* ξ *be an uncertain variable with regular uncertainty distribution* ϕ, *Then* $E[\xi] = \int_0^1 \varphi^{-1}(\alpha) d\alpha$.

Theorem 2.3. *[21] Let* $\xi_1, \xi_2, ..., \xi_n$ *be independent uncertain variables with regular uncertainty distributions* $\varphi_1, \varphi_2, ..., \varphi_n$ *respectively. If* f *is a strictly increasing with respect to* $\xi_1, \xi_2, ..., \xi_m$ *and strictly decreasing with respect to* $\xi_{m+1}, \xi_{m+2}, ..., \xi_n$, *then* $\xi = f(\xi_1, \xi_2, ..., \xi_n)$ *has an expected value* $E[\xi] = \int_0^1 f(\varphi_1^{-1}(\alpha), ..., \varphi_m^{-1}(\alpha), \varphi_{m+1}^{-1}(1-\alpha), ..., \varphi_n^{-1}(1-\alpha)) d\alpha$.

Theorem 2.4. *[21] Let* ξ *and* η *be independent uncertain variables with finite expected value. Then, for any real numbers* a *and* b *have* $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$.

Theorem 2.5. *[21] Let* $\xi_1, \xi_2, ..., \xi_n$ *be independent uncertain variables with regular uncertainty distributions* $\varphi_1, \varphi_2, ..., \varphi_n$ *respectively. If the function* $f(\xi_1, \xi_2, ..., \xi_n)$ *is a strictly increasing with respect to* $\xi_1, \xi_2, ..., \xi_m$ *and strictly decreasing with respect to* $\xi_{m+1}, \xi_{m+2}, ..., \xi_n$, *then* $\mathcal{M}\{f(\xi_1, \xi_2, ..., \xi_n) \leq 0\} \geq \alpha$ *if and only if* $f(\varphi_1^{-1}(\alpha), ..., \varphi_m^{-1}(\alpha), \varphi_{m+1}^{-1}(1-\alpha), ..., \varphi_n^{-1}(1-\alpha)) \leq 0$.

Example 2.3. *Let* $\xi \sim \mathcal{L}(a, b)$ *be a variable with linear uncertainty. Then its inverse is showed with* $\phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$, *and its expected value is* $E[\xi] = \int_0^1 ((1-\alpha)a + \alpha b) d\alpha = \frac{a+b}{2}$.

Consider a set of $DMU_k (k = 1, 2, ..., n)$ with m positive inputs $x_{ik} (i = 1, 2, ..., m)$ and s positive outputs $y_{rk} (r = 1, 2, ..., s)$. Now for evaluate $DMU_o = (\mathbf{x_o}, \mathbf{y_o})$, the linear programming BCC model will be formulated as follows [2]:

$$
\begin{aligned}
&\max \ \sum_{r=1}^{s} u_r y_{ro} + u_0 \qquad (2.1)\\
&s.t\\
&\sum_{i=1}^{m} v_i x_{io} = 1\\
&\sum_{r=1}^{s} u_r y_{rk} - \sum_{i=1}^{m} v_i x_{ik} + u_o \leq 0, \ k = 1, 2, ..., n\\
&u_r \geq 0, v_i \geq 0, \quad r = 1, 2, ..., s, \ i = 1, 2, ..., m.
\end{aligned}
$$

3 Uncertain DEA model

The BCC model requires inputs and outputs equipped with precise data. Nevertheless in real-world situations inputs and outputs may be unstable and complicated and therefore cannot be measured in an accurate manner. Consequently this conflict results in the investigation of uncertain DEA models. Decision-makers in real-word situations make their decisions in the indeterminacy state. To model indeterminacy there exist two mathematical systems one the probability theory and the other the uncertainty theory [19]. If there exists frequency in phenomena the probability theory is employed; otherwise the uncertain theory can be a powerful technique for resolving the drawback with no sample using the personal belief degree. For this purpose skilled consultants and experts should be invited to measure the belief degree. Throughout this approach we aimed to introduce a BCC model with uncertain inputs and outputs referred to as the uncertain BCC model. First new symbols and notations are presented:

$\tilde{\mathbf{x}}_k = (\tilde{x}_{1k}, \tilde{x}_{2k}, ..., \tilde{x}_{mk})$: the uncertain inputs vector of DMU_k, $k = 1, 2, ..., n$;

$\tilde{\mathbf{y}}_k = (\tilde{y}_{1k}, \tilde{y}_{2k}, ..., \tilde{y}_{rk})$: the uncertain inputs output of DMU_k, $k = 1, 2, ..., n$;

$\varphi_{ik}(x)$: The uncertainty distribution of $\tilde{x}_{ik}$, k=1,2,...,n, i=1,2,...,m;

$\psi_{rk}(x)$: the uncertainty distribution of $\tilde{y}_{rk}$, k=1,2,...,n, r=1,2,...,s;

α: is a predetermined confidence level;

$\mathcal{M}$: the uncertainty measure expressed in section 2;

Second the uncertain BCC model with uncertain inputs $(\tilde{x}_{ik})$ and uncertain outputs $(\tilde{y}_{rk})$ expresses as fallow:

$$\begin{aligned} &\max \quad \sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_0 \\ &s.t \\ &\sum_{i=1}^{m} v_i \tilde{x}_{io} = 1 \\ &\sum_{r=1}^{s} u_r \tilde{y}_{rk} - \sum_{i=1}^{m} v_i \tilde{x}_{ik} + u_o \le 0, \;\; k = 1, 2, ..., n \\ &u_r \ge 0, v_i \ge 0, \qquad r = 1, 2, ..., s, \;\; i = 1, 2, ..., m. \end{aligned} \tag{3.2}$$

To encounter with the uncertainty problem in the uncertain BCC model we introduced two models to crisp it:

1. Expected Value and Chance-Constrained method (EVCC).
2. Expected Value method (EV).

4 Crisp equivalents of the model

In this section, the uncertain BCC model is transformed to a crisp model using EVCC method.

4.1 EVCC method

Using EVCC method, the uncertain BCC model 3.2 is converted into crisp model and the new crisp model are solved with the help of specific software.

$$\begin{aligned} &\theta = \max \quad E\left[\sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_0\right] \\ &s.t \\ &\mathcal{M}\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} = 1\right\} \ge \alpha \\ &\mathcal{M}\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rk} - \sum_{i=1}^{m} v_i \tilde{x}_{ik} + u_o \le 0\right\} \ge \alpha, \;\; \forall k \\ &u_r \ge 0, v_i \ge 0, \qquad r = 1, 2, ..., s, \;\; i = 1, 2, ..., m. \end{aligned} \tag{4.3}$$

Definition 4.1. *A vector (ν, u, u_0) is called a feasible solution to the uncertain programming model (4.3) if for $k = 1, 2, ..., n$:*

$$\begin{aligned} &\mathcal{M}\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} = 1\right\} \ge \alpha \\ &\mathcal{M}\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rk} - \sum_{i=1}^{m} v_i \tilde{x}_{ik} + u_0 \le 0\right\} \ge \alpha, \\ &u_r \ge 0, v_i \ge 0, \qquad r = 1, 2, ..., s, \;\; i = 1, 2, ..., m. \end{aligned}$$

Definition 4.2. *A feasible solution (ν^*, u^*, u_0^*) is called an expected optimal solution to the uncertain programming model (4.3) if for any solution (ν, u, u_0): $E\left[\sum_{r=1}^{s} u_r^* \tilde{y}_{ro} + u_o^*\right] \ge E\left[\sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_0\right]$.*

Definition 4.3. *The greater the optimal objective value is the more efficient DMU_o is ranked.*

Theorem 4.1. *Assume $\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{in}$ are independent uncertain inputs with uncertainty distribution $\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{in}$ for each $i = 1, 2, \cdots, m$, and $\tilde{y}_{r1}, \tilde{y}_{r2}, \cdots, \tilde{y}_{rn}$ are independent uncertain outputs with uncertainty distribution $\psi_{r1}, \psi_{r2}, \cdots, \psi_{rn}$, for each $r = 1, 2, \cdots, s$. Then the uncertain programming model (4.3) is equivalent to the following model:*

$$\begin{aligned} &\theta = \max \quad \sum_{r=1}^{s} u_r \int_0^1 \varphi_{ro}^{-1}(\alpha) d\alpha + u_0 \\ &s.t \\ &\sum_{i=1}^{m} v_i \varphi_{io}^{-1}(\alpha) = 1 \\ &\sum_{r=1}^{s} u_r \psi_{rk}^{-1}(\alpha) - \sum_{i=1}^{m} v_i \varphi_{ik}^{-1}(1-\alpha) + \\ &u_0 \le 0, \qquad\qquad k = 1, 2, \cdots, n \\ &u_r \ge 0, v_i \ge 0, \;\; r = 1, 2, ..., s, \;\; i = 1, 2, ..., m. \end{aligned} \tag{4.4}$$

Proof. First according to Theorem (2.4) the objective function in (4.3) is rewritten as follows:

$$E\left[\sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_0\right] = E\left[\sum_{r=1}^{s} u_r \tilde{y}_{ro}\right] + E\left[u_0\right] = \sum_{r=1}^{s} u_r E[\tilde{y}_{ro}] + u_0.$$

The function $E\left[\tilde{y}_{ro}\right]$ is strictly increasing with respect to $\tilde{y}_{ro}$, for each k, as it follows from Theorems (2.1) and (2.3).

$$E\left[\sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_0\right] = \sum_{r=1}^{s} u_r \int_0^1 \varphi_{ro}^{-1}(\alpha)d\alpha + u_0$$

and the objective value has proved. Now, let us to prove the chance constraint as follows:

The function $\sum_{i=1}^{m} v_i \tilde{x}_{io} - 1$ is strictly increasing with respect to $\tilde{x}_{ij}$ for each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. According to Theorem (2.5):

$$\mathcal{M}\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} - 1 = 0\right\} \geq \alpha \Leftrightarrow$$
$$\sum_{i=1}^{m} v_i \varphi_{io}^{-1}(\alpha) = 1, \quad i = 1, 2, \cdots, m.$$

Also, the function $\sum_{r=1}^{s} u_r \tilde{y}_{rk} - \sum_{i=1}^{m} v_i \tilde{x}_{ik} + u_0$ is strictly increasing with respect to $\tilde{y}_{rk}$ and strictly decreasing with respect to $\tilde{x}_{ik}$ for each i and r, $i = 1, 2, \cdots, m$, $r = 1, 2, \cdots, s$ and $j = 1, 2, \cdots, n$. According to Theorem (2.5):

$$\mathcal{M}\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rk} - \sum_{i=1}^{m} v_i \tilde{x}_{ik} + u_o \leq 0\right\} \geq \alpha, \ \forall k$$

if and only if

$$\sum_{r=1}^{s} u_r \psi_{rk}^{-1}(\alpha) - \sum_{i=1}^{m} v_i \varphi_{ik}^{-1}(1-\alpha) + u_0 \leq 0 \ \forall k$$

Therefore, the theorem is proved.

4.2 EV method

The uncertain BCC model (3.2) is introduced as follows using EV method:

$$\theta = \max \quad E\left[\sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_0\right] \tag{4.5}$$
$$s.t$$
$$E\left[\sum_{i=1}^{m} v_i \tilde{x}_{io} - 1\right] = 0$$
$$E\left[\sum_{r=1}^{s} u_r \tilde{y}_{rk} - \sum_{i=1}^{m} v_i \tilde{x}_{ik} + u_o\right] \leq 0 \ \forall k$$
$$u_r \geq 0, v_i \geq 0, \quad r = 1, 2, ..., s, \ i = 1, 2, ..., m.$$

Definition 4.4. *(Efficiency) DMU_o is said to be efficient if and only if θ can achieve 1 where θ is the optimal value of (4.5).*

Theorem 4.2. *Assume $\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{in}$ are independent uncertain inputs with uncertainty distribution $\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{in}$ for each $i = 1, 2, \cdots, m$, and $\tilde{y}_{r1}, \tilde{y}_{r2}, \cdots, \tilde{y}_{rn}$ are independent uncertain inputs with uncertainty distribution $\psi_{r1}, \psi_{r2}, \cdots, \psi_{rn}$ for each $r = 1, 2, \cdots, n$. Then the uncertain programming model (4.5) is equivalent to the following model:*

$$\theta = \max \sum_{r=1}^{s} u_r \int_0^1 \psi_{ro}^{-1}(\alpha)d\alpha + u_0 \tag{4.6}$$
$$s.t$$
$$\sum_{i=1}^{m} v_i \int_0^1 \varphi_{io}^{-1}(\alpha)d\alpha = 1$$
$$\sum_{r=1}^{s} u_r \int_0^1 \varphi_{rk}^{-1}(\alpha)d\alpha$$
$$- \sum_{i=1}^{m} v_i \int_0^1 \varphi_{ik}^{-1}(1-\alpha)d\alpha + u_0 \leq 0 \quad \forall k$$
$$u_r \geq 0, v_i \geq 0, r = 1, 2, ..., s, \ i = 1, 2, ..., m.$$

where $\varphi_{1o}, \cdots, \varphi_{mo}$ and $\psi_{1o}, \cdots, \psi_{mo}$ are the regular uncertainty distributions of $\tilde{x}_{1o}, \tilde{x}_{2o}, \cdots, \tilde{x}_{no}$ and $\tilde{y}_{1o}, \tilde{y}_{2o}, \cdots, \tilde{y}_{no}$ respectively.

Proof. First according to Theorem (2.4) the objective function in (4.5) is rewritten as follows:

$$E\left[\sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_0\right] = E\left[\sum_{r=1}^{s} u_r \tilde{y}_{ro}\right] + E[u_0]$$
$$\sum_{r=1}^{s} u_r E[\tilde{y}_{ro}] + u_0$$

The function $\sum_{r=1}^{s} u_r E[\tilde{y}_{ro}]$ is strictly increasing with respect to $\tilde{y}_{ro}$, as it follows from Theorems (2.2) and (2.3).

$$E\left[\sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_0\right] =$$
$$\sum_{r=1}^{s} u_r \int_0^1 \psi_{ro}^{-1}(\alpha)d\alpha + u_0$$

So, the objective value has proved. Now, let us to prove the constraint as follows:

According to Theorem (2.4) the constraint function in (4.5) is rewritten as follows:

$$E\left[\sum_{i=1}^{m} v_i\tilde{x}_{io}-1\right]=E\left[\sum_{i=1}^{m} v_i\tilde{x}_{io}\right]-1$$
$$\sum_{i=1}^{m} v_iE\left[\tilde{x}_{io}\right]-1=\sum_{i=1}^{m} v_i\int_0^1 \varphi_{io}^{-1}(\alpha)d\alpha-1$$

Also, the function $\sum_{r=1}^{s} u_r\tilde{y}_{rk}-\sum_{i=1}^{m} v_i\tilde{x}_{ik}+u_o$ is strictly increasing with respect to $\tilde{y}_{rk}$ and strictly decreasing with respect to $\tilde{x}_{ik}$ for each $i=1,2,..,m$, $r=1,2,...,m$ and $k=1,2,...,n$. According to Theorems (2.2) and (2.3):

$$\begin{aligned}&E\left[\sum_{r=1}^{s} u_r\tilde{y}_{rk}-\sum_{i=1}^{m} v_i\tilde{x}_{ik}+u_o\right]\\&=\sum_{r=1}^{s} u_rE\left[\tilde{y}_{rk}\right]-\sum_{i=1}^{m} v_iE\left[\tilde{x}_{ik}\right]+E[u_o]\\&=\sum_{r=1}^{s} u_r\int_0^1 \varphi_{rk}^{-1}(\alpha)d\alpha\\&-\sum_{i=1}^{m} v_i\int_0^1 \varphi_{ik}^{-1}(1-\alpha)d\alpha+u_0\end{aligned}$$

Therefore, the theorem is proved.

Corollary 4.1. *If the $\tilde{y}_{io},\tilde{y}_{ro}$ and $\tilde{y}_{rk}$ are the independent uncertain variables of the form $\mathcal{L}(a,b)$ with $a,b\in\mathbb{R}$ and $a<b$, then the crisp equivalent of model (4.5) can be written as shown in model (4.7):*

$$\theta=\max \quad \sum_{r=1}^{s} u_r\left(\frac{a_{ro}+b_{ro}}{2}\right)+u_0 \tag{4.7}$$
$$\begin{aligned}&s.t\\&\sum_{i=1}^{m} v_i\left(\frac{a_{io}+b_{io}}{2}\right)=1,\\&\sum_{\mathrm{r}=1}^{s} u_{\mathrm{r}}\left(\frac{a_{ro}+b_{ro}}{2}\right)-\sum_{i=1}^{\mathrm{m}} v_{\mathrm{i}}\left(\frac{a_{io}+b_{io}}{2}\right)\\&+u_0\le 0\\&u_r\ge 0,\quad r=1,...,s,\quad v_i\ge 0,\quad i=1,...,m.\end{aligned}$$

Proof. According to Theorem (2.2)

$$\begin{aligned}E[a\xi]&=\int_0^1 a\phi^{-1}(1-\alpha)d\alpha=a\int_0^1 \phi^{-1}(\alpha)d\alpha\\&=aE[\xi]\end{aligned}$$

Now according to Theorem (2.4) rewrite (4.5) as follows:

$$\theta=\max \quad \sum_{r=1}^{s} u_rE\left[\tilde{y}_{ro}\right]+u_0 \tag{4.8}$$
$$\begin{aligned}&s.t\\&\sum_{i=1}^{m} v_iE\left[\tilde{x}_{io}\right]-1=0\\&\sum_{r=1}^{s} u_rE\left[\tilde{y}_{rk}\right]-\sum_{i=1}^{m} v_iE\left[\tilde{x}_{ik}\right]+u_o\le 0 \quad \forall k\\&u_r\ge 0, v_i\ge 0, \quad r=1,2,...,s,\ i=1,2,...,m.\end{aligned}$$

From the linearity property of expected value operator the crisp equivalent of uncertain BCC model in model (4.8) can be formulated as follows:

$$\theta=\max \quad \sum_{r=1}^{s} u_r\left(\frac{a_{ro}+b_{ro}}{2}\right)+u_0 \tag{4.9}$$
$$\begin{aligned}&s.t\\&\sum_{i=1}^{m} v_i\left(\frac{a_{io}+b_{io}}{2}\right)=1,\\&\sum_{\mathrm{r}=1}^{s} u_{\mathrm{r}}\left(\frac{a_{ro}+b_{ro}}{2}\right)-\sum_{i=1}^{\mathrm{m}} v_{\mathrm{i}}\left(\frac{a_{io}+b_{io}}{2}\right)\\&+u_0\le 0\\&u_r\ge 0,\quad r=1,...,s,\quad v_i\ge 0,\quad i=1,...,m.\end{aligned}$$

5 Practical Example

In this section we applied the BCC uncertain models to an example of an Iranian bank to see how the target customer accepts a new banking product before the promotion and announcement phase. To this end, 20 branch managers (banking experts) were selected to answer to an eight-item questionnaire (Table 1) to produce the linear uncertain variable $\xi\sim\mathcal{L}(a,b)$ to test the newly implemented models (Tables 2 and 3).

Because of classical banking system in order to employ a new product in branches first a cost-benefit analysis and must be performed; if result is positive; it will be directly announced in branches while the customers' feedback on the product is not considered. In several cases they solely performed the cost-benefit analysis phase and then in practice customers did not accept the

Table 1: Input and output questions.

input question
To what percent do you predict the acceptability of this product would be for customers?
To what percent does this product have competitive power with the rivals?
To what percent does this product have the ability of WOM (word-of-mouth)?
Resources that this product managed to grab is a number between and ...
output question
To what percent will you predict the success of this product?
To what percent do you think this product will attract the customers?
To what percent will this product could grab rival banks' resources?
The loan which must be paid for the resources of this product is a number between ... and ...

product thereby wasting a considerable amount of money in the banking system. Results for model

Table 2: DMUs with four uncertain inputs.

DMU_i	1	2	3	4
1	$\mathcal{L}(60,70)$	$\mathcal{L}(60,70)$	$\mathcal{L}(60,70)$	$\mathcal{L}(20,25)$
2	$\mathcal{L}(70,80)$	$\mathcal{L}(60,70)$	$\mathcal{L}(40,50)$	$\mathcal{L}(50,120)$
3	$\mathcal{L}(80,90)$	$\mathcal{L}(70,80)$	$\mathcal{L}(70,80)$	$\mathcal{L}(50,80)$
4	$\mathcal{L}(70,100)$	$\mathcal{L}(50,100)$	$\mathcal{L}(40,100)$	$\mathcal{L}(1,3)$
5	$\mathcal{L}(10,15)$	$\mathcal{L}(10,15)$	$\mathcal{L}(5,10)$	$\mathcal{L}(0.3,0.5)$
6	$\mathcal{L}(30,60)$	$\mathcal{L}(30,40)$	$\mathcal{L}(30,50)$	$\mathcal{L}(30,40)$
7	$\mathcal{L}(50,100)$	$\mathcal{L}(50,100)$	$\mathcal{L}(30,100)$	$\mathcal{L}(1,2)$
8	$\mathcal{L}(70,90)$	$\mathcal{L}(60,70)$	$\mathcal{L}(30,60)$	$\mathcal{L}(70,80)$
9	$\mathcal{L}(30,70)$	$\mathcal{L}(40,70)$	$\mathcal{L}(40,80)$	$\mathcal{L}(20,50)$
10	$\mathcal{L}(80,90)$	$\mathcal{L}(60,80)$	$\mathcal{L}(40,80)$	$\mathcal{L}(7,10)$
11	$\mathcal{L}(5,10)$	$\mathcal{L}(10,20)$	$\mathcal{L}(10,15)$	$\mathcal{L}(1.5,2.5)$
12	$\mathcal{L}(10,30)$	$\mathcal{L}(17,49)$	$\mathcal{L}(11,35)$	$\mathcal{L}(20,30)$
13	$\mathcal{L}(20,40)$	$\mathcal{L}(20,30)$	$\mathcal{L}(20,40)$	$\mathcal{L}(10,15)$
14	$\mathcal{L}(50,80)$	$\mathcal{L}(70,80)$	$\mathcal{L}(50,60)$	$\mathcal{L}(20,30)$
15	$\mathcal{L}(70,100)$	$\mathcal{L}(70,100)$	$\mathcal{L}(80,100)$	$\mathcal{L}(10,30)$
16	$\mathcal{L}(60,80)$	$\mathcal{L}(80,100)$	$\mathcal{L}(70,80)$	$\mathcal{L}(60,70)$
17	$\mathcal{L}(20,30)$	$\mathcal{L}(30,50)$	$\mathcal{L}(10,30)$	$\mathcal{L}(10,30)$
18	$\mathcal{L}(50,70)$	$\mathcal{L}(40,50)$	$\mathcal{L}(20,50)$	$\mathcal{L}(10,30)$
19	$\mathcal{L}(20,40)$	$\mathcal{L}(10,20)$	$\mathcal{L}(60,90)$	$\mathcal{L}(10,30)$
20	$\mathcal{L}(50,80)$	$\mathcal{L}(30,60)$	$\mathcal{L}(30,50)$	$\mathcal{L}(30,60)$

(4.4) showed that 11 branches supported this new product as an attractive use case showing a 55 percent success likelihood for the product before the announcement phase (Table 4).

The result of model (4.6) is as same as model (4.4). According to the CPU times reported it can be observed that model (4.4) is run in 140 seconds depending on the combination of confidence levels while model (4.6) is run in 3455 seconds depending on the combination of confidence levels. It seems that model (4.4) has the best performance in terms of CPU running time in big data

Table 3: DMUs with four uncertain outputs.

DMU_i	1	2	3	4
1	$\mathcal{L}(50,70)$	$\mathcal{L}(70,80)$	$\mathcal{L}(40,50)$	$\mathcal{L}(14,17.5)$
2	$\mathcal{L}(50,70)$	$\mathcal{L}(50,60)$	$\mathcal{L}(40,50)$	$\mathcal{L}(35,84)$
3	$\mathcal{L}(80,90)$	$\mathcal{L}(80,90)$	$\mathcal{L}(80,90)$	$\mathcal{L}(35,56)$
4	$\mathcal{L}(50,100)$	$\mathcal{L}(70,100)$	$\mathcal{L}(80,100)$	$\mathcal{L}(0.7,2.1)$
5	$\mathcal{L}(10,20)$	$\mathcal{L}(10,15)$	$\mathcal{L}(5,10)$	$\mathcal{L}(0.21,0.35)$
6	$\mathcal{L}(30,50)$	$\mathcal{L}(30,40)$	$\mathcal{L}(30,60)$	$\mathcal{L}(21,28)$
7	$\mathcal{L}(50,100)$	$\mathcal{L}(30,100)$	$\mathcal{L}(30,100)$	$\mathcal{L}(0.7,1.4)$
8	$\mathcal{L}(70,90)$	$\mathcal{L}(70,90)$	$\mathcal{L}(20,30)$	$\mathcal{L}(49,56)$
9	$\mathcal{L}(20,60)$	$\mathcal{L}(30,70)$	$\mathcal{L}(20,50)$	$\mathcal{L}(14,35)$
10	$\mathcal{L}(50,80)$	$\mathcal{L}(50,70)$	$\mathcal{L}(40,70)$	$\mathcal{L}(4.9,7)$
11	$\mathcal{L}(10,15)$	$\mathcal{L}(10,20)$	$\mathcal{L}(10,20)$	$\mathcal{L}(1.05,1.7)$
12	$\mathcal{L}(15,20)$	$\mathcal{L}(13,20)$	$\mathcal{L}(34,51)$	$\mathcal{L}(14,21)$
13	$\mathcal{L}(20,30)$	$\mathcal{L}(10,20)$	$\mathcal{L}(10,15)$	$\mathcal{L}(7,10.5)$
14	$\mathcal{L}(50,90)$	$\mathcal{L}(60,80)$	$\mathcal{L}(1,50)$	$\mathcal{L}(14,21)$
15	$\mathcal{L}(80,100)$	$\mathcal{L}(90,100)$	$\mathcal{L}(70,100)$	$\mathcal{L}(7,21)$
16	$\mathcal{L}(20,70)$	$\mathcal{L}(80,90)$	$\mathcal{L}(80,90)$	$\mathcal{L}(42,49)$
17	$\mathcal{L}(30,60)$	$\mathcal{L}(10,30)$	$\mathcal{L}(10,20)$	$\mathcal{L}(7,21)$
18	$\mathcal{L}(20,70)$	$\mathcal{L}(30,80)$	$\mathcal{L}(20,50)$	$\mathcal{L}(7,21)$
19	$\mathcal{L}(20,50)$	$\mathcal{L}(20,40)$	$\mathcal{L}(10,20)$	$\mathcal{L}(7,21)$
20	$\mathcal{L}(30,70)$	$\mathcal{L}(50,80)$	$\mathcal{L}(30,60)$	$\mathcal{L}(21,42)$

Table 4: Results of evaluating the DMUs with $\alpha = 0.5$.

DMU_1	DMU_2	DMU_3	DMU_4
0.9126	0.9160	1	1
Inefficiency	Inefficiency	efficiency	efficiency
DMU_5	DMU_6	DMU_7	DMU_8
1	0.9160	1	1
efficiency	Inefficiency	efficiency	efficiency
DMU_9	DMU_{10}	DMU_{11}	DMU_{12}
1	0.8855	1	0.8891
efficiency	Inefficiency	efficiency	Inefficiency
DMU_{13}	DMU_{14}	DMU_{15}	DMU_{16}
0.7271	0.8623	1	1
Inefficiency	Inefficiency	efficiency	efficiency
DMU_{17}	DMU_{18}	DMU_{19}	DMU_{20}
1	0.8947	0.8613	1
efficiency	Inefficiency	Inefficiency	efficiency

whereas model (4.6) has the worst performance.

6 Conclusion

In this paper we aimed to explain an uncertain BCC model with inherent complexity for uncertain models. Then by use of two methods we attempted to crisp the uncertain model. Finally an applicable example regarding the Iranian banking system was proposed to document the new models. For this purpose we selected 20 branch managers (bank experts) to respond to an eight-item questionnaire to examine the success probability for a new product before the announcement phase. This method assisted bank CEOs in choosing the product which might be interesting to the costumer. Further studies can find a new method for saving cost in bank products before promotion and announcement phase. As a future study direction the proposed variance instead of the expected value in objective functions with uncertain variables can be taken into account. Also the problem of this paper with normal type uncertain variables can be studied.

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